Lecture 5 - Invertibility (cont.), Vector Spaces, Subspaces, nul and col

Wednesday, July 12, 2023 9:46 AM

HW1 due tonight at 11:59 pm

3×3 determinant

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$\det(A)$

$$:= a \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Prop: A 3×3 matrix A is invertible $\Leftrightarrow$ $\det(A) \neq 0$ $\square$

ex/ $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \\ 3 & 1 & 1 \end{bmatrix}$ $\qquad \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

$$\det(A) = 1\cdot\det\begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix} - 2\det\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} + 4\det\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$
$$= 1(1-5) - 2(2-15) + 4(2-3)$$
$$= -4 + 26 - 4 = 18 \neq 0 \Rightarrow A \text{ invertible.}$$

Theorem [Properties of Inverses]

- Let A, B be invertible $n\times n$ matrices, and let $c \in \mathbb{R}$, $c \neq 0$. Then,

(i) A^{-1} is invertible ... $(A^{-1})^{-1} = A$

and let $c \in \mathbb{R}$, $c \neq 0$. Then,

(i) A^{-1} is invertible, $(A^{-1})^{-1} = A$

(ii) AB is invertible, $(AB)^{-1} = B^{-1}A^{-1}$

(iii) A^T is invertible, $(A^T)^{-1} = (A^{-1})^T$

(iv) cA is invertible, $(cA)^{-1} = \frac{1}{c}A^{-1}$

proof:

(i) find C s.t. $CA^{-1} = I_n = A^{-1}C$

since A is invertible,

$$A^{-1}A = I_n = AA^{-1}$$

choose $C = A \Rightarrow (A^{-1})^{-1} = A$.

(ii)

$$(B^{-1}A^{-1})(AB) = B^{-1}\overbrace{(A^{-1}A)}^{I_n}B$$

$$= B^{-1}B = I_n$$

$$(AB)(B^{-1}A^{-1}) = A\overbrace{(BB^{-1})}^{I_n}A^{-1}$$

$$= AA^{-1} = I_n.$$

$$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$$

(iii) $A^{-1}A = I_n = AA^{-1}$

take transpose

$$(A^{-1}A)^T = I_n^T = (AA^{-1})^T$$

$$A^T(A^{-1})^T = I_n = (A^{-1})^TA^T$$

$$\Rightarrow (A^T)^{-1} = (A^{-1})^T$$

$\Rightarrow (A^T)^{-1} = (A^{-1})^T$

(iv) check. $\square$

<u>Theorem</u> [Invertible Matrix Theorem]

Let A be an $n \times n$ matrix.

The following are equivalent:

(i) A is invertible

(ii) A is row equivalent to I_n

(iii) A has n pivots

(iv) $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$

(v) Columns of A are lin. indep.

(vi) The linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ associated to A is injective.

(vii) $A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^n$

(viii) The columns of A span $\mathbb{R}^n$

(ix) T is surjective

(x) There exists $n \times n$ matrix C s.t. $CA = I_n$

(xi) There exists $n \times n$ matrix D s.t. $AD = I_n$

(xii) A^T is invertible.

ex/ Consider an upper triangular matrix, i.e., an $n \times n$ matrix where the entries below the diagonal are 0

$$A = \begin{bmatrix} \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet \\ 0 & 0 & \bullet \end{bmatrix}$$

What must be true about the entries A for it be invertible?

$\begin{bmatrix} 0 & 0 & \cancel{d} \end{bmatrix}$ the entries A for it be invertible?

A is invertible $\Leftrightarrow$ All of the diagonal entries are nonzero $\square$

ex/ Prove that if A & B are two $n \times n$ matrices s.t. AB is invertible, then so is B.

Can I solve $B\vec{x} = \vec{y}$ for all $\vec{y} \in \mathbb{R}^n$?

$$B\vec{x} = \vec{y}$$
$$\Rightarrow AB\vec{x} = A\vec{y}$$
$$\Rightarrow \vec{x} = (AB)^{-1}A\vec{y}$$
$\Rightarrow$ B is invertible. $\square$

(Chapter 4) Vector Spaces

<u>Def</u>:

A vector space is a nonempty set V, whose elements are called vectors, which has operations of addition & scalar multiplication (by real #s) satisfying:

For all $\vec{u}, \vec{v}, \vec{w} \in V$ and scalars $c, d \in \mathbb{R}$,

(i) $\vec{u} + \vec{v} \in V$

(ii) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(i) $\vec{u} + \vec{v} \in V$

(ii) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(iii) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

(iv) There is a zero vector, $\vec{0}$, s.t. $\vec{u} + \vec{0} = \vec{u}$

(v) For each $\vec{u} \in V$, there exists a vector $-\vec{u} \in V$ s.t. $\vec{u} + (-\vec{u}) = \vec{0}$

