

## Lecture 1 - Pre-calculus review

Wednesday, July 28, 2021 2:28 PM

Students: Read sections 1.1 - 1.6 of text

Functions : domain range

- A function  $f: A \rightarrow B$  is a map from a set  $A$  to a set  $B$ , i.e., to every  $x$  in  $A$  (denoted  $x \in A$ ),  $f$  assigns an element  $f(x)$  in  $B$  (i.e.,  $x \in A \mapsto f(x) \in B$ ).
- In this course,  $A$  &  $B$  will usually be some interval subset of the set of the real numbers, denoted  $\mathbb{R}$ .
- Interval notation:

Open:  $(a, b) = \{x \in \mathbb{R} : a < x < b\}$

$\overbrace{\quad}^{(a,b)}$

$a \text{-----} b \rightarrow \mathbb{R}$

Closed:  $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

$\overbrace{\quad}^{[a,b]}$

$a \text{-----} b \rightarrow \mathbb{R}$

Half-open  $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$

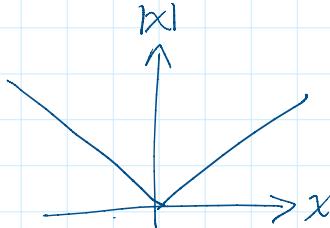
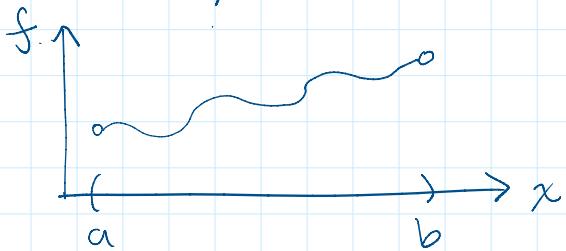
$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ .

In this notation,  $\mathbb{R} = (-\infty, \infty)$ .

$\downarrow$  (or some other type of interval)

- For a function  $f: (a, b) \rightarrow \mathbb{R}$ , its graph is the set of all ordered pairs  $\{(x, f(x)) : x \in (a, b)\}$ .

We usually draw this:



ex/ The absolute value function

$$|x|: \mathbb{R} \rightarrow [0, \infty)$$

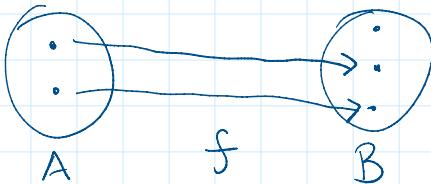
$|x| = \text{"distance between } 0 \text{ and } x\text{"} \geq 0$  (non-negative)

Note  $|x| = |-x|$ ,  $|xy| = |x||y|$

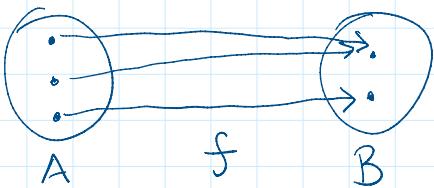
and triangle inequality  $|x+y| \leq |x| + |y|$

## Inverses of Functions

- Say  $f: A \rightarrow B$  is injective (or one-to-one)  
if: for  $x, y \in A$  and  $x \neq y$ , then  $f(x) \neq f(y)$ .



injective ✓

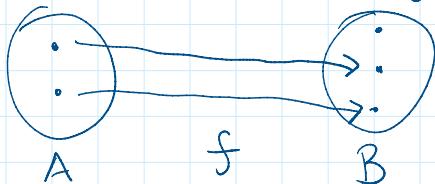


not injective X

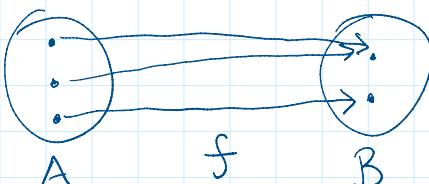
- Say  $f: A \rightarrow B$  is surjective (or onto)

if: for every  $y \in B$ , there is some  $x \in A$

such that  $f(x) = y$ . (using same examples as above:)



not surjective X



surjective ✓

- If  $f: A \rightarrow B$  is both injective and surjective, we say it is bijective.

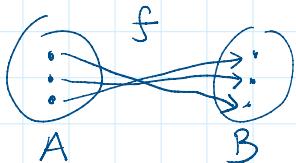
Bijective Functions are invertible..

