Lecture 11 - Eigenproblems for Linear Transformations, Dot Products and Orthogonality

Wednesday, July 26, 2023 9:56 AM

- HW3 due tonight at 11:59 pm
- HW4 posted; due next Wednesday at 11:59 pm
- Practice final will be posted this weekend
- The Final will be in this room (Center 216) at the time listed on the schedule of classes
- Student Evaluation of Teaching (SET), evaluation period from Fri July 28 - Fri Aug 4 (ends at 8 am). If >60% of the class fills out their SETs, I'll add 1% extra credit to everyone's grades.

Def: Let V be a vector space. An eigenvector of a linear transf $T: V \to V$ is a nonzero vector $\vec{x} \in V$ st.

$$T(\vec{x}) = \lambda \vec{x}$$

for some scalar λ called an eigenvalue of T.

ex/ $C^\infty(\mathbb{R}, \mathbb{R})$

the vector space of all infinitely differentiable functions

(e.g. contains sines, cosines, polynomials, exponential, ...)

lin. transf. $\frac{d}{dx}: C^\infty(\mathbb{R}, \mathbb{R}) \longrightarrow C^\infty(\mathbb{R}, \mathbb{R})$

$f(x) = e^x$ is an eigenvector of d/dx

$$\frac{d}{dx}(f(x)) = \frac{d}{dx} e^x = e^x = f(x)$$

eigenvalue $\lambda = 1$

more generally, $f_k(x) = e^{kx}$ $(k \neq 0, k \in \mathbb{R})$

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is an eigenvector of d/dx

$$\frac{d}{dx}(f_k(x)) = \frac{d}{dx} e^{kx} = k e^{kx} = k f_k(x)$$

eigenvalue $\lambda = k$.

* important in differential equations *

ex/ $T: \mathbb{P}_2 \to \mathbb{P}_2$ lin. transf.

$T(c_0 + c_1 t + c_2 t^2)$

$= (c_0 + c_2) + c_1 t + (c_0 + c_2) t^2$

let $p(t) = 1 + t^2$

$T(p(t)) = T(1+t^2) = 2 + 2t^2 = 2(1+t^2) = 2p(t)$

$\Rightarrow p(t)$ is an eigenvector of T
w/ eigenvalue 2.

Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis for a vector space V.

$\vec{x} \in V \Rightarrow$ there is a unique expansion

$$\vec{x} = c_1 \vec{b}_1 + \dots + c_n \vec{b}_n$$

coordinates $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$

mapping from V to $\mathbb{R}^n$ "coordinate map"

$C_{\mathcal{B}} : V \to \mathbb{R}^n$

$$C_{\mathcal{B}} : V \to \mathbb{R}^n$$
$$\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$$

invertible linear transf.

$$C_{\mathcal{B}}^{-1} : \mathbb{R}^n \to V$$
$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \mapsto c_1\vec{b}_1 + \dots + c_n\vec{b}_n$$

Let $T : V \to V$ be a lin. transf.

$$\vec{x} \mapsto T(\vec{x})$$

$$\underset{\mathbb{R}^n}{[\vec{x}]_{\mathcal{B}}} \longmapsto \underset{\mathbb{R}^n}{[T(\vec{x})]_{\mathcal{B}}}$$ is a lin. transf. from $\mathbb{R}^n$ to $\mathbb{R}^n$

$\Rightarrow$ we can represent this map as a matrix, call it $[T]_{\mathcal{B}}$

$$[T(\vec{x})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$$

$$\begin{tikzcd}
\vec{x} \arrow[r, mapsto, "T"] \arrow[d, mapsto, "C_{\mathcal{B}}"'] & T(\vec{x}) \arrow[d, mapsto, "C_{\mathcal{B}}"] \\
{[\vec{x}]_{\mathcal{B}}} \arrow[r, mapsto, "\text{multiply by } {[T]_{\mathcal{B}}}"'] & {[T(\vec{x})]_{\mathcal{B}}}
\end{tikzcd}$$

claim:

$$[T]_{\mathcal{B}} = \Big[\, [T(\vec{b}_1)]_{\mathcal{B}} \; \cdots \; [T(\vec{b}_n)]_{\mathcal{B}} \,\Big]$$ $n \times n$ matrix

$$[T]_{\mathcal{B}} = \begin{bmatrix} [T(\vec{b}_1)]_{\mathcal{B}} & \cdots & [T(\vec{b}_n)]_{\mathcal{B}} \end{bmatrix}$$

