Lecture 7 - Determinants

Monday, July 17, 2023 9:56 AM

- HW2 due Wednesday July 19, 11:59 pm
- Practice midterm is posted. I will post solutions tomorrow, but I highly encourage doing the practice midterm yourself before viewing the solutions.
- The midterm will be Friday July 21. It can be accessed on Gradescope from 12 pm to 11:59 pm. Once accessed, you will have 90 minutes to complete, scan, and upload your exam to Gradescope (as a single PDF file). Thus, you should begin your exam no later than 10:29 pm for the full time on the exam.
- The midterm review will be during Friday morning's lecture, through Zoom. I plan on going over HW1, HW2, and the practice midterm. If you have any questions, feel free to ask then as well.

Use rank-nullity,

1) Can there exist a surjective linear transf. $T: \mathbb{R}^p \to \mathbb{R}^q$, $p<q$?

2) Can there exist an injective linear transf. $T: \mathbb{R}^a \to \mathbb{R}^b$, $a>b$?

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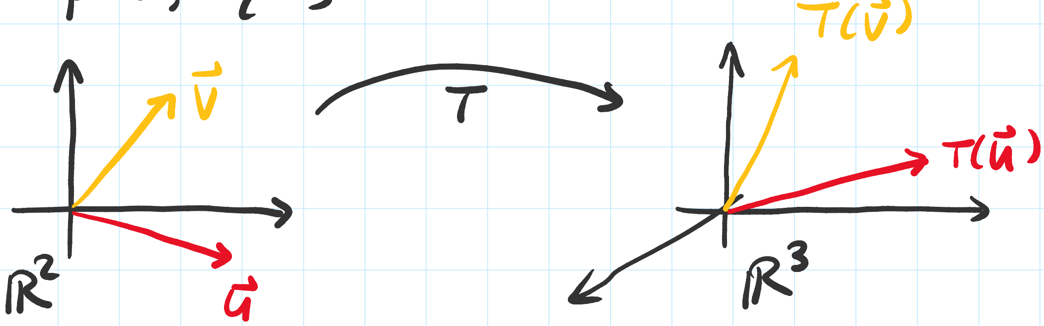
1) $T: \mathbb{R}^p \xrightarrow{\text{linear transf.}} \mathbb{R}^q$ $(p<q)$. Surjective possible?

ex $p=2$, $q=3$



Let A be the $q \times p$ matrix assoc. to T

$$\dim(\text{range}(T)) + \dim(\ker T) = \dim(\mathbb{R}^p)$$

$$\underbrace{\dim(\text{col}(A))}_{=q?\ \text{no}} + \underbrace{\dim(\text{nul}(A))}_{\geq 0} = p$$

$$\underbrace{\dim(\mathrm{col}(A))}_{=q?\ \text{no}} + \underbrace{\dim(\mathrm{nul}(A))}_{\geq 0} = p$$

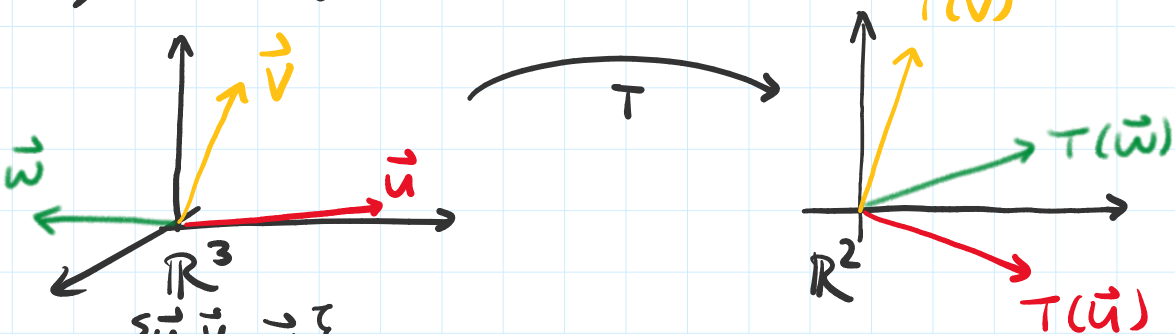
$$\dim(\mathrm{col}(A)) = p - \dim(\mathrm{nul}(A)) \leq p < q$$

$$\Rightarrow \dim(\mathrm{col}(A)) < q$$

$$\Rightarrow \dim(\mathrm{col}(A)) \neq q \quad \text{not surjective} \ \square$$

2) $T: \mathbb{R}^a \to \mathbb{R}^b$ (linear transf.), $a > b$, injective possible?