(vi) $c\vec{u} \in V$

(vii) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

(viii) $(c+d)\vec{u} = c\vec{u} + d\vec{u}$

(ix) $c(d\vec{u}) = (cd)\vec{u}$

(x) $1\vec{u} = \vec{u}$ "Vector space axiom"

ex/ For each positive integer n, $\mathbb{R}^n$ is a vector space.

ex/ Consider the space of polynomials of degree at most n

$$\mathbb{P}_n = \left\{ p(x) = c_0 + c_1 x + \dots + c_n x^n \; : \; \underset{i=1,\dots,n}{c_i \in \mathbb{R}} \right\}$$

$\mathbb{P}_n$ is a vector space

- $\vec{0}(x) = 0 \quad (c_0 = c_1 = \dots = c_n = 0)$
- let $p(x), q(x) \in \mathbb{P}_n$ and let $\alpha, \beta \in \mathbb{R}$. check $\alpha p(x) + \beta q(x) \in \mathbb{P}_n$.

$$p(x) = c_0 + c_1 x + \dots + c_n x^n$$

$$p(x) = c_0 + c_1 x + \dots + c_n x^n$$
$$q(x) = d_0 + d_1 x + \dots + d_n x^n$$

$$\alpha p(x) + \beta q(x) = \alpha(c_0 + c_1 x + \dots + c_n x^n) + \beta(d_0 + d_1 x + \dots + d_n x^n)$$

$$= \alpha c_0 + \beta d_0 + (\alpha c_1 + \beta d_1)x + \dots + (\alpha c_n + \beta d_n)x^n$$

$\Rightarrow \alpha p(x) + \beta q(x) \in \mathbb{P}_n$.

ex/ $\mathrm{Fun}(\mathbb{R}, \mathbb{R})$ = set of all functions from $\mathbb{R}$ to $\mathbb{R}$

$\Rightarrow \mathrm{Fun}(\mathbb{R}, \mathbb{R})$ is a vector space.

Def: A subspace of a vector space V is a subset $H \subseteq V$ satisfying:

(i) The zero vector of V is in H

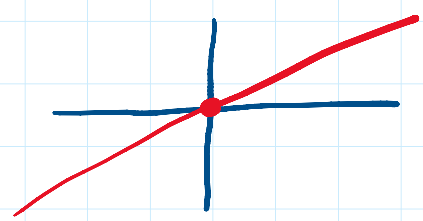
(ii) H is closed under addition, i.e.,

$$\vec{u}, \vec{v} \in H \Rightarrow \vec{u} + \vec{v} \in H$$

(iii) H is closed under scalar mult., i.e.,

$$c \in \mathbb{R}, \vec{u} \in H \Rightarrow c\vec{u} \in H$$

ex/ $V = \mathbb{R}^2$

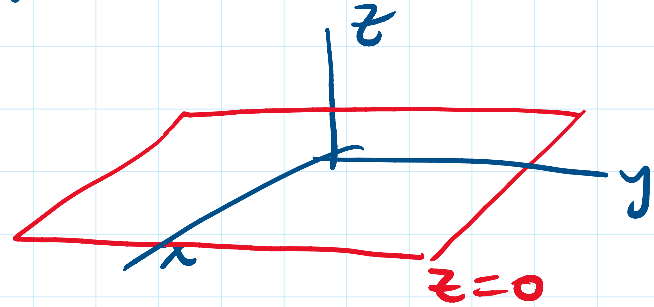


- any line passing through the origin is a subspace of $\mathbb{R}^2$
- $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$

· $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$

· $\{\vec{0}\}$ is a subspace of $\mathbb{R}^2$

ex/ $V = \mathbb{R}^3$



$z=0$ plane is a subspace.

$$H = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$\vec{0} \in H \quad (x = 0 = y)$

$\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ 0 \end{bmatrix} \in H, \quad a, b \in \mathbb{R}$

$$a\vec{u} + b\vec{v} = a\begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix} + b\begin{bmatrix} x_2 \\ y_2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + bx_2 \\ ay_1 + by_2 \\ 0 \end{bmatrix} \in H$$

$\square$

ex/ $\mathbb{P}_n$ polynomials of degree at most n $(n \geq 1)$

$\mathbb{P}_{n-1}$ polynomials of degree at most $n-1$

$\mathbb{P}_{n-1}$ is a subspace of $\mathbb{P}_n$.

For any integers $a \geq b \geq 0$,

$\mathbb{P}_b$ is a subspace of $\mathbb{P}_a$.

ex/ Fun$(\mathbb{R},\mathbb{R})$ functions from $\mathbb{R}$ to $\mathbb{R}$

Let $C(\mathbb{R},\mathbb{R})$ = set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$

$C(\mathbb{R},\mathbb{R})$ is a subspace of Fun$(\mathbb{R},\mathbb{R})$.

Theorem: If $\vec{v}_1,\dots,\vec{v}_p$ are in a vector space V, then $H = \text{span}\{\vec{v}_1,\dots,\vec{v}_p\}$ is a subspace of V.

