Lecture 3 - Linear Independence (cont.), Functions, and Linear Transformations

Friday, July 7, 2023 9:57 AM

(cont. discussion of lin. ind.)

Theorem: if $p>n$, then any list of p vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ must be linearly dependent.

proof: $A = [\vec{v}_1 \cdots \vec{v}_p]$ has linearly indep. cols if and only if A has full (p) pivots

$$n=2 \; \underbrace{\begin{bmatrix} \textcircled{\bullet} & \bullet & \bullet & \bullet \\ \bullet & \textcircled{\bullet} & \bullet & \bullet \end{bmatrix}}_{p=4}$$

□

$A = [\vec{v}_1 \cdots \vec{v}_p]$ has lin. indep. cols if & only if the only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$

$[A \;\; \vec{0}] = [\vec{v}_1 \cdots \vec{v}_p \;\; \vec{0}]$

only trivial sol'n when there are no free variables

$$n=2\left\{ \begin{bmatrix} \underbrace{\begin{matrix} \textcircled{\bullet} & \bullet & \bullet & \bullet \\ \bullet & \textcircled{\bullet} & \bullet & \bullet \end{matrix}}_{p=4} & \begin{matrix} 0 \\ 0 \end{matrix} \end{bmatrix}\right.$$

since # of cols of A > # of rows, at least one free var. □

ex/ $A = \begin{bmatrix} \boxed{0} & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix}$ Determine if the columns of A are lin. ind.

row reduce

$$A \xrightarrow{① \leftrightarrow ④} \begin{bmatrix} \boxed{1} & -3 & 2 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 0 & -8 & 5 \end{bmatrix}$$

$$\xrightarrow[③ + ① \to ③]{② - 3 \cdot ① \to ②} \begin{bmatrix} \boxed{1} & -3 & 2 \\ 0 & \boxed{2} & -2 \\ 0 & 2 & -2 \\ 0 & -8 & 5 \end{bmatrix}$$

$$\xrightarrow[④ + 4 \cdot ② \to ④]{③ - ② \to ③} \begin{bmatrix} \boxed{1} & -3 & 2 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\xrightarrow{③ \leftrightarrow ④} \begin{bmatrix} \boxed{1} & -3 & 2 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{-3} \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ cols of A are lin. indep.

□

Functions (not in textbook)

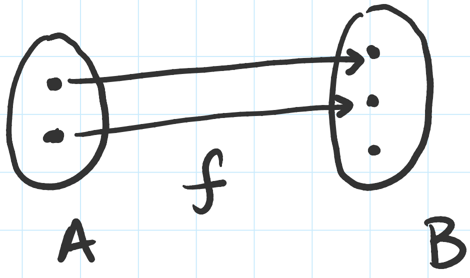
Def: A function $f : \overset{\text{domain}}{A} \longrightarrow \overset{\text{codomain}}{B}$ is a map from

Def: A function $f: A \longrightarrow B$ is a map from a set A to a set B, i.e., to each $x \in A$, it assigns an element $f(x) \in B$.

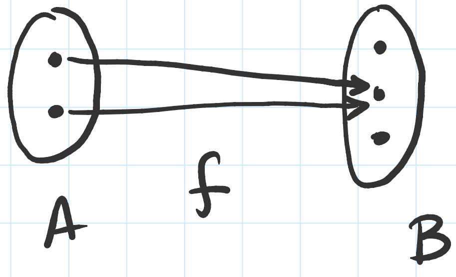
(in this class, we'll consider $f: \mathbb{R}^n \to \mathbb{R}^m$)

Def: We say a function $f: A \to B$ is injective (or one-to-one) when:

If whenever $x, y \in A$ s.t. $x \neq y$, then $f(x) \neq f(y)$.