ex/ $a = 3$, $b = 2$



$\{\vec{u}, \vec{v}, \vec{w}\}$ basis

Let A be the $b \times a$ matrix assoc. to T

$$\dim(\mathrm{range}(T)) + \underbrace{\dim(\ker(T))}_{0?} = a$$

$\mathrm{range}(T) \subseteq \mathbb{R}^b$

$\dim(\mathrm{range}(T)) \leq b$

$$\dim(\ker(T)) = a - \dim(\mathrm{range}(T)) \geq 1$$

No. $\square$

↓(not on midterm)

Determinants (ch 3)

2x2 determinant

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

3x3 determinant

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} - \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

- Let A be an $n \times n$ matrix.

Let A_{ij} denote the $(n-1)\times(n-1)$ matrix obtained from removing the i^{th} row & j^{th} column of A.

entries of A

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} = \sum_{j=1}^{n} a_{1j} C_{1j}$$

The (i,j) cofactor of A is the number $(-1)^{i+j} \det A_{ij} = C_{ij}$

<u>Thm</u>: the determinant of an $n\times n$ matrix can be computed down any column or across any row

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \text{(for any } i\text{)}$$

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \text{(for any } i\text{)}$$

across a row

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij} \quad \text{(for any } j\text{)}$$

down a column

cofactor expansion

$\blacksquare$

ex/

$$\det \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= -2 \det \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} + 2 \det \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + 0 \det \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$= -2 \cdot 2 + 2 \cdot 2 + 0 = 0.$$

$$\det \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= (-1) \det \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} - 0 \det \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} + 1 \det \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

$$= (-1)(-2) - 0 + (-2) = 0.$$

Thm: Let A be an $n \times n$ matrix.
A is invertible if and only if $\det A \neq 0$. $\square$

Properties of det

Prop: Let A be square.

(i) $(i \neq j)$ $A \xrightarrow{(i) + c(j) \to (i)} B$, $\det A = \det B$

(ii) $A \xrightarrow{(i) \leftrightarrow (j)} B$, $\det A = -\det B$

(iii) $A \xrightarrow{(i) \to k(i)} B$, $\det B = k \det A$

Thm: If A, B are $n \times n$ matrices, then

$$\det(AB) = \det(A) \cdot \det(B)$$

Thm: Let A be an $n \times n$ matrix. Then,

$$\det A = \det A^T.$$

Proof: Induction

- Base case $n=1$ trivial $[a]^T = [a]$
- Suppose it holds for $k \times k$ matrices. Want to show it holds for $(k+1) \times (k+1)$ matrices.
- The cofactor expansion $\{C_{1j}\}_{j=1}^n$ of A equals the cofactor expansion $\{C_{j1}\}_{j=1}^n$ of A^T

$\Rightarrow$ the determinants are the same for $(k+1) \times (k+1)$ matrices.

by induction, done $\square$

ex/ Show if A is triangular $(n \times n)$, its determinant is a product of its diagonal entries

$$\underbrace{\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}}_{\text{upper triangular}} \qquad \underbrace{\begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}}_{\text{lower triangular}}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

$$\det A = a_{11} \det \begin{bmatrix} a_{22} & & \\ 0 & \ddots & \\ 0 \cdots 0 & & a_{nn} \end{bmatrix}$$

$$= a_{11} \cdot a_{22} \cdot \det \begin{bmatrix} a_{33} & & \\ 0 & \ddots & \\ 0 \cdots 0 & & a_{nn} \end{bmatrix}$$

$$\cdots$$

$$= a_{11} \cdot a_{22} \cdot (\ldots) \cdot a_{nn}$$

also holds for lower Δ, since $\det A = \det A^T$

$\square$

Cramer's Rule:

Let A be an invertible $n \times n$ matrix.

- Consider $A\vec{x} = \vec{b}$.
- Let $A_i(\vec{b})$ be the matrix obtained

· Let $A_i(\vec{b})$ be the matrix obtained from replacing i^{th} column of A by $\vec{b}$

$$A_i(\vec{b}) = [\vec{a}_1 \cdots \vec{a}_{i-1} \ \vec{b} \ \vec{a}_{i+1} \cdots \vec{a}_n]$$

Then, for any $\vec{b} \in \mathbb{R}^n$, the unique solution $\vec{x}$ to $A\vec{x} = \vec{b}$ is given by

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{where} \quad x_i = \frac{\det A_i(\vec{b})}{\det A} \quad i = 1, \ldots, n.$$

