# Liouville's Note on Kelvin's Electric Images 

# A Translation of "Note au sujet de l'article précédent" by J. Liouville* 

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1. The Letter from Mr. Thomson ${ }^{\dagger}$ has suggested to me a few remarks which I should present here, because they will show, it seems to me, even more clearly the importance of the work of which this young geometer from Glasgow gave us a quick extract.

We first solve the following problem:
Problem.-Let $x, y, \ldots, z$ and $\xi, \eta, \ldots, \zeta$ be two groups containing an equal or unequal number of variables, the first $x, y, \ldots, z$ independent, the other $\xi, \eta, \ldots, \zeta$ functions of the first, so that

$$
\xi=f(x, y, \ldots, z), \quad \eta=F(x, y, \ldots, z), \ldots, \quad \zeta=\phi(x, y, \ldots, z)
$$

in addition,

$$
p=\psi(x, y, \ldots, z) .
$$

Let us denote by $\xi^{\prime}, \eta^{\prime}, \ldots, \zeta^{\prime}, p^{\prime}$ what the functions $\xi, \eta, \ldots, \zeta, p$ become when we replace $x, y, \ldots, z$ by $x^{\prime}, y^{\prime}, \ldots, z^{\prime}$. Now, we ask to determine the functions $f, F, \ldots, \phi, \psi$, so that in general

$$
\left(\xi^{\prime}-\xi\right)^{2}+\left(\eta^{\prime}-\eta\right)^{2}+\cdots+\left(\zeta^{\prime}-\zeta\right)^{2}=\frac{\left(x^{\prime}-x\right)^{2}+\left(y^{\prime}-y\right)^{2}+\cdots+\left(z^{\prime}-z\right)^{2}}{p^{2} p^{\prime 2}}
$$

[^0]To fix ideas, we limit ourselves to the case of three variables $x, y, z$ and three variables $\xi, \eta, \zeta$. The question then becomes to validate the equation

$$
\begin{equation*}
\left(\xi^{\prime}-\xi\right)^{2}+\left(\eta^{\prime}-\eta\right)^{2}+\left(\zeta^{\prime}-\zeta\right)^{2}=\frac{\left(x^{\prime}-x\right)^{2}+\left(y^{\prime}-y\right)^{2}+\left(z^{\prime}-z\right)^{2}}{p^{2} p^{2}} \tag{1}
\end{equation*}
$$

The same method would work for any two groups $x, y, \ldots, z$ and $\xi, \eta, \ldots, \zeta$. There would only be a change in a few details if the number of variables is different in the two groups. Besides, we will only need the case when this number is the same on both sides and does not exceed three, which will allow us to interpret the result of our analysis geometrically.

Give $x^{\prime}, y^{\prime}, z^{\prime}$ particular values $x_{0}, y_{0}, z_{0}$ at will, and represent the corresponding values of $p^{\prime}, \xi^{\prime}, \eta^{\prime}, \zeta^{\prime}$ by $p_{0}, \xi_{0}, \eta_{0}, \zeta_{0}$. Then Equation (1) gives us

$$
p^{2}=\frac{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}+\left(z-z_{0}\right)^{2}}{p_{0}^{2}\left[\left(\xi-\xi_{0}\right)^{2}+\left(\eta-\eta_{0}\right)^{2}+\left(\zeta-\zeta_{0}\right)^{2}\right]}
$$

For simplicity, we write $\xi+\xi_{0}, \eta+\eta_{0}, \zeta+\zeta_{0}, x+x_{0}, y+y_{0}, z+z_{0}$, in place of $\xi, \eta, \zeta, x, y, z$, and similarly $\xi^{\prime}+\xi_{0}, x^{\prime}+x_{0}$, etc., in place of $\xi^{\prime}, x^{\prime}$, etc., which does not change the differences $\xi^{\prime}-\xi, x^{\prime}-x$, etc. The value of $p^{2}$ becomes

$$
p^{2}=\frac{x^{2}+y^{2}+z^{2}}{p_{0}^{2}\left(\xi^{2}+\eta^{2}+\zeta^{2}\right)}
$$

and Equation (1) remains as it is.
By writing

$$
\begin{array}{ll}
x^{2}+y^{2}+z^{2}=r^{2}, & \xi^{2}+\eta^{2}+\zeta^{2}=\rho^{2} \\
x^{\prime 2}+y^{\prime 2}+z^{\prime 2}=r^{\prime 2}, & \xi^{\prime 2}+{\eta^{\prime}}^{2}+\zeta^{\prime 2}=\rho^{\prime 2}
\end{array}
$$