<u>proof:</u>

let $\vec{x} \in V \Rightarrow \vec{x} = c_1\vec{b}_1 + \ldots + c_n\vec{b}_n$

$$T(\vec{x}) = c_1 T(\vec{b}_1) + \ldots + c_n T(\vec{b}_n)$$

Apply $C_{\mathcal{B}}$ to both sides

$$[T(\vec{x})]_{\mathcal{B}} = c_1 [T(\vec{b}_1)]_{\mathcal{B}} + \ldots + c_n [T(\vec{b}_n)]_{\mathcal{B}}$$

$$= \underbrace{\begin{bmatrix} [T(\vec{b}_1)]_{\mathcal{B}} & \cdots & [T(\vec{b}_n)]_{\mathcal{B}} \end{bmatrix}} \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}_{[\vec{x}]_{\mathcal{B}}}$$

$$\Rightarrow \; = [T]_{\mathcal{B}}.$$

$\square$

We call $[T]_{\mathcal{B}}$ the matrix representation of $T: V \to V$ in the basis $\mathcal{B}$.

<u>Theorem</u> [Eigenproblems for Lin. Transfs $T: V \to V$]

- Let $T: V \to V$ where V has a basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$.

- Then, the eigenvalues of T are the same as the eigenvalues of its $\mathcal{B}$-matrix representation $[T]_{\mathcal{B}}$.

- Furthermore, if $\vec{x}$ is an eigenvector

- Furthermore, if $\vec{x}$ is an eigenvector of T w/ eigenvalue λ, then $[\vec{x}]_{\mathcal{B}}$ is an eigenvector of $[T]_{\mathcal{B}}$ w/ eigenval λ

- Conversely, if $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is an eigenvector of $[T]_{\mathcal{B}}$ w/ eigenvalue λ, then $c_1\vec{b}_1 + \dots + c_n\vec{b}_n$ is an eigenvector of T w/ eigenvalue λ.

proof:

$$T(\vec{x}) = \lambda\vec{x} \iff \underbrace{[T(\vec{x})]_{\mathcal{B}}}_{[T]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}} = \lambda[\vec{x}]_{\mathcal{B}}$$

□

ex/ $T: \mathbb{P}_2 \to \mathbb{P}_2$ lin. transf., find its eigenvals & eigenvector

$$T(c_0 + c_1 t + c_2 t^2)$$
$$= (c_0 + c_2) + c_1 t + (c_0 + c_2)t^2$$

$$c_0 + c_1 t + c_2 t^2 \xrightarrow{C_{\mathcal{B}}} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$$

monomial basis for $\mathbb{P}_2$ $\mathcal{B} = \{\underset{b_1(t)}{1}, \underset{b_2(t)}{t}, \underset{b_3(t)}{t^2}\}$.

$$[T]_{\mathcal{B}} = \left[[\underbrace{T(b_1(t))}_{T(1)}]_{\mathcal{B}} \quad [\underbrace{T(b_2(t))}_{T(t)}]_{\mathcal{B}} \quad [\underbrace{T(b_3(t))}_{T(t^2)}]_{\mathcal{B}} \right]$$

$$= \left[[1 + t^2]_{\mathcal{B}} \quad [t]_{\mathcal{B}} \quad [1 + t^2]_{\mathcal{B}} \right]$$

$$= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

compute & eigenvectors, but for example, you'll get $\lambda = 2$ with eigenvec $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow T$ has eigenvalue $\lambda = 2$ w/ eigenvec

$$C_{\mathcal{B}}^{-1}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = 1 + 0 \cdot t + 1 \cdot t^2 = 1 + t^2$$

$\square$

What about $T: \mathbb{R}^n \to \mathbb{R}^n$?

$$T(\vec{x}) = A\vec{x}$$

Suppose C is similar to A,

$$A = PCP^{-1} \text{ where } P \text{ invertible.}$$

Let $\mathcal{B} = \{\vec{p}_1, \dots, \vec{p}_n\}$ $\quad P = [\vec{p}_1 \cdots \vec{p}_n]$

Then, C is the representation of T in the basis $\mathcal{B}$, $C = [T]_{\mathcal{B}}$.

(whereas $A = [T]_{\text{standard basis}}$)

<u>proof</u>:

$$[T]_{\mathcal{B}} = \left[\ [T(\vec{p}_1)]_{\mathcal{B}} \cdots [T(\vec{p}_n)]_{\mathcal{B}}\ \right]$$

