Section 1.4: Matrix Equations

Let A be an $\underset{\text{rows}}{m} \times \underset{\text{cols}}{n}$ matrix and let $\vec{x} \in \mathbb{R}^n$, the (matrix-vector) product of A and $\vec{x}$, $A\vec{x}$, is defined by:

$$A = [\vec{a}_1 \cdots \vec{a}_n], \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$A\vec{x} := [\vec{a}_1 \cdots \vec{a}_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + \ldots + x_n\vec{a}_n \in \mathbb{R}^m$$

ex/

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$= 1\begin{bmatrix} 3 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 2 \end{bmatrix}.$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3\cdot 1 + 2\cdot 2 + 1\cdot 0 \\ 0\cdot 1 + 1\cdot 2 + (-1)0 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

Properties (Linearity)

For A $m\times n$, $\vec{u} \in \mathbb{R}^n$, $\vec{v} \in \mathbb{R}^n$, $c \in \mathbb{R}$

(i) $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$

(ii) $A(c\vec{v}) = cA\vec{v}$

proof:

$u_1 + v_1$

proof:

$$A(\vec{u}+\vec{v}) = [\vec{a}_1 \cdots \vec{a}_n]\begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}$$

$$= (u_1+v_1)\vec{a}_1 + \dots + (u_n+v_n)\vec{a}_n$$

$$= \underbrace{u_1\vec{a}_1 + \dots + u_n\vec{a}_n}_{A\vec{u}} + \underbrace{v_1\vec{a}_1 + \dots + v_n\vec{a}_n}_{A\vec{v}}$$

$$= A\vec{u} + A\vec{v}$$

Similarly $A(c\vec{u}) = cA\vec{u}$

$\square$

Theorem: Let A be $m\times n$ with cols $A=[\vec{a}_1 \cdots \vec{a}_n]$, $\vec{b}\in\mathbb{R}^m$, then the matrix equation

$$A\vec{x} = \vec{b}$$

unknown, n entries

is equivalent to the vector equation

$$x_1\vec{a}_1 + \dots + x_n\vec{a}_n = \vec{b}$$

(is equivalent to the linear system $[A \ \ \vec{b}]$)

proof:

$$A\vec{x} = x_1\vec{a}_1 + \dots + x_n\vec{a}_n$$

$\square$

Theorem: Let A be $m\times n$.

The following are equivalent:

(i) $A\vec{x}=\vec{b}$ has a solution for every $\vec{b}\in\mathbb{R}^m$

(ii) The columns of A span $\mathbb{R}^m$

(iii) A has a pivot in every row

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<u>Proof</u>:

- From last time, (i) $\Leftrightarrow$ (ii) since $\vec{b} \in \mathbb{R}^m$ arbitrary.
- (iii) $\Leftrightarrow$ (i)

(iii) $\Rightarrow$ (i)

Assume A has a pivot in every row.

Then, let $\vec{b} \in \mathbb{R}^m$, ask does linear system:

$[A \ \vec{b}]$ have a solution?

$$[A \ \vec{b}] \xrightarrow[\text{reduce}]{\text{row}} [U \ \vec{d}]$$

$$\left[\begin{array}{ccc|c} * & & & \\ 0 & * & & \vec{d} \\ 0 & 0 & * & \end{array}\right]$$

$\Rightarrow$ no row of the form $[0 \cdots 0 \ c]$, $c \neq 0$.

(i) $\Rightarrow$ (iii)

- contrapositive not (iii) $\Rightarrow$ not (i) is logically equivalent (warning: converse (iii) $\Rightarrow$ (i) is not logically equivalent)

- Assume that A does not have a pivot in every row.

$$[A \ \vec{b}] \xrightarrow[\text{reduce}]{\text{row}} [U \ \vec{d}]$$

last row is all 0's.

choose $\vec{d}$ such that $d_n \neq 0$.