we have

$$
p^{2}=\frac{r^{2}}{p_{0}^{2} \rho^{2}}, \quad p^{\prime 2}=\frac{r^{2}}{p_{0}^{2} \rho^{2}},
$$

and by inserting these values into Equation (1), we find

$$
\frac{1}{\rho^{2}}+\frac{1}{\rho^{\prime 2}}-2\left(\frac{\xi}{\rho^{2}} \frac{\xi^{\prime}}{\rho^{\prime 2}}+\frac{\eta}{\rho^{2}} \eta^{\prime} \rho^{2}+\frac{\zeta}{\rho^{2}} \frac{\zeta^{\prime}}{\rho^{\prime 2}}\right)
$$

$$
=p_{0}^{4}\left[\frac{1}{r^{2}}+\frac{1}{r^{\prime 2}}-2\left(\frac{x}{r^{2}} \frac{x^{\prime}}{r^{\prime 2}}+\frac{y}{r^{2}} \frac{y^{\prime}}{r^{\prime 2}}+\frac{z}{r^{2}} \frac{z^{\prime}}{r^{\prime 2}}\right)\right] .
$$

Now assign $x^{\prime}, y^{\prime}, z^{\prime}$ four sets of known values at will, each of which determines the values of $r^{\prime}, \xi^{\prime}, \eta^{\prime}, \zeta^{\prime}, \rho^{\prime}$, and we will thereby obtain four equations of first order that yields the values of

$$
\frac{\xi}{\rho^{2}}, \quad \frac{\eta}{\rho^{2}}, \quad \frac{\zeta}{\rho^{2}}, \quad \frac{1}{\rho^{2}},
$$

considered as the four unknowns, as a linear function of

$$
\frac{x}{r^{2}}, \quad \frac{y}{r^{2}}, \quad \frac{z}{r^{2}}, \quad \frac{1}{r^{2}} .
$$

By denoting by $A, B, C, D$ the constants and by $P, Q, R, S$ the first order polynomials in $x, y, z$, these values take the form

$$
\frac{\xi}{\rho^{2}}=A+\frac{P}{r^{2}}, \quad \frac{\eta}{\rho^{2}}=B+\frac{Q}{r^{2}}, \quad \frac{\zeta}{\rho^{2}}=C+\frac{R}{r^{2}}, \quad \frac{1}{\rho^{2}}=D+\frac{S}{r^{2}} .
$$

Summing the squares of the first three, we find a value of $\frac{1}{\rho^{2}}$ which must equal to that given by the fourth equation. So the two functions

$$
D+\frac{S}{r^{2}} \quad \text { and } \quad A^{2}+B^{2}+C^{2}+\frac{2(A P+B Q+C R)}{r^{2}}+\frac{P^{2}+Q^{2}+R^{2}}{r^{4}}
$$

must be equal. But the first becomes an entire function when it is multiplied by $r^{2}$. The second must also become so, and therefore $P^{2}+Q^{2}+R^{2}$ must be divisible by $r^{2}$. The quotient can obviously only be a constant, since the numerator and the denominator are of the same degree. Let $m^{2}$ be this constant, and

$$
P^{2}+Q^{2}+R^{2}=m^{2} r^{2}=m^{2}\left(x^{2}+y^{2}+z^{2}\right)
$$

$P, Q, R$ being first order polynomial, write

$$
\begin{aligned}
& P=m(a x+b y+c z+g), \\
& Q=m\left(a^{\prime} x+b^{\prime} y+c^{\prime} z+g^{\prime}\right), \\
& R=m\left(a^{\prime \prime} x+b^{\prime \prime} y+c^{\prime \prime} z+g^{\prime \prime}\right),
\end{aligned}
$$

and conclude from the comparing of the two sides, on the one hand,

$$
\begin{array}{ll}
a^{2}+a^{2}+a^{\prime \prime 2}=1, & a b+a^{\prime} b^{\prime}+a^{\prime \prime} b^{\prime \prime}=0 \\
b^{2}+b^{\prime 2}+b^{\prime \prime} 2=1, & a c+a^{\prime} c^{\prime}+a^{\prime \prime} c^{\prime \prime}=0 \\
c^{2}+c^{2}+c^{\prime \prime 2}=1, & b c+b^{\prime} c^{\prime}+b^{\prime \prime} c^{\prime \prime}=0
\end{array}
$$

