Math 20A Summer Bridge 2021: Homework 4

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Due Wednesday, September 1, 11:59 pm

Remark. Problems written as "Exercise X.Y.Z" are from the textbook, section X.Y exercise Z. For example, Exercise 2.2.8 denotes Exercise 8 of section 2.2. For problems referring to a figure, find the question in the textbook for the corresponding figure.

Remark. You can apply any theorem or rule (as long as the problem does not say explicitly to not use that theorem or rule), but make sure to show that the assumptions of said theorem or rule apply.

Problem 1 The Bohr Radius of the Hydrogen Atom

The Bohr radius a_0 of the hydrogen atom is the value of r which minimizes the energy

$$E(r) = \frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r},$$

where $\hbar, m, e, \epsilon_0$ are positive physical constants. The Bohr radius is the radius where you would classically "find an electron orbiting the nucleus" in a hydrogen atom. The first term above is the kinetic energy of the electron in a hydrogen atom and the second term is the potential energy due to the Coulomb electrostatic interaction. The energy function E has domain $(0, \infty)$.

Recall from our discussion on optimization on open domains that if we want to minimize a function on an open domain, we have to: first, find any local minima of our function; subsequently, if there is only one local minima, we just have to check that the local minima is less than the limit of the function as we approach the endpoints of the open interval.

(a) Show that the critical point of $E(r)$ is given by

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}.$$

Using the second derivative test, show that this value of r is a local minima; i.e., check that

$$E''(a_0) > 0,$$

at the above value of r.

(b) To show that this is an absolute minima of E, show that

$$E(a_0) < \lim_{r\to 0^+} E(r) \quad \text{and} \quad E(a_0) < \lim_{r\to\infty} E(r).$$

Thus, $a_0 = 4\pi\epsilon_0\hbar^2/(me^2)$ is the minimizer of E; i.e., a_0 is the Bohr radius.

Problem 2 Exercise 4.3.16

Determine the intervals on which f is increasing or decreasing, assuming that Figure 15 is the graph of f' (that is, figure 15 is the graph of the derivative of f, NOT the graph of f). Assume that f and f' are defined on $(-\infty, 6]$.

Problem 3 Exercise 4.3.28

Find the critical points and the intervals on which the function is increasing or decreasing. Use the first derivative test to determine whether each critical point yields a local min or max (or neither).

$$f(x) = 5x^2 + 6x - 4 \ (x \in \mathbb{R}).$$

Problem 4 Exercise 4.3.38

Find the critical points and the intervals on which the function is increasing or decreasing. Use the first derivative test to determine whether each critical point yields a local min or max (or neither).

$$f(x) = x^{5/2} - x^2 \ (x > 0).$$

Problem 5 Exercise 4.3.68

Show that if f is any quadratic polynomial (i.e., of the form $f(x) = c_2x^2 + c_1x + c_0$ for arbitrary $c_0, c_1, c_2 \in \mathbb{R}$), then the midpoint $c = \frac{a+b}{2}$ satisfies the conclusion of the MVT on $[a, b]$ for any a, b.

Problem 6 Exercise 4.4.48

Find the intervals on which f is concave up or down, the points of inflection, the critical points, and the local minima and maxima.

$$f(x) = x^2(x - 4).$$

Problem 7

Consider $f : [0, \pi] \to \mathbb{R}$ defined by $f(x) = \cos^2(x)$. Find the intervals on which f is concave up or down, the points of inflection, and the absolute minima and maxima of f.

Problem 8 Exercise 4.5.4

Evaluate the limit, using LH where it applies:

$$\lim_{x\to-1} \frac{x^4 + 2x + 1}{x^5 - 2x - 1}.$$

Problem 9 Exercise 4.5.8

Evaluate the limit, using LH where it applies:

$$\lim_{x\to 0} \frac{x^3}{\sin(x) - x}.$$

Problem 10 Exercise 4.5.52

Evaluate the limit

$$\lim_{x\to 0} \frac{\tan^{-1}(x)}{\sin^{-1}(x)}.$$

Problem 11 Exercise 4.6.18

Find the transition points (critical points and points of inflection), intervals of increase/decrease, intervals of concavity, and asymptotic behavior. Then, sketch the graph with this information indicated.

$$f(x) = \frac{1}{3}x^3 + x^2 + 3x.$$

Remark. For the subsequent optimization problems, do not just guess the correct answer; show that your answer is the absolute minima or maxima of the appropriate objective function.

Problem 12 Exercise 4.7.12

A flexible tube of length 4m is bent into an L shape (i.e., the tube is bent so that it makes a right angle at the point where it is bent). Where should the bend be made to minimize the distance between the two ends (by "ends", the textbook means the two end points of the tube)?

Problem 13 Exercise 4.7.16

What is the maximum area of a rectangle inscribed in a right triangle with legs of length 3 and 4 as in Figure 12? The sides of the rectangle are parallel to the legs of the triangle.

Problem 14 Finding the closest point on a line to a point off of the line.

Consider the line $y = x$. Find the point on the line that is closest to the point $(1, 0)$. Subsequently, show that the line connecting these two points is perpendicular to the line $y = x$.

(This is true more generally: if you want to find the point on a line that is closest to some point off of the line, draw a perpendicular line from the point off of the line and where it crosses the original line is the closest point.)

Problem 15 Exercise 4.7.28

A right circular cone (Figure 18) has volume

$$V = \frac{\pi}{3}r^2h$$

and surface area $S = \pi r\sqrt{r^2 + h^2}$. Find the dimensions of the cone with surface area $S = 1$ and maximal volume.

Problem 16 Exercise 5.1.17

Calculate the approximation R_6 for the given function and interval.

$$f(x) = 2x - x^2, \ [0, 2].$$

Problem 17 Exercise 5.2.14

Refer to Figure 15. Using geometry, evaluate

$$(a)\ \int_1^4 f(x)dx,$$
$$(b)\ \int_1^6 |f(x)|dx.$$

Problem 18 Exercise 5.2.49

Assume $a < b$ and H is the Heaviside function given by

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

Find an expression for $\int_a^b H(x)dx$ in terms of a, b.

Hint: Draw the graph of H to guide your intuition. Keeping in mind that $a < b$, consider separately the cases: (1) a and b are both negative; (2) a is negative and b is non-negative; (3) both a and b are nonnegative.

Problem 19 Exercise 5.2.50

By computing the limit of right-endpoint approximations, prove that (for $b > 0$),

$$\int_0^b x^3 dx = \frac{b^4}{4}.$$

Note: Do not use the fundamental theorem of calculus to answer this problem. You should directly show that the limit $\lim_{N\to\infty} R_N$ gives the above expression.

Hint: The following summation identity should be useful:

$$\sum_{j=1}^{N} j^3 = \frac{N^4}{4} + \frac{N^3}{2} + \frac{N^2}{4}.$$