Lecture 6 - col and nul (cont.), bases, dim, rank-nullity

Friday, July 14, 2023 9:52 AM

- HW 2 posted
- Practice midterm to be posted this weekend
- Next Friday's lecture (July 21st) will be through Zoom; it will be a review session

<u>Def</u>: A linear transformation between two vector spaces V and W is a map $T: V \to W$ satisfying

$$T(\alpha \vec{x} + \beta \vec{y}) = \alpha T(\vec{x}) + \beta T(\vec{y})$$

for all $\alpha, \beta \in \mathbb{R}$ and $\vec{x}, \vec{y} \in V$.

$$\ker(T) = \{\vec{x} \in V : T(\vec{x}) = \vec{0}\}$$

$$\text{range}(T) = \{T(\vec{x}) : \vec{x} \in V\} = T(V)$$

when T is a linear transf., $\ker(T)$ is a subspace of V and range(T) is a subspace of W.

ex/ $W = C(\mathbb{R}, \mathbb{R})$ cont. functions from $\mathbb{R}$ to $\mathbb{R}$

$V = C^1(\mathbb{R}, \mathbb{R})$ continuously differentiable functions from $\mathbb{R}$ to $\mathbb{R}$

$\frac{d}{dx} : C^1(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$ is a linear transformation.

$\ker(d/dx)$ = space of const. fcns

range$(d/dx) = C(\mathbb{R}, \mathbb{R})$

If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a lin. transf. and A is the associated matrix,

$$\ker(T) = \text{nul}(A)$$
$$\text{range}(T) = \text{col}(A)$$

vector space

Def: A collection of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ in V are linearly independent if the only solution to

$$c_1\vec{v}_1 + \dots + c_p\vec{v}_p = \vec{0}$$

is the trivial solution $c_i = 0$ for all i

· Otherwise, say they are linearly dependent.

Def: [Basis]

A basis for a vector space V is a set of vectors $\mathcal{B}$ in V s.t.

(i) $\mathcal{B}$ is linearly independent

(ii) $\mathcal{B}$ spans V (i.e., $\text{span}\,\mathcal{B} = V$).

ex/ standard basis in $\mathbb{R}^n$

$\mathcal{B} = \{\vec{e}_1, \dots, \vec{e}_n\}$ is a basis for $\mathbb{R}^n$

ex/ Invertible $n \times n$ matrices in $\mathbb{R}^n$

If $A = [\vec{a}_1 \cdots \vec{a}_n]$ is an invertible $n \times n$ matrix, then its columns

$\mathcal{B} = \{\vec{a}_1, \dots, \vec{a}_n\}$ is a basis for $\mathbb{R}^n$

ex/ $\mathbb{P}_n$ polynomials of degree at most n

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Claim: monomials $\{1, t, t^2, \ldots, t^n\} = \mathcal{B}$ is a basis for $\mathbb{P}_n$. "monomial basis"

- clear $\operatorname{span} \mathcal{B} = \mathbb{P}_n$ ✓
- Linearly independent:

suppose

$$c_0 \cdot 1 + c_1 t + c_2 t^2 + \cdots + c_n t^n = 0$$

evaluate @ $t = 0 \Rightarrow c_0 = 0$

differentiate & evaluate @ $t=0 \Rightarrow c_1 = 0$

etc. $\Rightarrow$ all $c_i = 0$

□

<u>Def</u>: A basis for a subspace H of V is a collection of vectors $\mathcal{B}$ in V s.t.

(i) $\mathcal{B}$ is linearly indep.

(ii) $\mathcal{B}$ spans H.

[Spanning Set Theorem]

- Let $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ in V and let $H = \operatorname{span} S$.

(i) If one of the $\vec{v}_j$ can be written as a linear combination of the others, it can be removed from S and that new set still spans H.

(ii) (If $H \neq \{\vec{0}\}$) some subset of S is a basis for H.

a basis for H.

$\square$

Basis: • a spanning set that is as small as possible

OR

• a linearly indep. set that is as big as possible

ex/

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

$$\xrightarrow[\text{reduce}]{\text{row}} \quad B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for nul(A) & col(A)

$\text{nul}(A) = \text{nul}(B)$

$$[B \ \vec{0}] = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $x_1 = -4x_2 - 2x_4$

x_2 free

$x_3 = x_4$

x_4 free

$x_5 = 0$

$$\left\{ x_2 \begin{bmatrix} -4 \\ 1 \\ \vdots \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ \vdots \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\}$$

$x_5 = 0$

$$\mathrm{nul}(A) = \mathrm{nul}(B) = \left\{ x_2 \underbrace{\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{= \vec{v}_1} + x_4 \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{= \vec{v}_2} : x_2, x_4 \in \mathbb{R} \right\}$$

$\Rightarrow \{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathrm{nul}(A)$.

