Lecture 10 - Characteristic Polynomial; Diagonalization

Monday, July 24, 2023 9:58 AM

- HW3 due Wednesday July 26th at 11:59 pm
- Midterm solutions will be posted tomorrow.

λ is an eigenvalue of A

$\iff$ $\text{nul}(A-\lambda I)$ is non-trivial

$\iff$ $A-\lambda I$ is not invertible

$\iff$ $\det(A-\lambda I)=0$ characteristic equation of A

ex/ Find the eigenvalues of $A=\begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix}$

$$A-\lambda I=\begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}=\begin{bmatrix}2-\lambda & 1\\ 0 & 1-\lambda\end{bmatrix}$$

$0=\det(A-\lambda I)=(2-\lambda)(1-\lambda)$ $\leftarrow$ polynomial of degree 2

$\Rightarrow$ eigenvalues 2, 1

Def: The polynomial of degree n in λ for an $n\times n$ matrix A,

$$p(\lambda)=\det(A-\lambda I),$$

is called the characteristic polynomial of A.

λ is an eigenvalue of A

$\iff$ λ is a root of the char. poly.

$\iff$ λ is a root of the char. poly. of A.

[Fundamental Theorem of Algebra]
Let $q(t)$ be a polynomial of degree n.
Then, there exists n roots of $q(t)$, including multiplicity, $t_1, \dots, t_n \in \mathbb{C}$ ← complex numbers

s.t. $q(t_j) = 0 \quad j = 1, \dots, n.$

$\mathbb{C} = \{x + iy : x, y \in \mathbb{R},\ i = \sqrt{-1}\}$

$(t-2)^2 = t^2 - 4t + 4$ has root $t = 2$ with multiplicity 2

$(t-3)^3(t-1)(t+1)^2$ roots $t=3$ mult. 3
$t=1$ mult. 1
$t=-1$ mult. 2

ex/ Find the eigenvalues
$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix}\right)$

$= (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 1 + 1$
$= \lambda^2 - 2\lambda + 2$ ←

quadratic formula

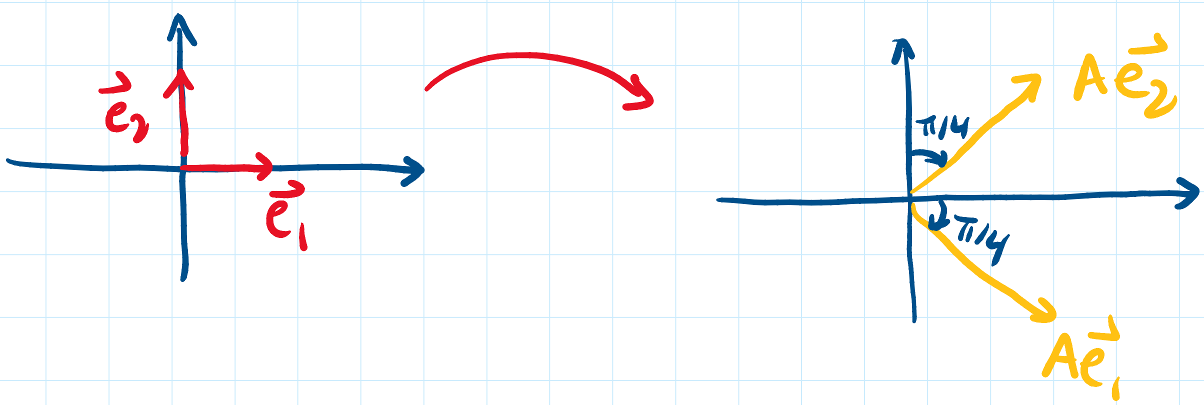
quadratic formula

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



A scales by $\sqrt{2}$ but also rotates clockwise by $\pi/4$

$\Rightarrow$ complex eigenvalues correspond to rotation

ex/ Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$ & a basis for each eigenspace

$\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$ each eigenspace.

$$0 = \det(A-\lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{bmatrix}\right)$$

$$= (2-\lambda)\left((1-\lambda)^2-1\right) = (2-\lambda)(\lambda^2-2\lambda)$$

$$= (2-\lambda)(\lambda-2)\lambda = (2-\lambda)(2-\lambda)(0-\lambda)$$

eigenvalues $\lambda=0$ multiplicity 1

$\lambda=2$ multiplicity 2

$(\lambda=0)$ eigenspace $\text{nul}(A-0I) = \text{nul}(A)$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_3$

$\Rightarrow x_2 = -x_3/2 \Rightarrow$ basis $\left\{\begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}\right\}$.

x_3 free

$(\lambda=2)$ eigenspace $\text{nul}(A-2I)$

$$A-2I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$
x_2 free $\Rightarrow$ basis $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.
$x_3 = 0$

Theorem:
Let $\vec{v}_1, \ldots, \vec{v}_r$ be eigenvectors of an $n \times n$ matrix A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$. Then, $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is linearly independent.

