Math 20A Summer Bridge 2021: Homework 1

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Due Tuesday, August 10, 11:59 pm.

Remark. Problems written as "Exercise X.Y.Z" are from the textbook, section X.Y exercise Z. For example, Exercise 2.2.8 denotes Exercise 8 of section 2.2. For problems referring to a figure, find the question in the textbook for the corresponding figure.

Problem 1 Injectivity, Surjectivity, and Bijectivity

Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$.

(a) Injectivity Show that f is injective. To show this, assume that $f(x) = f(y)$, then show that one must have $x = y$. Graph f and explain intuitively (i.e., in words) why f is injective.

(b) Surjectivity Show that f is surjective. To show this, let $y \in \mathbb{R}$. Show there exists some $x \in \mathbb{R}$ such that $f(x) = y$. Using the graph of f from part (a), explain intuitively why f is surjective.

(c) Bijectivity Since f is both injective and surjective, it is bijective and hence invertible. Thus, there exists an inverse $f^{-1} : \mathbb{R} \to \mathbb{R}$ satisfying the inverse properties. What is the inverse of f (to find this, let $f(x) = y$ and solve for x as a function of y; the inverse is this function of y; i.e., $x = f^{-1}(y)$)? Check that the inverse satisfies the inverse properties: $f^{-1}(f(x)) = x$ for all $x \in \mathbb{R}$ and $f(f^{-1}(y)) = y$ for all $y \in \mathbb{R}$.

Problem 2 Exercise 2.2.8

Determine $\lim_{x\to 0.5} g(x)$ for g as in Figure 11.

Problem 3 Exercise 2.2.18

Verify the following limit using the limit definition

$$\lim_{x\to 0}(3x^2 - 9) = -9.$$

(Recall the limit definition: $\lim_{x\to c} f(x) = L$ if $|f(x) - L|$ can be made arbitrarily small by taking x sufficiently close to c).

Problem 4 Exercise 2.2.56

Determine the infinite one- and two-sided limits in Figure 14.

Problem 5 Exercise 2.2.20

Using the limit laws, evaluate the limit

$$\lim_{w\to 7}\frac{\sqrt{w+2}+1}{\sqrt{w-3}-1}.$$

(Note: for such questions, you will not receive credit for just evaluating the function at $x = 7$. Show your work, including denoting which limit laws you use, that their assumptions hold in this situation, and precisely where you use them).

Problem 6 Exercise 2.3.32

Assuming that $\lim_{x\to -4} f(x) = 3$ and $\lim_{x\to -4} g(x) = 1$, evaluate the following limit (again, stating what limit laws you are using, that the assumptions hold, and where)

$$\lim_{x\to -4}\frac{f(x)+1}{3g(x)-9}.$$

Problem 7 Exercise 2.3.28

Assuming that $\lim_{x\to 6} f(x) = 4$, evaluate the following limits (again, stating what limit laws you are using, that the assumptions hold, and where)

(a) $\lim_{x\to 6} f(x)^2$

(b) $\lim_{x\to 6}[1/f(x)]$

(c) $\lim_{x\to 6} x\sqrt{f(x)}$

Problem 8 Exercise 2.3.38

Give an example where $\lim_{x\to 0}(f(x)g(x))$ exists but neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exists.

Problem 9 Exercise 2.4.4

Refer to the function g whose graph appears in Figure 16. Find the point c_1 at which g has a jump discontinuity but is left-continuous. How should $g(c_1)$ be redefined to make g right-continuous at $x = c_1$?

Problem 10 Exercise 2.4.12

Using the laws of continuity, show that the function is continuous (stating which laws you are using, that the assumptions hold, and where)

$$f(x) = \frac{x^2-\cos(x)}{3+\cos(x)}.$$

Problem 11 Evaluating using substitution

Using the laws of continuity, show that the following functions are continuous at the limit point. Subsequently, evaluate the limit using substitution.

(a)

$$\lim_{x \to 1} \frac{3x^2 - 4}{x}$$

(b)

$$\lim_{x \to \pi/2} \frac{\sin(x)}{x^2}$$

Problem 12 (Dis)continuities

For each of the following, does the limit exist? If the limit exists, is the function continuous at the limit point and why? If the function is not continuous at the limit point, what kind of discontinuity does it have and why?

(a) $\lim_{x \to 0} f(x)$ where

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x = 0 \\ x, & x > 0 \end{cases}$$

(b) $\lim_{x \to 0} g(x)$ where

$$g(x) = \begin{cases} \sin(x), & x \leq 0 \\ x + 1, & x > 0 \end{cases}$$

Problem 13 Exercise 2.5.8

Evaluate the limit, if it exists. If not, determine whether the one-sided limits exist (finite or infinite)

$$\lim_{x \to 8} \frac{x^3 - 64x}{x - 8}.$$

Problem 14 Exercise 2.5.22

Evaluate the limit, if it exists. If not, determine whether the one-sided limits exist (finite or infinite)

$$\lim_{x \to 8} \frac{\sqrt{x - 4} - 2}{x - 8}.$$

Problem 15 Indeterminate form

For which sign $+$ or $-$ does the following limit exist (and what is that limit?):

$$\lim_{x \to 0} \frac{1}{2x} \pm \frac{3}{2x(x - 3)}.$$

For the sign above where the limit does exist, what type of indeterminant form is this?

Problem 16 Exercise 2.7.30

Evaluate the limit (showing your work)

$$\lim_{x \to -\infty} \frac{4x-3}{\sqrt{25x^2+4x}}.$$

Problem 17 Limit at ∞

Evaluate the limit (showing your work)

$$\lim_{x \to \infty} \frac{2x+5}{\sqrt{9x^2-4x}}.$$

Problem 18 Horizontal Asymptotes

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 - \frac{1}{x}, & x \leq -1 \\ -\frac{1}{2}x + \frac{3}{2}, & x \in (-1, 1) \\ \frac{1}{x}, & x \geq 1. \end{cases}$$

Is this function continuous on all of $\mathbb{R}$? If not, where is it discontinuous? What are the horizontal asymptotes of f?