Careful: $\mathrm{col}(A) \neq \mathrm{col}(B)$.

basis for $\mathrm{col}(A)$ given by the pivot columns $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}$

observe $\vec{b}_2 = 4\vec{b}_1 \quad \vec{b}_4 = 2\vec{b}_1 - \vec{b}_3$

also $\vec{a}_2 = 4\vec{a}_1 \quad \vec{a}_4 = 2\vec{a}_1 - \vec{a}_3 \Leftarrow$

row reduction : E invertible

$$EA = E[\vec{a}_1 \cdots \vec{a}_n] = [\overset{\vec{b}_1 \,=}{E\vec{a}_1} \cdots \overset{\vec{b}_n \,=}{E\vec{a}_n}]$$

e.g. $E\vec{a}_4 = 2E\vec{a}_1 - E\vec{a}_3$

Thm: the pivot columns of a matrix A form a basis for $\mathrm{col}(A)$. □

Coordinate Systems (section 4.4)

Thm: Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis for a vector space V. Then, for each

Thm: Let $\mathcal{B} = \{b_1, \dots, b_n\}$ be a basis for a vector space V. Then, for each $\vec{x} \in V$, there exists unique weights $\{c_1, \dots, c_n\}$ s.t. $\vec{x} = \sum_{i=1}^{n} c_i \vec{b}_i$

proof: existence: $\mathcal{B}$ spans V.

uniqueness: suppose

$$\sum_{i=1}^{n} c_i \vec{b}_i = \vec{x} = \sum_{i=1}^{n} d_i \vec{b}_i$$

$$\sum_{i=1}^{n} \underbrace{(c_i - d_i)}_{=0} \vec{b}_i = \vec{0} \leftarrow \text{dependence relation}$$

$$\Rightarrow c_i = d_i \text{ for all } i.$$

$\square$

Def: We call the weights $\{c_1, \dots, c_n\}$ of $\vec{x} = \sum_{i=1}^{n} c_i \vec{b}_i$ the coordinates of $\vec{x}$ relative to $\mathcal{B}$.

- We call the vector in $\mathbb{R}^n$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$ the coordinate vector of $\vec{x}$ relative to $\mathcal{B}$.

ex/ $\mathbb{P}_3$, monomial basis $\{1, t, t^2, t^3\} = \mathcal{B}$

$q(t) = 3 + 2t + 5t^2 - t^3$

$p(t) = -1 + 4t - 3t^2 + 2t^3$

$$[q]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}, \quad [p]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$$

$$[q]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \quad [p]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$$

$$q(t) + p(t) = 2 + 6t + 2t^2 + t^3$$

$$[q+p]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 6 \\ 2 \\ 1 \end{bmatrix} = [q]_{\mathcal{B}} + [p]_{\mathcal{B}}.$$

<u>Thm</u>: Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis for V. Then, the coordinate mapping $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is a bijective linear transf $V \to \mathbb{R}^n$

<u>pf</u>: check $[a\vec{x} + b\vec{y}]_{\mathcal{B}} = a[\vec{x}]_{\mathcal{B}} + b[\vec{x}]_{\mathcal{B}}$

injective ~ linear independence

surjective: $\vec{y} \in \mathbb{R}^n \Rightarrow \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$$\vec{x} = \sum_{i=1}^{n} y_i \vec{b}_i \qquad [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \vec{y}$$

$\square$

Coordinates in $\mathbb{R}^n$

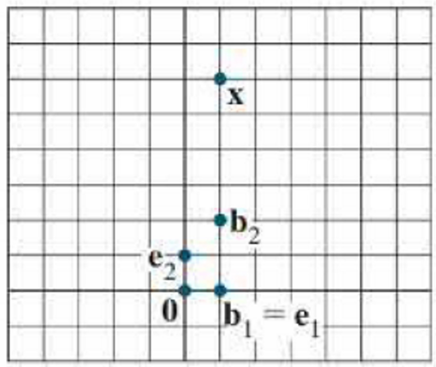


FIGURE 1 Standard graph paper.

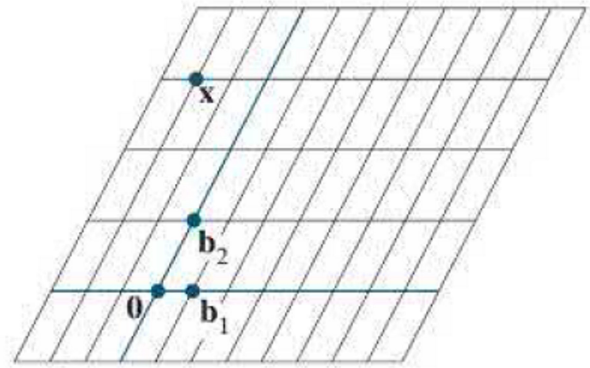


FIGURE 2 $\mathcal{B}$-graph paper.