- If  $f: A \rightarrow B$  is bijective, there exists  $f^{-1}: B \rightarrow A$  such that

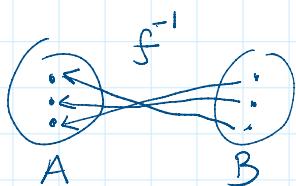
$$f^{-1}(f(x)) = x \text{ for all } x \in A,$$

$$f(f^{-1}(y)) = y \text{ for all } y \in B.$$

ex/



injective ✓  
surjective ✓  $\Rightarrow$  bijective.



## Examples

- Linear functions  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = ax + b$ .

If  $a \neq 0$ , then  $f$  is bijective:

$\Rightarrow$  injective: If  $x \neq y$ , then  $f(x) \neq f(y)$   
since  $ax + b \neq ay + b$ .

Alternative rel. acc. to  $f(x) = f(y)$  i.e.  $7$  . . . . .

• injective: if  $x \neq y$ , then  $f(x) \neq f(y)$   
since  $ax+b \neq ay+b$ .

Alternatively, assume  $f(x) = f(y)$ , i.e,

$$ax+b = ay+b$$

$$\Rightarrow ax = ay$$

$$\Rightarrow x = y \quad (\text{need } a \neq 0!)$$

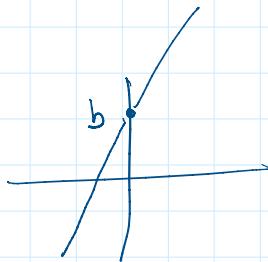
} explain this type  
of proof of  
injectivity, as the  
contrapositive of the  
definition.

→ surjective: let  $y \in \mathbb{R}$ . Want to find

$x$  such that  $f(x) = y$ .

$$\Rightarrow ax+b = y$$

$$\Rightarrow x = \frac{y-b}{a} \quad (\text{again, need } a \neq 0).$$



Intuitively, we can see both from its graph

(injective: passes horizontal line test)

(surjective: covers all vertical values in range)

$\Rightarrow f^{-1}$  exists,  $f^{-1}(y) = \frac{y-b}{a}$ ,

[check yourself:

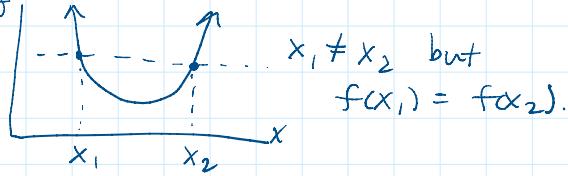
$$f(f^{-1}(y)) = y,$$

$$f^{-1}(f(x)) = x.$$

### Quadratic Functions:

$f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = ax^2 + bx + c$  ( $a \neq 0$ )

$f$  is not injective



$f$  is not surjective onto  $\mathbb{R}$ , but if we restrict to its image  $f(\mathbb{R}) = \{f(x) : x \in \mathbb{R}\}$ , then  $f: \mathbb{R} \rightarrow f(\mathbb{R})$  is surjective.  
e.g.  $f(x) = x^2$ ,  $f: \mathbb{R} \rightarrow [0, \infty)$  is surjective.

\* Explain this more generally for  $f: A \rightarrow B$ .

Given any  $f$ ,  $f: A \rightarrow f(A)$  is surjective.

Thus, if  $f: A \rightarrow B$  is injective,

$f: A \rightarrow f(A)$  is bijective and thus,  
there exists an inverse

$$f^{-1}: f(A) \rightarrow A.$$

## Common functions:

### Trig. functions

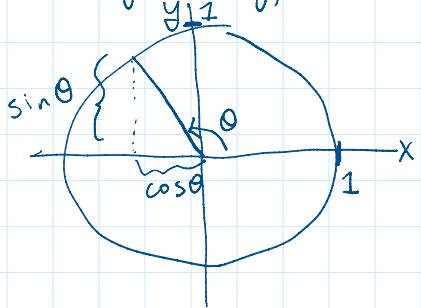
$$x^2 + y^2 = h^2.$$

$$\cos^2 \theta + \sin^2 \theta = 1.$$

$$\cos \theta = x/h, \quad \sin \theta = y/h,$$

$90^\circ$

As expressed,  $\cos, \sin: (0, \pi/2) \rightarrow (0, 1)$ .  
More generally, values given on unit circle



$$\cos: \mathbb{R} \rightarrow [-1, 1]$$

$$\sin: \mathbb{R} \rightarrow [-1, 1].$$

- $2\pi$ -periodic, so not injective.
- As defined, is surjective.

### Other trig. functions:

$$\tan \theta = \sin \theta / \cos \theta, \quad \cot \theta = \cos \theta / \sin \theta$$

$$\sec \theta = 1 / \cos \theta, \quad \csc \theta = 1 / \sin \theta$$

(where defined, i.e. denominator  $\neq 0$ )

• See text for graphs & properties of trig. functions.