injective ✓

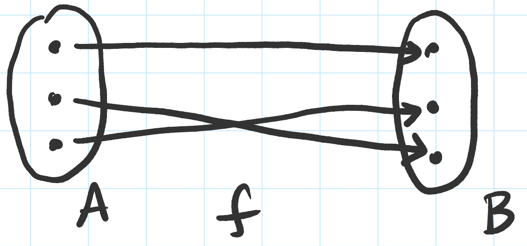


not injective ✗

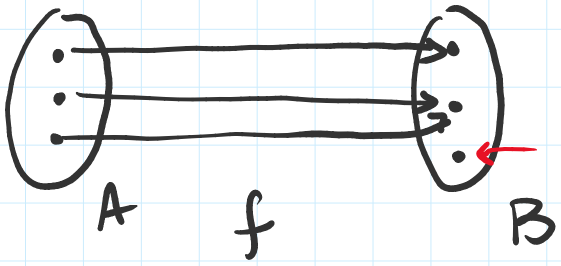
We say a function $f: A \to B$ is surjective (or onto) when:

for every $y \in B$, there exists $x \in A$ s.t. $f(x) = y$.

$x \in A$ s.t. $f(x) = y$.



surjective ✓



not surjective ✗

ex/ Consider $f(x) = x^2$.

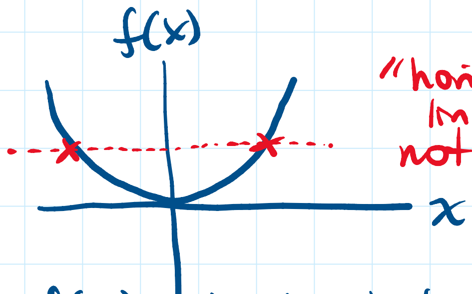
Is it injective / surjective?

Doesn't make sense; must specify domain / codomain.

case 1

- $f : \mathbb{R} \to \mathbb{R}$

  $f(x) = x^2$



"horizontal line test" not injective

Is it injective?

no, e.g. $f(1) = 1 = f(-1)$ but $1 \neq -1$

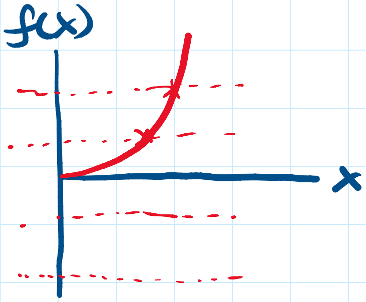
Is it surjective?

is it surjective?

no, let $y<0$. Then, there does exist $x \in \mathbb{R}$ s.t. $f(x)=y$.

Case 2

- $f:[0,\infty) \longrightarrow \mathbb{R}$



injective: yes

proof:

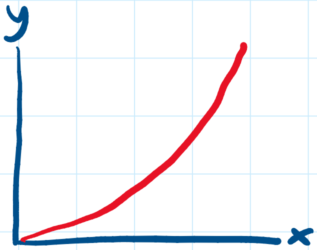
let $x,y \in [0,\infty)$ s.t. $x \neq y$. $f:[0,\infty) \to \mathbb{R}$

WLOG $x>y$. Then, since $f(x)=x^2$ is a strictly monotone increasing function, therefore $\underset{f(x)}{\underset{\shortparallel}{x^2}} > \underset{f(y)}{\underset{\shortparallel}{y^2}}$

In particular, $f(x) \neq f(y)$. $\square$

surjective: no (see prev. case)

Case 3:



$f:[0,\infty) \longrightarrow [0,\infty)$
$f(x)=x^2$

injective: yes (see case 2)

surjective: yes (proof IVT from calc
$f(0)=0$,

calC

$$\left(\begin{array}{l} f(0) = 0 \\ \lim_{x \to \infty} f(x) = \infty \end{array} \right)$$

Theorem (Invertibility)

Let $f : A \to B$ be bijective (means both injective & surjective), then, there exists a function

$$f^{-1} : B \to A$$

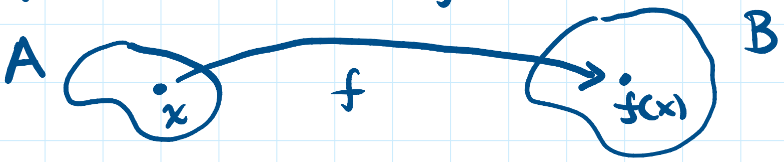
such that $f^{-1}(f(x)) = x$ for all $x \in A$
$f(f^{-1}(y)) = y$ for all $y \in B$