ex/ Write $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in coordinates relative $\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{"}\vec{b}_1}, \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{"}\vec{b}_2} \right\}$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2$$

$$= \underbrace{[\vec{b}_1 \;\; \vec{b}_2]}_{=B} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \leftarrow [\vec{x}]_{\mathcal{B}}$$

$$B[\vec{x}]_{\mathcal{B}} = \vec{x}$$

$$[\vec{x}]_{\mathcal{B}} = B^{-1}\vec{x}$$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \det B = -1-1 = -2 \neq 0$$

$$B^{-1} = \frac{1}{\det B}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{-1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[\vec{x}]_{\mathcal{B}} = \frac{-1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

$$\vec{x} = \frac{3}{2}\vec{b}_1 - \frac{1}{2}\vec{b}_2 \quad \text{(check).}$$

More generally in $\mathbb{R}^n$, if $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ and let $B = [\vec{b}_1 \cdots \vec{b}_n]$,

then for any $\vec{x} \in \mathbb{R}^n$,

$$[\vec{x}]_{\mathcal{B}} = B^{-1}\vec{x}.$$

$$\left(\text{pf: } \begin{array}{l} \vec{x} = \sum_{i=1}^{n} c_i \vec{b}_i = [\vec{b}_1 \cdots \vec{b}_n]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\ \vec{x} = B\,[\vec{x}]_{\mathcal{B}} \end{array} \right)$$

The dimension of a vector space (section 4.5)
(* last section tested on midterm)

<u>Prop:</u> Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis for V. Then, any set of vectors in V $\{\vec{u}_1, \dots, \vec{u}_p\}$ $p > n$ must be linearly dependent.

<u>proof:</u> consider the coord. vectors

$$\{[\vec{u}_1]_{\mathcal{B}}, \dots, [\vec{u}_p]_{\mathcal{B}}\}$$

consider the matrix

$$\underbrace{\begin{bmatrix} [\vec{u}_1]_{\mathcal{B}} & \cdots & [\vec{u}_p]_{\mathcal{B}} \end{bmatrix}}_{p \text{ cols}} \Big\} n \text{ rows}$$

$p > n$

not enough rows to have p pivot columns $\Rightarrow$ these coordinate vectors are linearly dependent.

columns ⇒ these coordinate vectors are linearly dependent.

⇒ $\{\vec{u}_1, \dots, \vec{u}_p\}$ are linearly dep.

since $\vec{u} \mapsto [u]_{\mathcal{B}}$ is invertible lin. transf. $\square$

<u>Thm:</u> If a vector space V has a basis of n elements, then any other basis of V also has n elements.

$\square$

<u>Def:</u> If there is a basis for a vector space V with finitely many vectors, we say V is finite-dimensional. The dimension of V is the # of vectors in any basis for V, denoted $\dim(V)$.

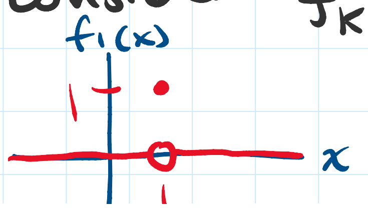
Otherwise, we say V is infinite-dimensional.
(similar subspace)

ex/ $\mathbb{R}^n$ basis $\{\vec{e}_1, \dots, \vec{e}_n\}$ ⇒ $\dim(\mathbb{R}^n) = n$

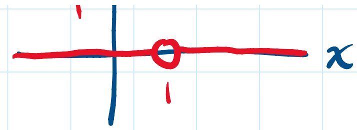
ex/ $\mathbb{P}_n$ basis $\{1, t, t^2, \dots, t^n\}$
⇒ $\dim(\mathbb{P}_n) = n+1$

ex/ $\mathrm{Fun}(\mathbb{R}, \mathbb{R})$ is infinite-dimensional

consider $f_k(x) = \begin{cases} 1 & \text{if } x = k \\ 0 & \text{if } x \neq k \end{cases}$



$\{f_k(x)\}_{k=-\infty}^{\infty}$ are



$\{f_k(x)\}_{k=-\infty}^{\infty}$ are linearly independent.

Rank-Nullity

- Let A be an $m\times n$ matrix

  $\text{rank}(A) := \dim(\text{col}(A))$

  $\text{nullity}(A) := \dim(\text{nul}(A))$

<u>Theorem</u> [Rank-Nullity Theorem]

$$\text{rank}(A) + \text{nullity}(A) = n \quad (\#\text{ of cols of } A)$$

<u>Pf:</u>

$$\underbrace{\#\text{ of pivot cols}}_{\text{rank}(A)} + \underbrace{\#\text{ of non-pivot cols}}_{\text{nullity}(A)} = \#\text{ of cols} \qquad \square$$

ex/ Invertible $n\times n$ matrix A

$\text{nullity}(A) = \dim(\text{nul}(A)) = 0$

$\text{rank}(A) = \dim(\text{col}(A)) = n$

ex/ Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a surjective linear transformation. What is the dimension of its kernel?

Let A be the assoc. 2×3 matrix.

$\text{col}(A) = \text{range}(T) = \mathbb{R}^2$

$\text{rank}(A) = 2$

$\text{rank}(A) + \text{nullity}(A) = 3$

$\text{nullity}(A) = 1$

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = 3$$

$$\mathrm{nullity}(A) = 1$$

$$\dim(\mathrm{nul}(A)) = \dim(\ker(T))$$

Use rank-nullity,

Can there exist a surjective linear transf. $T: \mathbb{R}^p \to \mathbb{R}^q$, $p<q$?

Can there exist an injective linear transf. $T: \mathbb{R}^a \to \mathbb{R}^b$, $a>b$?