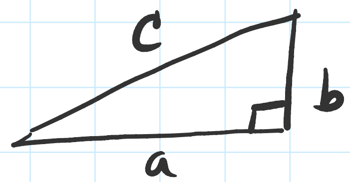
$$= \left[\ [A\vec{p}_1]_{\mathcal{B}} \cdots [A\vec{p}_n]_{\mathcal{B}}\ \right]$$

$$= [P^{-1}A\vec{p}_1 \cdots P^{-1}A\vec{p}_n]$$

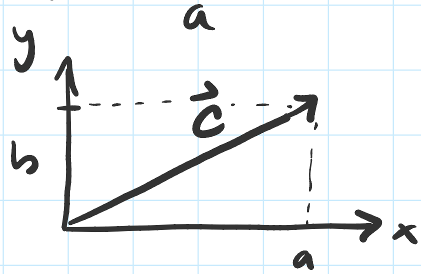
$$= [P^{-1}A\vec{p}_1 \cdots P^{-1}A\vec{p}_n]$$
$$= P^{-1}A[\vec{p}_1 \cdots \vec{p}_n] = P^{-1}AP = C \quad \square$$

Chapter 6: Dot Products & Orthogonality

Pythagorean Theorem in $\mathbb{R}^2$



$$c^2 = a^2 + b^2$$



$$\vec{c} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\text{length}(\vec{c})^2 = a^2 + b^2$$
$$= c_1^2 + c_2^2$$

Definition: The dot product (or inner product) of two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, denoted $\vec{u} \cdot \vec{v}$,

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$$

If $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, then $\vec{u} \cdot \vec{v} = u_1 v_1 + \ldots + u_n v_n$
$$= \sum_{j=1}^{n} u_j v_j$$

Properties:

symmetric $\quad \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

linear $\quad \vec{x} \cdot (a\vec{u} + b\vec{v}) = a\vec{x} \cdot \vec{u} + b\vec{x} \cdot \vec{v}$

positive-definite

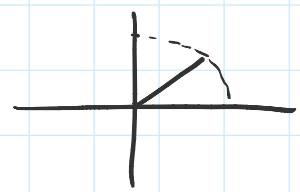
$\vec{u} \cdot \vec{u} \geq 0$ and equals zero if & only if $\vec{u} = \vec{0}$

The length (or norm) of a vector $\vec{u} \in \mathbb{R}^n$ is defined as $\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}}$

ex/ Find length of $\vec{u} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

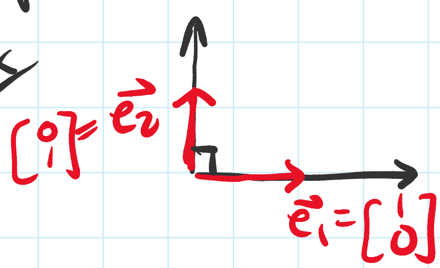
$$\vec{u}\cdot\vec{u} = u_1^2 + u_2^2 = \frac{1}{2} + \frac{1}{2} = 1$$

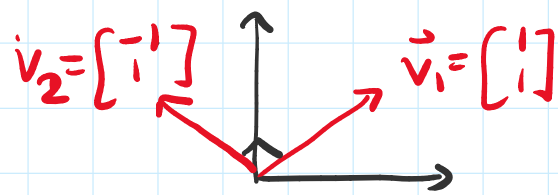
$$\Rightarrow \|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{1} = 1$$



Dot products between $\vec{u} \neq \vec{v}$?

ex/



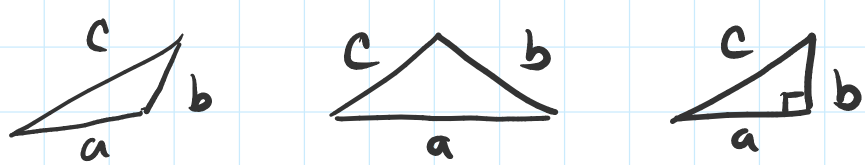


$$\vec{e}_1\cdot\vec{e}_2 = 1\cdot 0 + 0\cdot 1 = 0$$

$$\vec{v}_1\cdot\vec{v}_2 = (-1)1 + 1\cdot 1 = 0$$

Def: Say two vectors $\vec{u}$ and $\vec{v}$ are orthogonal when $\vec{u}\cdot\vec{v} = 0$

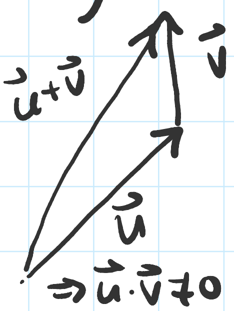
↳ perpendicular

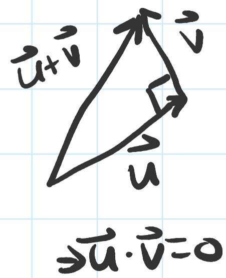


$a^2 + b^2 = c^2$ if & only if it's a right triangle

Thm: [Pythagorean Theorem in $\mathbb{R}^n$]
Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Then, the Pythagorean theorem $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$ holds if & only if $\vec{u}\cdot\vec{v}=0$