ex/ case 3 $f : [0, \infty) \to [0, \infty)$
$f(x) = x^2$

$$f^{-1}(y) = \sqrt{y}$$

Def: $f : A \to B$

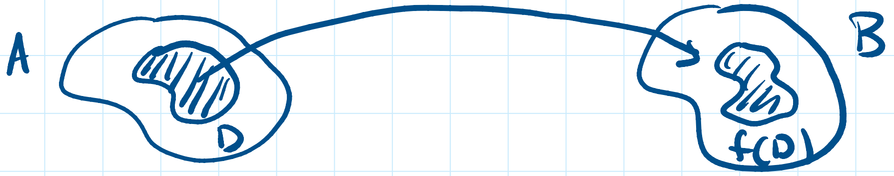
- Say, for $x \in A$, $f(x)$ is the "image of x under f"



subset

subset

- For $D \subseteq A$, call $f(D)$ the image of D under f



$$f(D) = \{ f(x) : x \in D \}$$

- The range of a function of $f: A \to B$ is the image of the domain, $f(A)$

$\Rightarrow$ A function is surjective precisely when its range agrees w/ its codomain, i.e., $f(A) = B$.

Linear Transformations (sect. 1.8 & 1.9)

- In $\mathbb{R}^n$, we have two algebraic operations:

vector addition : $\vec{u} + \vec{v}$ $\quad \vec{u}, \vec{v} \in \mathbb{R}^n$

scalar mult. : $c\vec{u}$ $\quad c \in \mathbb{R}$

Def: A function $T: \mathbb{R}^n \to \mathbb{R}^m$ is

Def: A function $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be a linear transformation when

"the function is compatible with the operations of vector add. & scalar mult."

$$T(c\vec{u} + d\vec{v}) = c\,T(\vec{u}) + d\,T(\vec{v})$$

for all $\vec{u}, \vec{v} \in \mathbb{R}^n$, $c, d \in \mathbb{R}$.

ex/ Let A be an $m \times n$ matrix

(e.g. $\begin{matrix} m=3 \\ n=2 \end{matrix}$ $\quad \underset{A}{\begin{bmatrix} \vdots & \vdots \end{bmatrix}} \underset{\vec{x} \in \mathbb{R}^2}{\begin{bmatrix} \vdots \end{bmatrix}} = \underset{A\vec{x} \in \mathbb{R}^3}{\begin{bmatrix} \vdots \end{bmatrix}}$)

The formula $T(\vec{x}) = A\vec{x}$ $(x \mapsto A\vec{x})$ defines a function $T: \mathbb{R}^{n} \to \mathbb{R}^{m}$

cols rows

This is a linear transformation:

$$\begin{aligned} T(c\vec{u} + d\vec{v}) &= A(c\vec{u} + d\vec{v}) \\ &= cA\vec{u} + dA\vec{v} \\ &= c\,T(\vec{u}) + d\,T(\vec{v}) \end{aligned}$$

$\square$

Theorem [Representation of Linear Transformations as Matrices]

- Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transf. Then, there exists an $m \times n$ matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

proof:

- Let $\vec{e}_i$ denote the i^{th} standard basis vector in $\mathbb{R}^n$

$$\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ entry}$$

$$\left(\text{in } \mathbb{R}^3, \quad \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

- In $\mathbb{R}^n$: $\{\vec{e}_i\}_{i=1}^{n} = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$

Spans $\mathbb{R}^n$

proof: consider $\vec{v} \in \mathbb{R}^n \Rightarrow \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$$\vec{v} = v_1 \vec{e}_1 + v_2 \vec{e}_2 + \dots + v_n \vec{e}_n$$

$$= \sum_{k=1}^{n} v_k \vec{e}_k$$

$$\begin{bmatrix} \textcircled{1} & & & \\ & \textcircled{1} & & \\ & & \ddots & \\ & & & \textcircled{1} \end{bmatrix}$$

