

Lecture 12 - Approximating Area & Definite Integrals

Tuesday, August 24, 2021 11:41 PM

Midterm 2

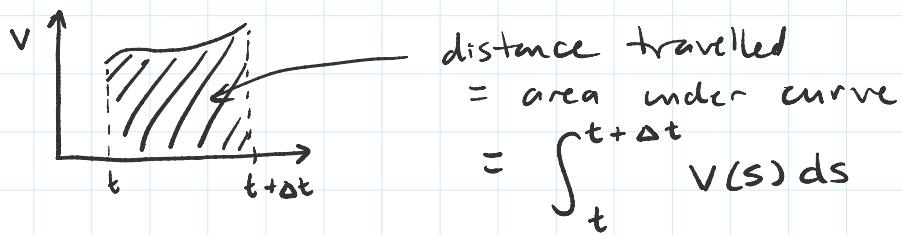
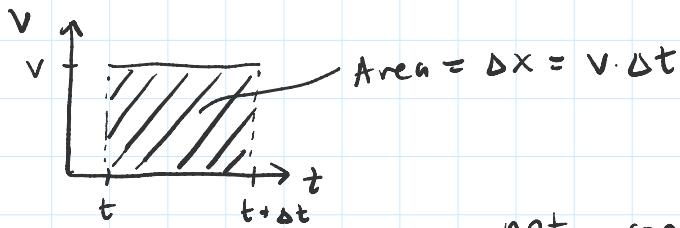
- Monday 8/30/21 available 1:30 pm - 1:30 am i.e. 12 hours.
90 mins to complete (start before 11:59 pm)
- Review lecture 9:30 am to 10:50 am
- Review OH 11 am to 11:59 am (recorded) \rightarrow hw 4 problems

Chapter 5 Integration

- Read sections 5.1 & 5.2
(this material is not on MT2)

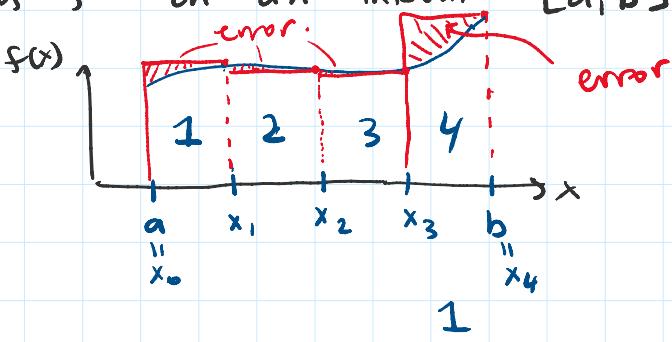
5.1 Approximating area

- Move with constant velocity v for a time Δt ,
distance travelled is $\Delta x = v \cdot \Delta t$



$$\begin{aligned}\text{distance travelled} \\ &= \text{area under curve} \\ &= \int_t^{t+\Delta t} v(s) ds\end{aligned}$$

- Say we want to approximate area under graph of f on an interval $[a, b]$.



- $[x_0, x_1]$ choose $f(x_1)$
- $[x_1, x_2]$ choose $f(x_2)$
- $[x_2, x_3]$ choose $f(x_3)$
- $[x_3, x_4]$ choose $f(x_4)$

$$\begin{array}{c}
 x_0 \quad x_4 \\
 \text{Area under } f \text{ on } [a, b] \approx \underbrace{f(x_1) \cdot (x_1 - x_0)}_1 + \underbrace{f(x_2) \cdot (x_2 - x_1)}_2 \\
 \quad \quad \quad + \underbrace{f(x_3) \cdot (x_3 - x_2)}_3 + \underbrace{f(x_4) \cdot (x_4 - x_3)}_4 \\
 \text{choose } x_3, x_4
 \end{array}$$