Pf: $A \, I_i(\vec{x})$ $\qquad I_i(\vec{x}) = [\vec{e}_1 \cdots \vec{e}_{i-1} \ \vec{x} \ \vec{e}_{i+1} \cdots \vec{e}_n]$

$$\begin{aligned} &\parallel \\ &A[\vec{e}_1 \cdots \vec{e}_{i-1} \ \vec{x} \ \vec{e}_{i+1} \cdots \vec{e}_n] \\ &= [A\vec{e}_1 \cdots A\vec{e}_{i-1} \ A\vec{x} \ A\vec{e}_{i+1} \cdots A\vec{e}_n] \\ &= [\vec{a}_1 \cdots \vec{a}_{i-1} \ \vec{b} \ \vec{a}_{i+1} \cdots \vec{a}_n] \\ &= A_i(\vec{b}) \end{aligned}$$

$$I_2(\vec{x}) \quad \begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix}$$

$$\begin{aligned} \det A_i(\vec{b}) &= \det(A \, I_i(\vec{x})) \\ &= \det(A) \underbrace{\det I_i(\vec{x})}_{= x_i.} \leftarrow \end{aligned}$$

□

ex/ $A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$ Solve $A\vec{x} = \vec{b}$ where $\vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\det A = 2 - (-3) = 5 \neq 0$

$A_1(\vec{b}) = [\vec{b} \ \vec{a}_2] = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ $\quad \det A_1(\vec{b}) = 1$

$$A_1(\vec{b}) = [\vec{b}\ \ \vec{a}_2] = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \qquad \det A_1(\vec{b}) = 1$$

$$A_2(\vec{b}) = [\vec{a}_1\ \ \vec{b}] = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \qquad \det A_2(\vec{b}) = 3$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad x_1 = \frac{\det A_1(\vec{b})}{\det A} = 1/5 \qquad x_2 = \frac{\det A_2(\vec{b})}{\det A} = 3/5 \qquad \Rightarrow \vec{x} = \begin{bmatrix} 1/5 \\ 3/5 \end{bmatrix}$$

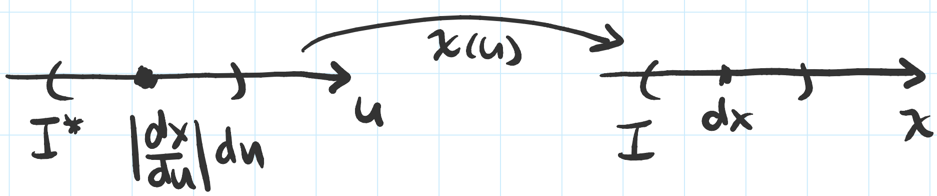
Determinants as Area/Volume

- Aside (calculus)

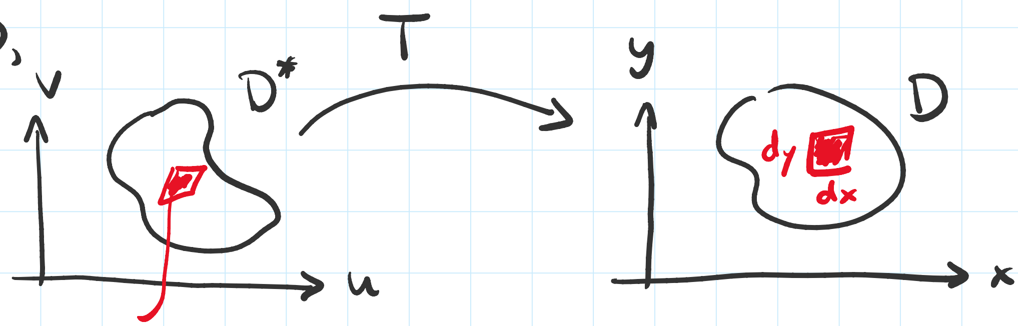
  let $x: u \mapsto x(u)$ be a continuously differentiable bijection between two intervals $x: I^* \to I$

  Change of variables (u-sub.)

$$\int_{I^*} f(x)\,dx = \int_I f(x(u)) \left|\frac{dx}{du}(u)\right| du$$



In 2D,



$|\det DT(u,v)|\, du\, dv$

Thm: If A is a 2×2 matrix, then the area of the parallelogram determined by its columns is $|\det(A)|$.

- If A is a 3×3 matrix, then the volume of the parallelepiped determined by its columns is $|\det(A)|$.
- If A is an $n \times n$ matrix, then the n-volume of the n-parallelotope determined by its columns is $|\det(A)|$.

(proof next lecture)

$A = [\vec{a}_1 \ \vec{a}_2]$