- **Claim:** T is completely determined by its action on the standard basis vectors $T(\vec{e}_1), \dots, T(\vec{e}_n)$

proof: given $\vec{v} \in \mathbb{R}$, $\vec{v} = \sum_{k=1}^{n} v_k \vec{e}_k$

$$T(\vec{v}) = T\left(\sum_{k=1}^{n} v_k \vec{e}_k\right)$$

$$= \sum_{k=1}^{n} v_k T(\vec{e}_k) \checkmark$$

$$T(\vec{v}) = \sum_{k=1}^{n} v_k T(\vec{e}_k)$$

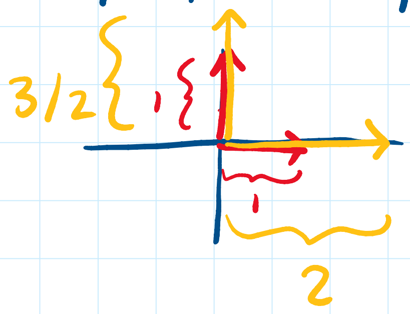
$$= \underbrace{\begin{bmatrix} T(\vec{e}_1) & \cdots & T(\vec{e}_n) \end{bmatrix}}_{=A} \underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}}_{\vec{v}}$$

$$= A\vec{v}$$

$\square$

Def: We call the matrix A constructed above "the matrix associated to the linear transformation" T.

ex/ $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by "dilation"



Find the matrix associated to T

$$T(\vec{e}_1) = 2\vec{e}_1$$
$$T(\vec{e}_2) = \frac{3}{2}\vec{e}_2$$

$$A = \left[T(\vec{e}_1) \quad T(\vec{e}_2) \right] = \left[2\vec{e}_1 \quad \frac{3}{2}\vec{e}_2 \right]$$
$$= \begin{bmatrix} 2 & 0 \\ 0 & 3/2 \end{bmatrix}$$

ex/ $T\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$ $a,b,c,d \in \mathbb{R}$

Find the matrix associated to this linear transf.

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} a \cdot 1 + b \cdot 0 \\ c \cdot 1 + d \cdot 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} a \cdot 0 + b \cdot 1 \\ c \cdot 0 + d \cdot 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$
$$A = \left[T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\square$

$$A\vec{x} = \vec{b}$$

$$\Big\updownarrow$$

$$T(\vec{x}) = \vec{b}$$

If T were a bijection, it would be invertible

$$\vec{x} = T^{-1}(b)$$

Theorem: [Injectivity & Surjectivity of Linear Transformations]

- Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transf.

(Injectivity) T is injective if and only if the columns of the associated matrix A are linearly independent
($\iff$ the only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$)

(Surjectivity) T is surjective if and only if the columns of the associated matrix span $\mathbb{R}^m$

proof:

- T is injective when:

whenever $\vec{x}, \vec{y} \in \mathbb{R}^n$, $\vec{x} \neq \vec{y}$, then $T(\vec{x}) \neq T(\vec{y})$

$\iff$ " $\vec{x}, \vec{y} \in \mathbb{R}^n$

whenever $\vec{x},\vec{y}\in\mathbb{R}^n$, $\vec{x}\neq\vec{y}$, then $T(\vec{x})\neq T(\vec{y})$

$\iff$ " $\vec{x},\vec{y}\in\mathbb{R}^n$, $\vec{x}-\vec{y}\neq\vec{0}$, then $T(\vec{x})-T(\vec{y})\neq\vec{0}$

$\iff$ " $\vec{x},\vec{y}\in\mathbb{R}^n$, $\vec{x}-\vec{y}\neq\vec{0}$, then $\underset{A(\vec{x}-\vec{y})}{\underset{\shortparallel}{T(\vec{x}-\vec{y})}}\neq\vec{0}$ $\square$

- Surjective

$$
\begin{aligned}
\mathbb{R}^m &\overset{\text{when}}{=} T(\mathbb{R}^n)\\
&= \{T(\vec{x}) : \vec{x}\in\mathbb{R}^n\}\\
&= \{A\vec{x} : \vec{x}\in\mathbb{R}^n\}\\
&= \left\{[\vec{a}_1 \cdots \vec{a}_n]\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} : x_1,\ldots,x_n\in\mathbb{R}\right\}\\
&= \{x_1\vec{a}_1+\ldots+x_n\vec{a}_n : x_1,\ldots,x_n\in\mathbb{R}\}\\
&= \text{Span of the columns of } A. \quad \square
\end{aligned}
$$