In general, subdivide interval $[a, b]$ into

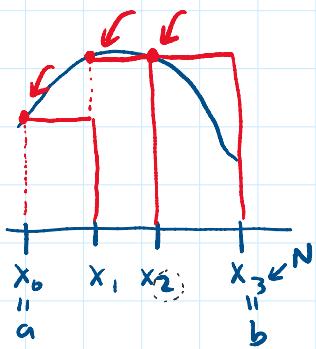
$$\begin{array}{l}
 \text{partition of } [a, b] \left\{ \begin{array}{l} [x_0, x_1], [x_1, x_2], \dots, [x_{N-2}, x_{N-1}], [x_{N-1}, x_N] \\ a \qquad \qquad \qquad b \end{array} \right\} N \text{ subintervals} \\
 x_0 < x_1 < x_2 < \dots < x_{N-2} < x_{N-1} < x_N
 \end{array}$$

assume intervals have same width $\Delta x = \frac{b-a}{N}$

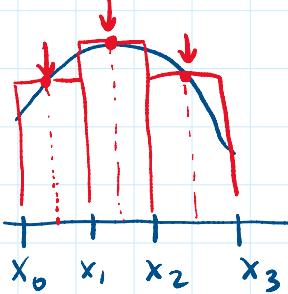
$$X_j = a + \Delta x j = a + \frac{b-a}{N} j \quad j = 0, 1, 2, \dots, N$$

$$x_0 = a + \left(\frac{b-a}{N} \right) \cdot 0 = a \quad \checkmark$$

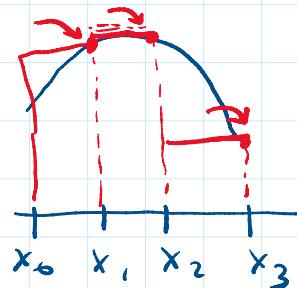
$$x_N = a + \left(\frac{b-a}{N} \right) \cdot N = a + (b-a) = b \quad \checkmark.$$



Left



Middle

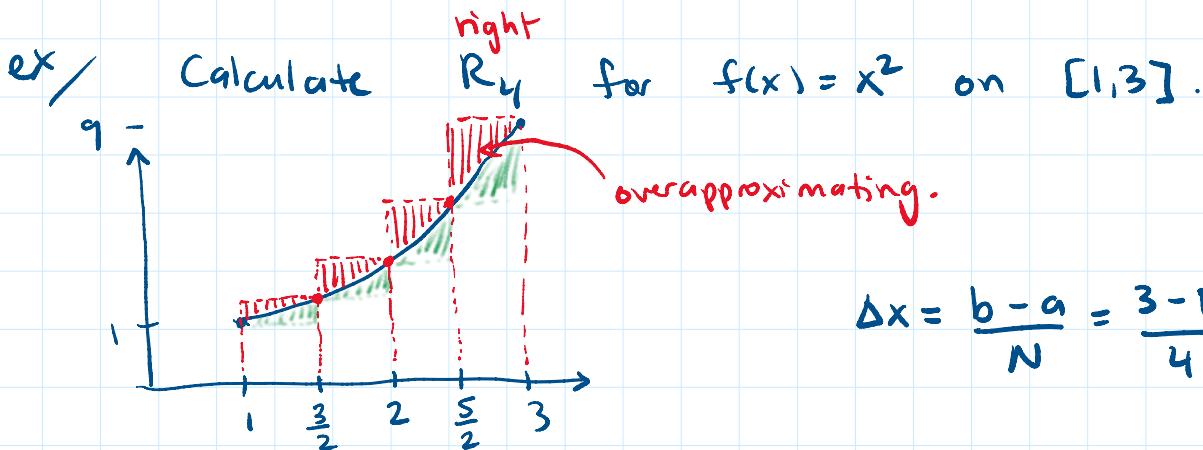


Right

$$\begin{aligned}
 \text{Left } L_N &= f(x_0) \cdot \Delta x + f(x_1) \cdot \Delta x + \dots + f(x_{N-1}) \cdot \Delta x \\
 &= \sum_{j=0}^{N-1} f(x_j) \Delta x
 \end{aligned}$$