which are equations that result in, as we know, the inverse equations

$$
\begin{array}{ll}
a^{2}+b^{2}+c^{2}=1, & a a^{\prime}+b b^{\prime}+c c^{\prime}=0 \\
a^{2}+b^{\prime 2}+c^{\prime 2}=1, & a a^{\prime \prime}+b b^{\prime \prime}+c c^{\prime \prime}=0 \\
a^{\prime \prime 2}+b^{\prime \prime 2}+c^{\prime \prime 2}=1, & a^{\prime} a^{\prime \prime}+b^{\prime} b^{\prime \prime}+c^{\prime} c^{\prime \prime}=0
\end{array}
$$

and on the other hand

$$
\begin{array}{ll}
a g+a^{\prime} g^{\prime}+a^{\prime \prime} g^{\prime \prime}=0, & c g+c^{\prime} g^{\prime}+c^{\prime \prime} g^{\prime \prime}=0 \\
b g+b^{\prime} g^{\prime}+b^{\prime \prime} g^{\prime \prime}=0, & g^{2}+g^{\prime 2}+g^{\prime \prime 2}=0
\end{array}
$$

If we admit that $g, g^{\prime}, g^{\prime \prime}$ are real constants, the equation $g^{2}+g^{2}+g^{\prime \prime 2}$ would give us $g=0, g^{\prime}=0, g^{\prime \prime}=0$. In any case, we would arrive at the same result using the previous three equations, having regard to the condition equations between $a, b, c$, etc. To prove, for example, that $g=0$, it suffices to add together the mentioned three equations after having them multiplied by the respective factors $a, b, c$. Hence, we are left with

$$
\begin{aligned}
& P=m(a x+b y+c z) \\
& Q=m\left(a^{\prime} x+b^{\prime} y+c^{\prime} z\right) \\
& R=m\left(a^{\prime \prime} x+b^{\prime \prime} y+c^{\prime \prime} z\right)
\end{aligned}
$$

where $a, b, c$, etc., satisfy the above condition equations, which are the same as those encountered in transforming rectangular coordinates into other rectangular ones. And as the equations

$$
\frac{\xi}{\rho^{2}}=A+\frac{P}{r^{2}}, \quad \frac{\eta}{\rho^{2}}=B+\frac{Q}{r^{2}}, \quad \frac{\zeta}{\rho^{2}}=C+\frac{R}{r^{2}}
$$

give

$$
\frac{1}{\rho^{2}}=\left(A+\frac{P}{r^{2}}\right)^{2}+\left(B+\frac{Q}{r^{2}}\right)^{2}+\left(C+\frac{R}{r^{2}}\right)^{2}
$$

we conclude the following formulas:

$$
\begin{aligned}
& \xi=\frac{A+\frac{P}{r^{2}}}{\left(A+\frac{P}{r^{2}}\right)^{2}+\left(B+\frac{Q}{r^{2}}\right)^{2}+\left(C+\frac{R}{r^{2}}\right)^{2}} \\
& \eta=\frac{B+\frac{Q}{r^{2}}}{\left(A+\frac{P}{r^{2}}\right)^{2}+\left(B+\frac{Q}{r^{2}}\right)^{2}+\left(C+\frac{R}{r^{2}}\right)^{2}}, \\
& \zeta=\frac{C+\frac{R}{r^{2}}}{\left(A+\frac{P}{r^{2}}\right)^{2}+\left(B+\frac{Q}{r^{2}}\right)^{2}+\left(C+\frac{R}{r^{2}}\right)^{2}}
\end{aligned}
$$

Now we restore $\xi-\xi_{0}, \eta-\eta_{0}, \zeta-\zeta_{0}$ in place of $\xi, \eta, \zeta$ and $x-x_{0}, y-y_{0}, z-z_{0}$ in place of $x, y, z$. With this change made, we obtain the most general formulas that satisfy Equation (1). Therefore we have the following theorem:

The general formulas that satisfy Equation (1) are obtained by first setting

$$
\begin{aligned}
& \mathrm{X}=a\left(x-x_{0}\right)+b\left(y-y_{0}\right)+c\left(z-z_{0}\right), \\
& \mathrm{Y}=a^{\prime}\left(x-x_{0}\right)+b^{\prime}\left(y-y_{0}\right)+c^{\prime}\left(z-z_{0}\right), \\
& \mathrm{Z}=a^{\prime \prime}\left(x-x_{0}\right)+b^{\prime \prime}\left(y-y_{0}\right)+c^{\prime \prime}\left(z-z_{0}\right),
\end{aligned}
$$

where the coefficients $a, b$, etc., validate the condition equations

$$
\begin{array}{ll}
a^{2}+a^{2}+a^{\prime \prime 2}=1, & a b+a^{\prime} b^{\prime}+a^{\prime \prime} b^{\prime \prime}=0 \\
b^{2}+b^{2}+b^{\prime \prime} 2=1, & a c+a^{\prime} c^{\prime}+a^{\prime \prime} c^{\prime \prime}=0 \\
c^{2}+c^{\prime 2}+c^{\prime \prime 2}=1, & b c+b^{\prime} c^{\prime}+b^{\prime \prime} c^{\prime \prime}=0
\end{array}
$$

then taking

$$
u=A+\frac{m \mathrm{X}}{\mathrm{X}^{2}+\mathrm{Y}^{2}+\mathrm{Z}^{2}}, \quad v=B+\frac{m \mathrm{Y}}{\mathrm{X}^{2}+\mathrm{Y}^{2}+\mathrm{Z}^{2}}, \quad w=C+\frac{m \mathrm{Z}}{\mathrm{X}^{2}+\mathrm{Y}^{2}+\mathrm{Z}^{2}},
$$

and finally

$$
\xi-\xi_{0}=\frac{u}{u^{2}+v^{2}+w^{2}}, \quad \eta-\eta_{0}=\frac{v}{u^{2}+v^{2}+w^{2}}, \quad \zeta-\zeta_{0}=\frac{w}{u^{2}+v^{2}+w^{2}} .
$$

Conversely, we can demonstrate that Equation (1) is satisfied in this manner, and find the appropriate value of $p$.

First, from the last three formulas we easily conclude

$$
\left(\xi^{\prime}-\xi\right)^{2}+\left(\eta^{\prime}-\eta\right)^{2}+\left(\zeta^{\prime}-\zeta\right)^{2}=\frac{\left(u^{\prime}-u\right)^{2}+\left(v^{\prime}-v\right)^{2}+\left(w^{\prime}-w\right)^{2}}{\left(u^{2}+v^{2}+w^{2}\right)\left(u^{2}+v^{2}+w^{\prime 2}\right)}
$$

the previous three give the same:

$$
\left(u^{\prime}-u\right)^{2}+\left(v^{\prime}-v\right)^{2}+\left(w^{\prime}-w\right)^{2}=m^{2} \frac{\left(\mathrm{X}^{\prime}-\mathrm{X}\right)^{2}+\left(\mathrm{Y}^{\prime}-\mathrm{Y}\right)^{2}+\left(\mathrm{Z}^{\prime}-\mathrm{Z}\right)^{2}}{\left(\mathrm{X}^{2}+\mathrm{Y}^{2}+\mathrm{Z}^{2}\right)\left(\mathrm{X}^{\prime 2}+\mathrm{Y}^{\prime 2}+\mathrm{Z}^{\prime 2}\right)}
$$

finally, from the condition equations for $a, b$, etc., we find

$$
\left(\mathrm{x}^{\prime}-\mathrm{X}\right)^{2}+\left(\mathrm{y}^{\prime}-\mathrm{Y}\right)^{2}+\left(\mathrm{Z}^{\prime}-\mathrm{z}\right)^{2}=\left(x^{\prime}-x\right)^{2}+\left(y^{\prime}-y\right)^{2}+\left(z^{\prime}-z\right)^{2} .
$$

Therefore, indeed,

$$
\left(\xi^{\prime}-\xi\right)^{2}+\left(\eta^{\prime}-\eta\right)^{2}+\left(\zeta^{\prime}-\zeta\right)^{2}=\frac{\left(x^{\prime}-x\right)^{2}+\left(y^{\prime}-y\right)^{2}+\left(z^{\prime}-z\right)^{2}}{p^{2} p^{\prime 2}}
$$

with the value of $p^{2}$ being

$$
p^{2}=\frac{\left(\mathrm{x}^{2}+\mathrm{Y}^{2}+\mathrm{Z}^{2}\right)\left(u^{2}+v^{2}+w^{2}\right)}{m}
$$