"Say H is the subspace generated or spanned by $\underbrace{\vec{v}_1,\dots,\vec{v}_p}_{\text{call these a generating (or spanning) set for } H.}$."

proof: $$H = \left\{ \sum_{k=1}^{p} c_k \vec{v}_k : \underset{k=1,\dots,p}{c_k \in \mathbb{R}} \right\} \subseteq V$$

$\vec{0} \in H \quad (c_k = 0,\ k=1,\dots,p)$

$\vec{u},\vec{w} \in H \qquad \vec{u} = \sum_{k=1}^{p} c_k \vec{v}_k,\ \vec{w} = \sum_{k=1}^{p} d_k \vec{v}_k$

let $\alpha,\beta \in \mathbb{R}$

$$\alpha\vec{u} + \beta\vec{w} = \alpha \sum_{k=1}^{p} c_k \vec{v}_k + \beta \sum_{k=1}^{p} d_k \vec{v}_k$$
$$= \sum_{k=1}^{p} (\alpha c_k + \beta d_k)\vec{v}_k \in H$$

$\square$

ex/ ... $H = \left\{ \begin{bmatrix} a+b+c \\ \dots \end{bmatrix} \dots \mathbb{R} \right\}$

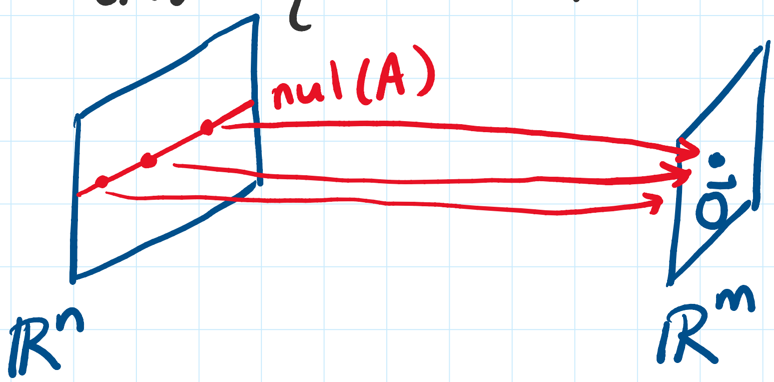
ex/ Show that $H = \left\{ \begin{bmatrix} a+b+c \\ a-c \\ b \\ 2a+b-c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^4$.

$$H = \left\{ a \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}}_{\vec{v}_1} + b \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{\vec{v}_2} + c \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}}_{\vec{v}_3} : a, b, c \in \mathbb{R} \right\}$$

$$= \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}.$$

__Def:__ The null space of an $m \times n$ matrix A, $\operatorname{nul}(A)$, is the set of all solutions to homog. eqn. $A\vec{x} = \vec{0}$

$$\operatorname{nul}(A) = \{ \vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0} \}$$



nul(A) characterizes the injectivity of the lin. transf. associated to A.

__Thm:__ $\operatorname{nul}(A)$ is a subspace of $\mathbb{R}^n$, where A is an $m \times n$ matrix.

Proof: $\vec{x} \in \operatorname{nul}(A)$

where A is an $m \times n$ matrix.

<u>proof:</u> $\vec{0} \in \text{nul}(A)$

let $\vec{x}, \vec{y} \in \text{nul}(A)$, $a, b \in \mathbb{R}$

$$A(a\vec{x} + b\vec{y}) = a\underbrace{A\vec{x}}_{=\vec{0}} + b\underbrace{A\vec{y}}_{=\vec{0}} = \vec{0}$$

$$\Rightarrow a\vec{x} + b\vec{y} \in \text{nul}(A) \qquad \square$$

ex/

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Find generating set for $\text{nul}(A)$

$$[A \;\; \vec{0}] \xrightarrow[\text{reduce}]{\text{row}} \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_2 free $\quad x_4$ free $\quad x_5$ free

$$\left\{ \begin{bmatrix} x_1 = 2x_2 + x_4 - 3x_5 \\ x_2 \text{ free} \\ x_3 = -2x_4 + 2x_5 \\ x_4 \text{ free} \\ x_5 \text{ free} \end{bmatrix} \right\} = \text{nul}(A)$$

$$= \left\{ x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} : x_2, x_4, x_5 \in \mathbb{R} \right\}$$

$$\Rightarrow \left\{ x_2 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{=\vec{v}_1} + x_4 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{=\vec{v}_2} + x_5 \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}_{=\vec{v}_3} : x_2, x_4, x_5 \in \mathbb{R} \right\} = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ generate $\operatorname{nul}(A)$.

Def: The column space of an $m \times n$ matrix A is the span of its cols; i.e. with $A = [\vec{a}_1 \cdots \vec{a}_n]$,

$$\operatorname{col}(A) = \operatorname{span}\{\vec{a}_1, \ldots, \vec{a}_n\}.$$

* Column space characterizes surjectivity of the associated linear transf.

Thm: $\operatorname{Col}(A)$ is a subspace of $\mathbb{R}^m$