$$\begin{aligned}
 \text{Middle } M_N &= f\left(\frac{x_0+x_1}{2}\right) \cdot \Delta x + \dots + f\left(\frac{x_{N-1}+x_N}{2}\right) \Delta x \\
 &= \sum_{j=0}^{N-1} f\left(\frac{x_j+x_{j+1}}{2}\right) \cdot \Delta x
 \end{aligned}$$

$$\text{Right } R_N = f(x_1) \Delta x + \dots + f(x_N) \Delta x \\ = \sum_{j=0}^{N-1} f(x_{j+1}) \Delta x.$$



$$R_4 = f(3/2) \cdot \Delta x + f(2) \cdot \Delta x + f(5/2) \cdot \Delta x + f(3) \cdot \Delta x \\ = \left(\frac{3}{2}\right)^2 \cdot \frac{1}{2} + (2)^2 \cdot \frac{1}{2} + \left(\frac{5}{2}\right)^2 \cdot \frac{1}{2} + (3)^2 \cdot \frac{1}{2} \\ = 10.75$$

actual area under curve $\frac{26}{3} \approx 8.7$

R_N overapproximates monotone increasing funcs

R_N underapproximates monotone decreasing funcs.

L_N overapproximates decreasing funcs

L_N underapproximates increasing funcs.

Summation Notation: $\sum_{j=n}^m a_j = a_n + a_{n+1} + \dots + a_m$

e.g. $\sum_{j=1}^4 j^2 = (1)^2 + (2)^2 + (3)^2 + (4)^2$

$$\sum_{j=n}^m (a_j + b_j) = \left(\sum_{j=n}^m a_j \right) + \left(\sum_{j=n}^m b_j \right)$$

} linearity

$$\sum_{j=n}^m C a_j = C \sum_{j=n}^m a_j$$

$$\sum_{j=n}^m C a_j = C \sum_{j=n}^m a_j$$

$$\sum_{j=n}^m C = C \sum_{j=n}^m 1 = C(n-m+1)$$

power sum laws

$$(1) \sum_{j=1}^M j = \frac{M(M+1)}{2} \xrightarrow{(M \rightarrow \infty)} \frac{M^2}{2}$$

$$\text{e.g. } \sum_{j=1}^{10} j = \overbrace{1+2+3+4+5+6+7+8+9+10} = 55.$$

$$\sum_{j=1}^M 1 = M$$

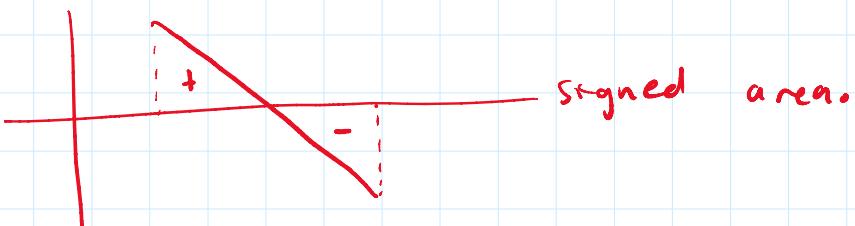
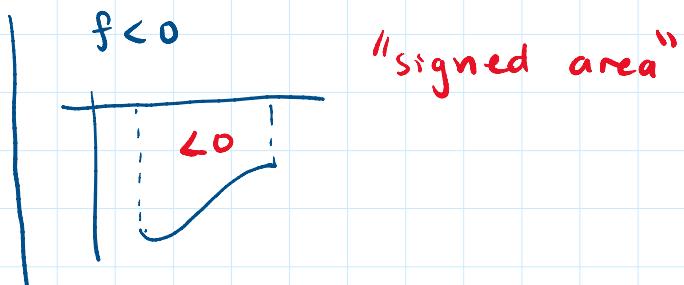
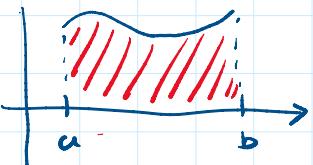
$$(2) \sum_{j=1}^M j^2 = \frac{M(M+1)(2M+1)}{6} \xrightarrow{(M \rightarrow \infty)} \frac{2M^3}{6} = \frac{M^3}{3}$$

$$(3) \sum_{j=1}^M j^3 = \frac{M^2(M+1)^2}{4}$$

Thm: If f is continuous on $[a, b]$, then the approximations L_N, M_N, R_N converge to the same limit as $N \rightarrow \infty$,