a value that can easily be expressed in $x, y, z$, as one observes that the product $\left(\mathrm{x}^{2}+\mathrm{Y}^{2}+\mathrm{z}^{2}\right)\left(u^{2}+v^{2}+w^{2}\right)$ is equal to

$$
\left(A^{2}+B^{2}+C^{2}\right)\left(\mathrm{x}^{2}+\mathrm{Y}^{2}+\mathrm{z}^{2}\right)+2 A m \mathrm{x}+2 B m \mathrm{Y}+2 C m \mathrm{z}+m^{2}
$$

and that $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ are known as functions of $x, y, z$. The value thus found can be put in the form

$$
m p^{2}=\left(A^{2}+B^{2}+C^{2}\right)\left[\left(x-x_{1}\right)^{2}+\left(y-y_{1}\right)^{2}+\left(z-z_{1}\right)^{2}\right],
$$

where $x_{1}, y_{1}, z_{1}$ are constants whose values are given by

$$
x_{1}=x_{0}-\frac{m\left(A a+B a^{\prime}+C a^{\prime \prime}\right)}{A^{2}+B^{2}+C^{2}}
$$

$$
\begin{aligned}
& y_{1}=y_{0}-\frac{m\left(A a+B a^{\prime}+C a^{\prime \prime}\right)}{A^{2}+B^{2}+C^{2}} \\
& z_{1}=z_{0}-\frac{m\left(A a+B a^{\prime}+C a^{\prime \prime}\right)}{A^{2}+B^{2}+C^{2}}
\end{aligned}
$$

Therefore, as we later view $x, y, z$ as the rectangular coordinates of any point, we see that the quantity $p$ will be proportional to the distance from this point $(x, y, z)$ to a fixed point $\left(x_{1}, y_{1}, z_{1}\right)$. It is also easy to check that

$$
\frac{d^{2} \frac{1}{p}}{d x^{2}}+\frac{d^{2} \frac{1}{p}}{d y^{2}}+\frac{d^{2} \frac{1}{p}}{d z^{2}}=0
$$

2. To get $\xi, \eta, \zeta$ explicitly in $x, y, z$, it suffices to replace $u, v, w, \mathrm{x}, \mathrm{Y}, \mathrm{Z}$ by their values. The first substitution yields

$$
\xi-\xi_{0}=\frac{A\left(\mathrm{x}^{2}+\mathrm{Y}^{2}+\mathrm{z}^{2}\right)+m \mathrm{x}}{\left(A^{2}+B^{2}+C^{2}\right)\left(\mathrm{x}^{2}+\mathrm{Y}^{2}+\mathrm{z}^{2}\right)+2 A m \mathrm{x}+2 B m \mathrm{Y}+2 C m \mathrm{Z}+m^{2}}
$$

The denominator is precisely the value of $m p^{2}$ which we have just given the expression in $x, y, z$, namely,

$$
m p^{2}=\left(A^{2}+B^{2}+C^{2}\right)\left[\left(x-x_{1}\right)^{2}+\left(y-y_{1}\right)^{2}+\left(z-z_{1}\right)^{2}\right]
$$

All that remains is to find the numerator. The calculation also becomes very simple if we subtract from both sides the quantity

$$
\frac{A}{A^{2}+B^{2}+C^{2}}
$$

because then the right-hand side can be reduced to a fraction having the numerator being a polynomial of first order on $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$, and consequently also in $x, y, z$. So by denoting by $X$ such a polynomial, and setting, for abbreviation,

$$
\xi_{0}+\frac{A}{A^{2}+B^{2}+C^{2}}=\xi^{0}
$$

we can write

$$
\xi-\xi^{0}=\frac{X}{p^{2}}
$$

and similarly

$$
\eta-\eta^{0}=\frac{Y}{p^{2}}, \quad \zeta-\zeta^{0}=\frac{Z}{p^{2}}
$$

where $\eta^{0}, \zeta^{0}$ are constants, and $Y, Z$ are linear functions of $x, y, z$. The polynomials $X, Y, Z$ would be obtained without difficulty by what we have just said; but we find them in a more convenient form by operating as follows. It is easy to see that by assigning an infinite value to one or more of the quantities $x, y, z$, or, if you will, by making

$$
x^{2}+y^{2}+z^{2}=\infty,
$$

we have

$$
\xi=\xi^{0}, \quad \eta=\eta^{0}, \quad \zeta=\zeta^{0}, \quad \frac{p^{2}}{x^{2}+y^{2}+z^{2}}=\frac{A^{2}+B^{2}+C^{2}}{m}
$$