Proof:
Suppose for contradiction $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is linearly dependent.

Let $j < r$ be the maximal index s.t. $\vec{v}_1, \ldots, \vec{v}_j$ is independent.

$\Rightarrow \vec{v}_{j+1} = c_1 \vec{v}_1 + \ldots + c_j \vec{v}_j$ for some scalars $c_1, \ldots, c_j$ (at least one nonzero)

$$A\vec{v}_{j+1} = c_1 A\vec{v}_1 + \ldots + c_j A\vec{v}_j$$
$$= c_1 \lambda_1 \vec{v}_1 + \ldots + c_j \lambda_j \vec{v}_j$$

$$= c_1\lambda_1\vec{v}_1 + \dots + c_j\lambda_j\vec{v}_j$$

$$A\vec{v}_{j+1} = \lambda_{j+1}\vec{v}_{j+1}$$

$$= c_1\lambda_{j+1}\vec{v}_1 + \dots + c_j\lambda_{j+1}\vec{v}_j$$

subtract

$$\Rightarrow \vec{0} = c_1\underbrace{(\lambda_1-\lambda_{j+1})}_{\neq 0}\vec{v}_1 + \dots + c_j\underbrace{(\lambda_j-\lambda_{j+1})}_{\neq 0}\vec{v}_j$$

$\Rightarrow$ non-trivial dependence relation between $\{\vec{v}_1,\dots,\vec{v}_j\}$. contradiction

$\square$

Thm: [Characteristic Polynomial]

Let $\lambda_1,\dots,\lambda_n$ be the eigenvalues of A $(n\times n)$ (including multiplicity). Then,

$$p(\lambda) = \det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)(\cdots)(\lambda_n-\lambda)$$

proof: a polynomial $f(t)$ of degree n is determined by its n roots $t_1,\dots,t_n$ up to a constant factor $c\neq 0$

$$f(t) = c(t_1-t)(t_2-t)(\cdots)(t_n-t)$$
$$= c(-1)^n t^n + \text{lower degree terms}$$

for $p(\lambda)$, we can compute leading factor c

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$$A-\lambda I = \begin{bmatrix} a_{11}-\lambda & & & \\ & a_{22}-\lambda & & \\ & & \ddots & \\ & & & a_{nn}-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (-1)^n \lambda^n + \text{lower order terms}$$

$\Rightarrow c = 1$ $\square$

<u>Corollary</u>: The determinant of a square matrix is the product of its eigenvalues

<u>pf</u>:

$$\underset{\det(A-\lambda I)}{\overset{p(\lambda)}{\parallel}} = (\lambda_1-\lambda)(\cdots)(\lambda_n-\lambda)$$

plug in $\lambda=0 \Rightarrow \det(A) = \lambda_1 \cdots \lambda_n$

ex A & A^T have same eigenvalues

$$(A-\lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$$

$$\det((A-\lambda I)^T) = \det(A^T-\lambda I) = p_{A^T}(\lambda)$$

$\parallel \leftarrow \det B^T = \det B$

$\det(A-\lambda I)$

$\parallel$ $p_A(\lambda)$

$\Rightarrow A$ & A^T have same char. poly.

$\Rightarrow$ A & A^T have same char. poly.

$\Rightarrow$ A & A^T have same eigenvalues

$\Rightarrow$ A inv. if & only if A^T is. $\square$

Diagonalization (section 5.3)

<u>Def</u>: Two $n \times n$ matrices A & B are said to be similar if there exists an invertible matrix P s.t.

$$A = PBP^{-1}$$

<u>Properties</u>: let A & B be similar. Then,

$$\det(B) = \det(A)$$

$$\det(B - \lambda I) = \det(A - \lambda I)$$

they have the same eigenvalues.

<u>proof</u>: hw 3. $\square$

justification:

let A & B be similar,

$$A = PBP^{-1}$$

$P = [\vec{p}_1 \cdots \vec{p}_n]$

$\mathcal{P} = \{\vec{p}_1, \ldots, \vec{p}_n\}$ is a basis for $\mathbb{R}^n$

$\begin{pmatrix}\text{mult. by} \\ P^{-1}\end{pmatrix} : \vec{x} \mapsto [\vec{x}]_{\mathcal{P}}$

$$A\vec{x} = PBP^{-1}\vec{x}$$

$\begin{pmatrix}\text{mult. by} \\ \ldots\end{pmatrix} : [\vec{x}]_{\mathcal{P}} \mapsto \vec{x}$

$$A\vec{x} = PBP^{-1}\vec{x}$$
$$= PB[\vec{x}]_{\mathcal{P}}$$

$\begin{pmatrix}\text{mult. by} \\ P\end{pmatrix}: [\vec{x}]_{\mathcal{P}} \mapsto \vec{x}$

Def: An $n\times n$ matrix A is said to be diagonalizable if it is similar to a diagonal matrix.

