# Math 18 Summer Session 1 2023: Homework 3

Instructor: Brian Tran

Due Wednesday, July 26, 11:59 pm.

**Remark.** Problems written as "Exercise X.Y.Z" are from the textbook, section X.Y exercise Z. For example, Exercise 1.2.4 denotes exercise 4 of section 1.2. For problems referring to a figure, find the question in the textbook for the corresponding figure. Make sure to show all of your work and steps; credit will not be given for just stating an answer.

## Problem 1  Practice with Polynomial Interpolation

Find the unique polynomial of degree 3

$$p(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$$

that passes through the points

$$(t_1, y_1) = (-1, 3),\ (t_2, y_2) = (0, 2),\ (t_3, y_3) = (1, 1),\ (t_4, y_4) = (2, 6),$$

by solving the associated linear system

$$V\vec{c} = \vec{y}$$

where $V$ is the $4 \times 4$ Vandermonde matrix associated to $(t_1, t_2, t_3, t_4)$ and

$$\vec{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix},\ \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}.$$

## Problem 2  Exercise 3.1.2

Compute the determinant of $A$ using a cofactor expansion across the first row and also down the second column (your answers should be the same).

$$A = \begin{bmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 4 & 1 \end{bmatrix}.$$

## Problem 3  Exercise 3.2.24

Use determinants to decide if the set of vectors is linearly independent.

$$\begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix},\ \begin{bmatrix} -6 \\ 0 \\ 6 \end{bmatrix},\ \begin{bmatrix} -3 \\ -5 \\ -2 \end{bmatrix}.$$

## Problem 4 The Determinant of the Inverse is the Inverse of the Determinant

Let $A$ be an invertible matrix, so that $\det(A) \neq 0$. Show that

$$\det(A^{-1}) = \frac{1}{\det(A)},$$

i.e., the determinant of the inverse of an invertible matrix is equal to the inverse of the determinant of that matrix (where the inverse of a nonzero real number $c$ is just $1/c$).

**Hint.** Consider $\det(A^{-1}A)$. What do we know about the determinant of a product of matrices? What do we know about $A^{-1}A$?

## Problem 5 Similar Matrices Have the Same Determinant

Next week in lecture, we will study "similar matrices".

**Definition.** Two $n \times n$ matrices $A$ and $B$ are said to be similar if there exists an invertible matrix $P$ such that

$$B = PAP^{-1}.$$

Let $A$ and $B$ be similar matrices. Show that $\det(A) = \det(B)$.

**Hint.** The previous problem should be useful.

## Problem 6 Exercise 3.3.24

Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices as $(1, 3, 0)$, $(-2, 0, 2)$, $(-1, 3, -1)$.

## Problem 7 A Formula for the Volume of an Ellipsoid

Let $B$ denote the unit ball in $\mathbb{R}^3$, which consists of all points in $\mathbb{R}^3$ that have distance less than or equal to one from the origin,

$$B = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x^2 + y^2 + z^2 \leq 1 \right\}.$$

Let $E$ denote the ellipsoid with semi-axes $a > 0, b > 0, c > 0$, which you can think of as arising from the unit ball by scaling each of the $x, y, z$ axes by $a, b, c$ respectively. We will call the coordinates on the scaled axes $u = ax, v = by, w = cz$. In these coordinates, the ellipsoid can be described as

$$E = \left\{ \begin{bmatrix} u \\ v \\ w \end{bmatrix} : \left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 + \left(\frac{w}{c}\right)^2 \leq 1 \right\}.$$

A plot of the unit ball, an ellipsoid, and sketches of these are shown in Figures 1 and 2 on the next page.

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by scaling the axes as above, with corresponding matrix

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Since $a > 0, b > 0, c > 0$, this matrix is invertible.

Show that the image of $B$ under $T$ is $E$, i.e., $T(B) = E$. Then, using the fact that the volume of the unit ball is $\text{Vol}(B) = 4\pi/3$, use $T(B) = E$ to derive a formula for the volume of the ellipsoid $\text{Vol}(E)$ in terms of $a, b, c$.