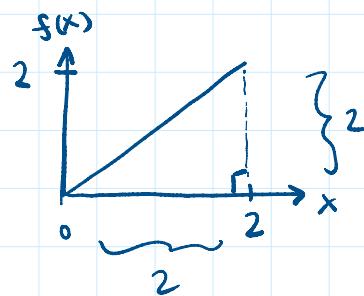
$$\lim_{N \rightarrow \infty} L_N = \lim_{N \rightarrow \infty} M_N = \lim_{N \rightarrow \infty} R_N = \int_a^b f(x) dx.$$

$f > 0$
"area under graph of f "



ex/ Find the area under graph of $f(x) = x$
on $[0, 2]$.

(i) Geometry



$$A = \frac{1}{2}bh = \frac{1}{2}(2)(2) \\ = 2.$$

(ii) Using L_N as $N \rightarrow \infty$

$$L_N = \sum_{j=0}^{N-1} f(x_j) \Delta x$$

$$= \Delta x \sum_{j=0}^{N-1} f(x_j)$$

$$= \Delta x \sum_{j=0}^{N-1} x_j$$

$$= \frac{2}{N} \sum_{j=0}^{N-1} \frac{2}{N} j = \frac{2}{N} \cdot \frac{2}{N} \sum_{j=1}^{N-1} j$$

$$= \frac{2}{N} \cdot \frac{2}{N} \cdot \frac{(N-1)(N)}{2}$$

power sum law (1) with $M = N - 1$

$$\lim_{N \rightarrow \infty} L_N = 2 \frac{(N-1)(N)}{N^2} = \boxed{2}$$

$$\left(\lim_{N \rightarrow \infty} \frac{N^2 - N}{N^2} = \lim_{N \rightarrow \infty} \left(1 - \frac{1}{N}\right) = 1 \right)$$

5.2 The Definite Integral

- Remove assumption of middle/left/right points for rectangle height & remove assumption of equal width subintervals
↳ Riemann Sums
- Partition P of $[a, b]$

• Partition P of $[a, b]$

$$a = x_0 < x_1 < x_2 < \dots < x_{N-1} < x_N = b$$

→ Subintervals $[x_{j-1}, x_j]$ $j = 1, \dots, N$

→ Width $\Delta x_j = x_j - x_{j-1}$ $j = 1, \dots, N$

→ Inside each $[x_{j-1}, x_j]$, choose a sample $C = \{c_1, \dots, c_N\}$
point $c_j \in [x_{j-1}, x_j]$ for rectangle height $f(c_j)$

→ The norm of partition

$$\|P\| = \max_{1 \leq j \leq N} \{\Delta x_j\} = \max \{\Delta x_1, \Delta x_2, \dots, \Delta x_N\}.$$

To define definite integrals, not sufficient $N \rightarrow \infty$ ($\Delta x = \frac{b-a}{N}$) unless const. width

$$\|P\| \rightarrow 0.$$

Riemann Sum

$$\begin{aligned} R(f, P, C) &= \sum_{j=1}^N f(c_j) \cdot (x_j - x_{j-1}) \\ &= \sum_{j=1}^N f(c_j) \downarrow \Delta x_j \end{aligned}$$

Def:

The definite integral of f over $[a, b]$, denoted $\int_a^b f(x) dx$,
is the limit (if it exists) of Riemann sums
as $\|P\| \rightarrow 0$

$$\int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} R(f, P, C) = \lim_{\substack{N \rightarrow \infty \\ (\|P\| \rightarrow 0)}} \sum_{j=1}^N f(c_j) \Delta x_j$$

ex/

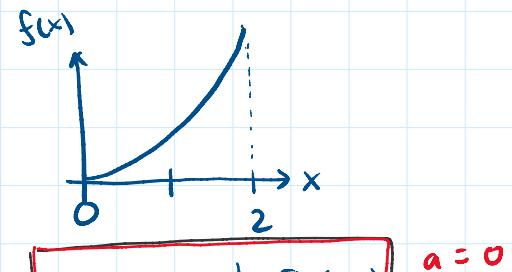
$$f(x) = x^2$$

compute

$$\int_0^2 x^2 dx$$

• Riemann sum

$$\frac{1}{N-1}$$



• Riemann sum

$$\cdot R_N = \sum_{j=0}^{N-1} f(x_{j+1}) \cdot \Delta x \quad \Delta x = \frac{2}{N}$$

$$0 \xrightarrow{\Delta x} 2$$

$$x_{j+1} = a + \frac{b-a}{N}(j+1) = \frac{2}{N} \cdot (j+1)$$

$a=0$
 $b=2$

$$= \Delta x \sum_{j=0}^{N-1} \underline{\underline{f(x_{j+1})}}$$

$$= \frac{2}{N} \sum_{j=0}^{N-1} (x_{j+1})^2 = \frac{2}{N} \sum_{j=0}^{N-1} \left(\frac{2}{N} (j+1) \right)^2$$

$$= \frac{8}{N^3} \sum_{j=0}^{N-1} (j+1)^2$$

$$= \frac{8}{N^3} \sum_{j=0}^{N-1} (j^2 + 2j + 1)$$

$$= \frac{8}{N^3} \left[\underbrace{\left(\sum_{j=0}^{N-1} j^2 \right)}_{\sim \frac{N^3}{3}} + \underbrace{\left(\sum_{j=0}^{N-1} 2j \right)}_{\sim \frac{N^2}{2} \cdot 2} + \underbrace{\left(\sum_{j=0}^{N-1} 1 \right)}_{\sim N} \right]$$

$$\int_0^2 x^2 dx = \lim_{N \rightarrow \infty} R_N$$

$$= \lim_{N \rightarrow \infty} \frac{8}{N^3} \left[\frac{N^3}{3} + N^2 + N \right]$$

$$= \lim_{N \rightarrow \infty} \left[\frac{8}{3} + \frac{8}{N} + \frac{8}{N^2} \right]$$

$$= \frac{8}{3}.$$

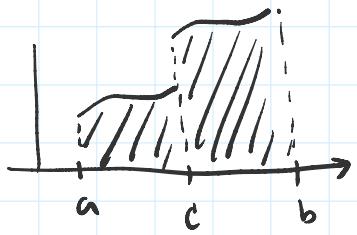
If the limit of Riemann sums for f exists on $[a, b]$, then say f is integrable on $[a, b]$.

Thm

If f is continuous on $[a, b]$ (except at finitely many jump discontinuities), then f is integrable on $[a, b]$.



1



Notation

Riemann sum

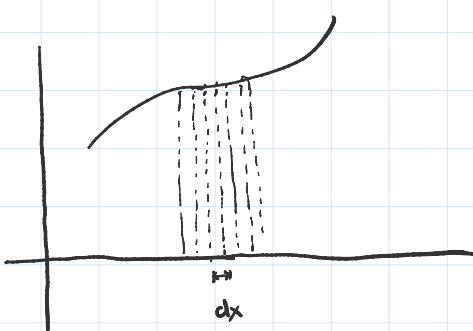
$$\sum_{j=1}^N f(c_j) \Delta x_j$$

Integral

$$\int_a^b f(x) dx$$

$dx : \Delta x \rightarrow 0$

$$\int \leftarrow \sum$$



adding up infinitesimally thin
rectangles gives full
area.

OH tomorrow at 11 am.