proof:





$$\begin{aligned}\|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \\ &= \underbrace{\vec{u}\cdot\vec{u}} + \underbrace{\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u}} + \underbrace{\vec{v}\cdot\vec{v}} \\ &= \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2\end{aligned}$$

pythag. thm holds $\Leftrightarrow$ $\vec{u}\cdot\vec{v} = 0$ $\square$

Def: Let W be a subspace of $\mathbb{R}^n$. The orthogonal complement of W, denoted $W^\perp$, is the set of all vectors orthogonal to W:

$$W^\perp = \{\vec{u} \in \mathbb{R}^n : \vec{u}\cdot\vec{w} = 0 \text{ for all } \vec{w} \in \mathbb{R}^n\}.$$

Thm: Let W be a subspace of $\mathbb{R}^n$.

(i) $W^\perp$ is a subspace of $\mathbb{R}^n$

(ii) $\vec{u} \in W^\perp \Leftrightarrow \vec{u}$ is orthogonal to every vector in a set of vectors that spans W

to every vector in a set of vectors that spans W.

<u>proof</u>: (i) HW4

(ii) ($\Rightarrow$) easy

($\Leftarrow$) let $\{\vec{w}_1, \dots, \vec{w}_p\}$ span W.

let $\vec{u}$ be orthogonal to $\vec{w}_1, \dots, \vec{w}_p$

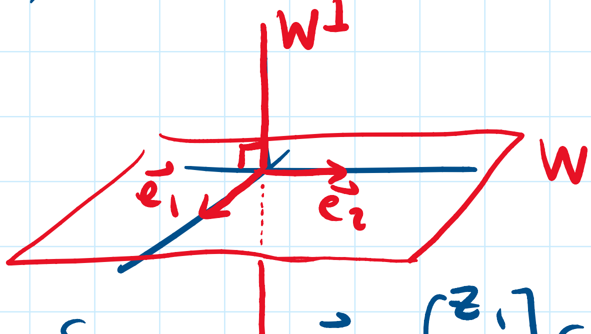
i.e. $\vec{w}_j \cdot \vec{u} = 0$ for all $j = 1, \dots, p$

let $\vec{w} \in W \Rightarrow \vec{w} = c_1 \vec{w}_1 + \dots + c_p \vec{w}_p$

$$\vec{w} \cdot \vec{u} = c_1 \underbrace{\vec{w}_1 \cdot \vec{u}}_{=0} + \dots + c_p \underbrace{\vec{w}_p \cdot \vec{u}}_{=0} = 0.$$

$\square$

ex/ consider $z=0$ xy plane in $\mathbb{R}^3$, call it W.



What is $W^\perp$?

$W = \text{span}\{\vec{e}_1, \vec{e}_2\}$

$= \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$

Suppose $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in W^\perp$.

$$0 = \vec{z} \cdot \vec{e}_1 = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = z_1$$

$$0 = \vec{z} \cdot \vec{e}_2 = z_2$$

$\Rightarrow z_1 = 0 = z_2$.

z_3 can be anything

$$\Rightarrow W^\perp = \left\{ \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} : c \in \mathbb{R} \right\} \leftarrow \text{z-axis}$$

note: $\dim(W) = 2$ $\dim(\mathbb{R}^3) = 3$

note $\dim(W) = 2$ $\quad \dim(\mathbb{R}^3) = 3$
$\dim(W^\perp) = 1$

Def: An orthogonal set of vectors in $\mathbb{R}^n$ $\{\vec{v}_1, \dots, \vec{v}_p\}$ is a set of vectors that are mutually orthogonal $\vec{v}_j \cdot \vec{v}_i = 0$, $i \neq j$

ex) standard basis in $\mathbb{R}^n$
is an orthogonal set

Thm: If S is an orthogonal set of nonzero vectors, then S is linearly independent.

proof:
$S = \{\vec{v}_1, \dots, \vec{v}_p\}$
Dependence relation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p = \vec{0}$$

dot both sides with $\vec{v}_1$

$$c_1\vec{v}_1\cdot\vec{v}_1 + c_2\underbrace{\vec{v}_2\cdot\vec{v}_1}_{=0} + \dots + c_p\underbrace{\vec{v}_p\cdot\vec{v}_1}_{=0} = \vec{0}\cdot\vec{v}_1 = 0$$

$$\Rightarrow c_1\underbrace{\vec{v}_1\cdot\vec{v}_1}_{\|\vec{v}_1\|^2 > 0} = 0 \Rightarrow c_1 = 0.$$

Similarly, $c_2 = 0, \dots, c_p = 0$
$\Rightarrow$ trivial dep. relation $\square$