**Hint.** This problem is the 3-dimensional version of the exercise we did in lecture, for finding the formula for the area of an ellipse from the formula for the area of the unit disk.

To show two sets $Q$ and $R$ are the same, it is equivalent to showing that each one is a subset of the other $Q \subseteq R$ and $R \subseteq Q$. Thus, we can show $T(B) = E$ by showing that $T(B) \subseteq E$ and $E \subseteq T(B)$.

To show $T(B) \subseteq E$, let $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in B$. Then, we know that $x^2 + y^2 + z^2 \leq 1$. Compute $T(\vec{x})$ and show that its components $u, v, w$ satisfy the relation defining the ellipsoid $E$. This shows that every point in $B$ gets mapped to some point in $E$ under $T$, i.e., the image $T(B)$ is a subset of $E$, $T(B) \subseteq E$.

To show $E \subseteq T(B)$, since $T$ is invertible, this is equivalent to showing that $T^{-1}(E) \subseteq B$. Now, let $\vec{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in E$. Then, we know that $(u/a)^2 + (v/b)^2 + (w/c)^2 \leq 1$. Compute $T^{-1}(\vec{u})$ and show that its components $x, y, z$ satisfy the relation defining the unit ball $B$. This shows that every point in $E$ gets mapped to some point in $B$ under $T^{-1}$, i.e., $T^{-1}(E) \subseteq B$. Since $T$ is invertible, this is equivalent to $E \subseteq T(B)$.

Thus, we have proved $T(B) \subseteq E$ and $E \subseteq T(B)$, so $T(B) = E$.

To derive the formula for the volume of the ellipsoid, note that

$$\text{Vol}(T(B)) = |\det(A)| \, \text{Vol}(B).$$

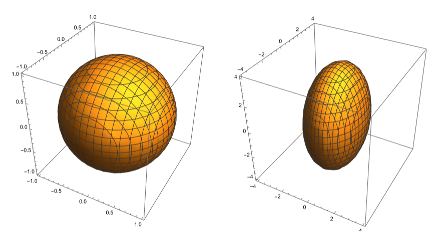

Figure 1: Left: the unit ball. Right: an ellipsoid, with $a = 2, b = 3, c = 4$. (Note the two images are not in the same scale. The axes for the ball plot are all from $-1$ to 1; the axes for the ellipsoid plot are all from $-4$ to 4)

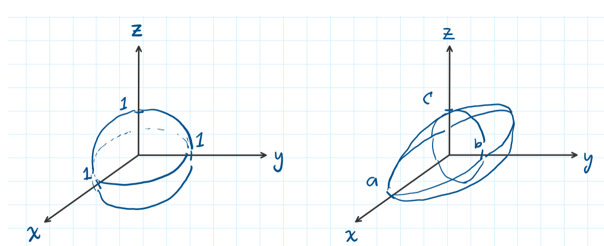


Figure 2: Left: sketch of the unit ball. Right: sketch of an ellipsoid with semi-axes $a, b, c$.

## Problem 8 Exercise 5.1.8

Is $\lambda = 3$ an eigenvalue of the matrix

$$\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} ?$$

If so, find one corresponding eigenvector.

## Problem 9 Exercise 5.1.14

Find a basis for the eigenspace corresponding to the eigenvalue $\lambda = -4$ for the matrix

$$A = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 3 & 3 \\ 6 & 6 & 2 \end{bmatrix}.$$

## Problem 10 Exercise 5.2.6

Find the characteristic polynomial and the eigenvalues of the matrix

$$\begin{bmatrix} 1 & -4 \\ 4 & 6 \end{bmatrix}.$$

## Problem 11 More practice with eigenvalues and eigenvectors

Find the three eigenvalues and three corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

You should find that the three eigenvalues are distinct; thus, we know that the three corresponding eigenvectors should be linearly independent. Verify that the three eigenvectors that you found are linearly independent.