$\Rightarrow [U \ \vec{d}]$ has a row $[0 \cdots 0 \ d_n]$ $d_n \neq 0$

$\Rightarrow A\vec{x} = \vec{b}$ is inconsistent, i.e. not (i). $\square$

* Note: If A satisfies (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii), then $n \geq m$.

$$\underbrace{\begin{bmatrix} x & x & x & x \\ x & x & x & x \end{bmatrix}}_{n} \Big\} m$$

$$\underbrace{\begin{bmatrix} x & x \\ x & x \\ x & x \\ x & x \end{bmatrix}}_{n} \Bigg\} m$$

$n < m$
not possible to pivot on every row!

$$\begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{③ - \frac{1}{2}① \to ③} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1/2 & 1/2 \end{bmatrix} \to \text{etc...}$$

ex/ Determine whether the matrix
$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$
has a solution to $A\vec{x} = \vec{b}$ for all $\vec{b} \in \mathbb{R}^3$.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow[③ - ① \to ③]{② - 2 \cdot ① \to ②} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & -1 \end{bmatrix}$$

$\square$

Section 1.5: Solution sets of linear system

Def:

Def:

A linear system is homogeneous if it is of the form $A\vec{x} = \vec{0}$, where $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ is the zero vector in $\mathbb{R}^m$.

A: $m \times n$

$\vec{x}$: unknown, n variables

$\vec{0}$: $\in \mathbb{R}^m$

Clearly, $\vec{x} = \vec{0} \in \mathbb{R}^n$ satisfies the homog. eqn.

$A\vec{0} = \vec{0}$. We call this the trivial solution.

Are they are any nontrivial solutions?

<u>Theorem</u>: $A\vec{x} = \vec{0}$ has a nontrivial solution if and only if ($\Leftrightarrow$) the system has at least one free variable.

<u>proof</u>: $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

($\Leftarrow$) Assume the system has at least one free variable. Without loss of generality (WLOG), just assume one free variable.

WLOG assume it's x_n.

$$\begin{aligned} x_1 &= y_1 + c_1 x_n \\ &\vdots \\ x_{n-1} &= y_{n-1} + c_{n-1} x_n \end{aligned}$$

$y_i \in \mathbb{R}$

$c_i \in \mathbb{R}$ ← at least one of them $\neq 0$.

x_n free

$\Rightarrow$ existence of nontrivial solution.

($\Rightarrow$) Proof the contrapositive.

Assume the system has no free variables.

$$[A \;\; \vec{0}] \xrightarrow[\text{reduce}]{\text{row}} [U \;\; \vec{0}]$$

$x_1 = 0$

$\vdots$

$\left[\ \cdots\ \right]$ reduce $\left[\ \cdots\ \right]$ full of pivots $\quad \vdots \quad x_n = 0$

$\Rightarrow$ only trivial solution $\square$

ex/ Determine if $A\vec{x} = \vec{0}$ has any nontrivial solutions where $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

$$A\vec{x} = \vec{0} \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{③-①\to③} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{③-②\to③} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x = -z \\ y = 0 \\ z \text{ free} \end{array}$$

e.g. $z = 1 \Rightarrow x = -1$

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{x} \neq 0} = (-1)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{0}}$$

$\square$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ (shorthand)}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x + z \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

__Theorem:__

Suppose $A\vec{x} = \vec{b}$ is consistent, for some given $\vec{b}$.

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Suppose $A\vec{x}=\vec{b}$ is consistent, for some given $\vec{b}$.