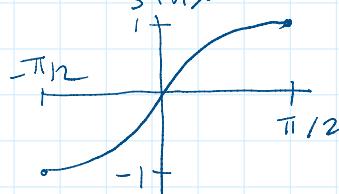
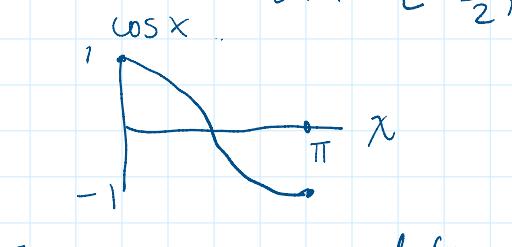
### Inverse Trig. Functions

Since  $\cos: \mathbb{R} \rightarrow [-1, 1]$ ,  $\sin: \mathbb{R} \rightarrow [-1, 1]$  are not injective, can't define an inverse. However, we can restrict the domain so they are:

check

$$\cos: [0, \pi] \rightarrow [-1, 1] \quad \text{are bijective}$$

$$\sin: [-\pi/2, \pi/2] \rightarrow [-1, 1]$$





Thus, we can define

$$\cos^{-1} \text{ or } \arccos : [-1, 1] \rightarrow [0, \pi]$$

$$\sin^{-1} \text{ or } \arcsin : [-1, 1] \rightarrow [-\pi/2, \pi/2]$$

$\theta = \cos^{-1}(x)$  is the unique angle in  $[0, \pi]$  such that  
 $x = \cos \theta$  (for  $x \in [-1, 1]$ )

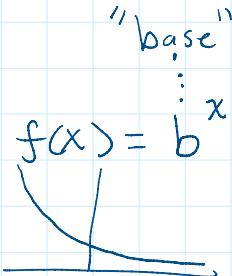
$\theta = \sin^{-1}(y)$  is the unique angle in  $[-\pi/2, \pi/2]$   
such that  $y = \sin \theta$  (for  $y \in [-1, 1]$ ).

We can similarly define the inverses of the other trig. functions, by appropriately restricting the domain (see text).

## Exponentials

$$f : \mathbb{R} \rightarrow (0, \infty), \quad f(x) = b^x$$

- $b < 1$  "exp. decay"



$$(b > 0)$$

Properties:

$$b^x b^y = b^{x+y}$$

$$b^x / b^y = b^x b^{-y} = b^{x-y}$$

$$(b^x)^y = b^{xy}$$

- $b > 1$  "exp. growth"



- $b = 1$ , constant,  $f(x) = 1$ .

For the above  $f$  and  $b \neq 1$ , it is bijective

$\Rightarrow$  there exists an inverse.

Call the inverse of  $f(x) = b^x$

$\log_b : (0, \infty) \rightarrow \mathbb{R}$  "Logarithm, base  $b$ "

$$x = \log_b(y) \Leftrightarrow b^x = y$$

$$\begin{pmatrix} b^{\log_b y} = y \\ \log_b(b^x) = x \end{pmatrix}$$

Properties:  $\log_b(a) + \log_b(c) = \log_b(ac)$

$$\log_b(x^y) = y \log_b(x)$$

## Common Bases:

$b=2$  (binary, useful for computers)

$b=10$  (decimal system, useful for how we write numbers, e.g.  $1,000,000 = 10^6$ ;  $0.002 = 2 \times 10^{-3}$ )

\*  $b=e$ . Euler's number  $e \approx 2.718$ .

Can be defined a few different ways

$$\text{e.g., } e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}$$

We will see it is useful for calculus!

For  $f(x) = e^x$ , call  $\log_e = \ln$  "natural logarithm".

Can always switch bases  $b \rightarrow a$ :

$$b^x = a^{x \log_a b} \quad \leftarrow$$
$$\log_b x = \frac{\log_a x}{\log_a b} \quad \leftarrow$$
$$x = b^y \Leftrightarrow y = \log_b x$$
$$\Rightarrow \log_a x = \log_a(b^y) = y \log_a b$$
$$\Rightarrow \frac{\log_a x}{\log_a b} = y = \log_b x \quad \square$$

In this course, we will almost always use base  $e$ .

\* End of pre-calc review \*

- You should be familiar with functions ( $f: A \rightarrow B$ ) & their properties, and trig.-/exp.-/logarithms, since we will use them throughout the course
- If you are having any issues with these concepts, please come to my or your TA's office hours.