I.e., there exists P invertible & D diagonal s.t. $A = PDP^{-1}$.

Thm: [Diagonalization]

An $n\times n$ matrix A is diagonalizable

$\Leftrightarrow$ A has n linearly indep eigenvectors

proof:

($\Rightarrow$) Assume A is diagonalizable

$$A = \underset{\text{inv.}}{P}\,\underset{\text{diagonal}}{D}\,P^{-1}$$

$$\Rightarrow AP = PD$$

$$A[\vec{p}_1 \cdots \vec{p}_n] = [\vec{p}_1 \cdots \vec{p}_n]\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & d_n \end{bmatrix}$$

$$\Rightarrow [A\vec{p}_1 \cdots A\vec{p}_n] = [d_1\vec{p}_1 \cdots d_n\vec{p}_n]$$

$$\Rightarrow A\vec{p}_j = d_j\vec{p}_j \quad j=1,\dots,n.$$

$\Rightarrow$ $\vec{p}_j$ are eigenvectors of A w/ eigenvalues d_j $j=1,\dots,n$

w/ eigenvalues λ_j $j=1,\dots,n$

$\Rightarrow$ A has n lin. indep eigenvectors.

($\Leftarrow$) Suppose A has n lin. indep. eigenvectors $\{\vec{v}_1,\dots,\vec{v}_n\}$ w/ corresp. eigenvalues $\lambda_1,\dots,\lambda_n$

$P=[\vec{v}_1 \cdots \vec{v}_n]$ is invertible since n lin. indep eigenvects

$$AP=[A\vec{v}_1 \cdots A\vec{v}_n]=[\lambda_1\vec{v}_1 \cdots \lambda_n\vec{v}_n]$$

$$=\underbrace{[\vec{v}_1 \cdots \vec{v}_n]}_{P}\underbrace{\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}}_{D}=PD$$

$\Rightarrow A=PDP^{-1}$

$\square$

As a result of the prev. thm.

an $n\times n$ matrix A is diagonalizable

$\Longleftrightarrow$ the dimension of each eigenspace equals the multiplicity of the corresponding eigenvalue

$\Longleftrightarrow$ the sum of all of the dimensions of the eigenspaces $=n$.

ex/ Any $n \times n$ matrix w/ n distinct eigenvalues is diagonalizable.

ex/ Determine if one can diagonalize

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} ?$$

If so, diagonalize, i.e. find P invertible & D diagonal $A = PDP^{-1}$.

find eigenvalues

$$0 = \det(A - \lambda I) = -(\lambda-1)(\lambda+2)^2$$

$\Rightarrow$ $\lambda = -2$ multiplicity 2

$\lambda = 1$ multiplicity 1

$(\lambda=1)$ eigenspace $\mathrm{nul}(A-I)$

$$A - I = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}$$

$$\xrightarrow{(1)\leftrightarrow(3)} \begin{bmatrix} 3 & 3 & 0 \\ -3 & -6 & -3 \\ 0 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 3 & 0 \\ 0 & -3 & -3 \\ 0 & 3 & 3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 3 & 3 & 0 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$\uparrow x_3$ free

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= -x_3 \\ x_3 &\text{ free} \end{aligned}$$

$\lambda=1$ eigenspace basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ $= \vec{v}_1$

$(\lambda=-2)$ eigenspace $\mathrm{nul}(A-(-2)I)$

$$A+2I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}$$

$$\to \begin{bmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2 & x_3 free

$$\begin{aligned} x_1 &= -x_2-x_3 \\ x_2 &\text{ free} \\ x_3 &\text{ free} \end{aligned}$$

$\Rightarrow$ $\lambda=-2$ eigenspace basis $\left\{ \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\dim(\mathrm{nul}(A-(-2)I)) = 2$$
$$= (\text{multiplicity of } \lambda=-2)$$

$$P = [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \ldots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ \ldots \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} & & \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

<u>Check</u>: $\det(P) \neq 0$

$$AP \underset{\uparrow \text{check}}{=} PD$$

$\square$

ex/ Determine if $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is diagonalizable?

Eigenvalues are $1, 0, -1$.
All distinct and therefore diagonalizable.

* An $n \times n$ matrix A is diagonalizable $\Leftrightarrow$ it has n eigenvectors that form a basis for $\mathbb{R}^n$ *