Let $\vec{p}$ be some solution to $A\vec{p}=\vec{b}$. Then, the solution set to $A\vec{x}=\vec{b}$ is the set of all vectors of the form

$$\vec{x}=\vec{p}+\vec{v}_h$$

where $\vec{v}_h$ is a solution to $A\vec{v}_h=\vec{0}$.

proof:

- Check $\vec{x}=\vec{p}+\vec{v}_h$ is a solution

$$A\vec{x}=A(\vec{p}+\vec{v}_h)=\underbrace{A\vec{p}}_{\vec{b}}+\underbrace{A\vec{v}_h}_{\vec{0}}=\vec{b}+\vec{0}=\vec{b}$$

- check any solution is of this form

Let $\vec{w}$ be a solution $A\vec{w}=\vec{b}$.

$$\vec{v}_h=\vec{w}-\vec{p} \qquad (\vec{p}+\vec{v}_h=\vec{p}+\vec{w}-\vec{p}=\vec{w})$$

defined such that $\vec{w}=\vec{p}+\vec{v}_h$

$$A\vec{v}_h=A(\vec{w}-\vec{p})=\underbrace{A\vec{w}}_{\vec{b}}-\underbrace{A\vec{p}}_{\vec{b}}=\vec{b}-\vec{b}=\vec{0}$$

$\square$

Section 1.7 : Linear Independence

1   0 ∞

1 $\qquad\qquad$ 0 $\qquad\qquad$ ∞

$$2x + 4y = \#$$
$$4x + 8y = \#$$

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} x + \begin{bmatrix} 4 \\ 8 \end{bmatrix} y = \vec{\#}$$

Think of homog. eqn. $A\vec{x} = \vec{0}$

$\Leftrightarrow \quad x_1 \vec{a}_1 + \dots + x_n \vec{a}_n = \vec{0}$

__Def__: A list of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ is linearly independent if there is only the trivial solution to $x_1 \vec{a}_1 + \dots + x_p \vec{v}_p = \vec{0}$.

Otherwise, if there is a nontrivial solution, say that they are linearly dependent.

__Proposition__:

A set of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if and only if one of the vectors can be written as a linear comb. of the others.

__proof__:

nontrivial solution

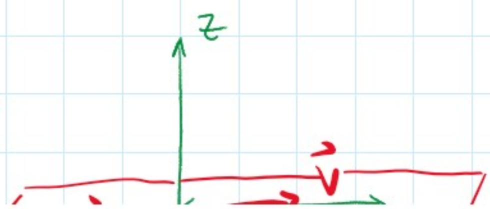
$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots + x_p \vec{v}_p = \vec{0}$$

where one of the $x_i \neq 0$, WLOG $x_1 \neq 0$.

$$x_1 \vec{v}_1 = -x_2 \vec{v}_2 - \dots - x_p \vec{v}_p$$
$$\vec{v}_1 = -\frac{x_2}{x_1} \vec{v}_2 - \dots - \frac{x_p}{x_1} \vec{v}_p$$

$\square$

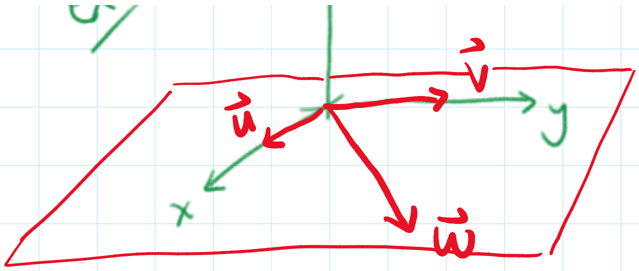
ex/



$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\qquad$ $\vec{w} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} 0 \\ 2 \\ \dots \end{bmatrix}$

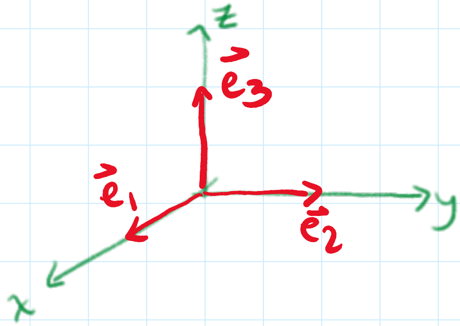
ex/



$\vec{v} = \begin{bmatrix}0\\2\\0\end{bmatrix}$ $\quad w = \begin{bmatrix}\ \\5\\0\end{bmatrix}$

linearly dependent

ex/



$\vec{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $\quad \vec{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\quad \vec{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

linearly independent