Therefore if we introduce this hypothesis of $x^{2}+y^{2}+z^{2}=\infty$ in the general equation

$$
\left(\xi^{\prime}-\xi\right)^{2}+\left(\eta^{\prime}-\eta\right)^{2}+\left(\zeta^{\prime}-\zeta\right)^{2}=\frac{\left(x^{\prime}-x\right)^{2}+\left(y^{\prime}-y\right)^{2}+\left(z^{\prime}-z\right)^{2}}{p^{2} p^{\prime 2}}
$$

it will become

$$
\left(\xi^{\prime}-\xi^{0}\right)^{2}+\left(\eta^{\prime}-\eta^{0}\right)^{2}+\left(\zeta^{\prime}-\zeta^{0}\right)^{2}=\frac{m}{\left(A^{2}+B^{2}+C^{2}\right) p^{\prime 2}}
$$

and hence, by erasing the accents,

$$
\left(\xi-\xi^{0}\right)^{2}+\left(\eta-\eta^{0}\right)^{2}+\left(\zeta-\zeta^{0}\right)^{2}=\frac{m}{\left(A^{2}+B^{2}+C^{2}\right) p^{2}}
$$

But on the other hand,

$$
\left(\xi-\xi^{0}\right)^{2}+\left(\eta-\eta^{0}\right)^{2}+\left(\zeta-\zeta^{0}\right)^{2}=\frac{X^{2}+Y^{2}+Z^{2}}{p^{4}}
$$

thus

$$
X^{2}+Y^{2}+Z^{2}=\frac{m p^{2}}{A^{2}+B^{2}+C^{2}}
$$

that is,

$$
X^{2}+Y^{2}+Z^{2}=\left(x-x_{1}\right)^{2}+\left(y-y_{1}\right)^{2}+\left(z-z_{1}\right)^{2}
$$

Therefore, by a calculation similar to the one we carried out in the previous section for the equation

$$
P^{2}+Q^{2}+R^{2}=m^{2}\left(x^{2}+y^{2}+z^{2}\right)
$$

we conclude that by representing by $\alpha, \beta, \gamma, \alpha^{\prime}$, etc., constants subject to the condition equations

$$
\begin{array}{ll}
\alpha^{2}+\alpha^{2}+\alpha^{\prime \prime 2}=1, & \alpha \beta+\alpha^{\prime} \beta^{\prime}+\alpha^{\prime \prime} \beta^{\prime \prime}=0 \\
\beta^{2}+\beta^{2}+\beta^{\prime \prime 2}=1, & \alpha \gamma+\alpha^{\prime} \gamma^{\prime}+\alpha^{\prime \prime} \gamma^{\prime \prime}=0 \\
\gamma^{2}+\gamma^{\prime 2}+\gamma^{\prime \prime 2}=1, & \beta \gamma+\beta^{\prime} \gamma^{\prime}+\beta^{\prime \prime} \gamma^{\prime \prime}=0
\end{array}
$$

which are of the same kind as those for $a, b$, etc., we must have

$$
\begin{aligned}
& X=\alpha\left(x-x_{1}\right)+\beta\left(y-y_{1}\right)+\gamma\left(z-z_{1}\right) \\
& Y=\alpha^{\prime}\left(x-x_{1}\right)+\beta^{\prime}\left(y-y_{1}\right)+\gamma^{\prime}\left(z-z_{1}\right), \\
& Z=\alpha^{\prime \prime}\left(x-x_{1}\right)+\beta^{\prime \prime}\left(y-y_{1}\right)+\gamma^{\prime \prime}\left(z-z_{1}\right)
\end{aligned}
$$

And, conversely, it is easy to verify that by adopting these values of $X, Y, Z$, the formulas

$$
\xi-\xi^{0}=\frac{n X}{X^{2}+Y^{2}+Z^{2}}, \quad \eta-\eta^{0}=\frac{n Y}{X^{2}+Y^{2}+Z^{2}}, \quad \zeta-\zeta^{0}=\frac{n Z}{X^{2}+Y^{2}+Z^{2}}
$$

which come from our analysis by letting, to abbreviate,

$$
\frac{m}{A^{2}+B^{2}+C^{2}}=n, \text { and thus } p^{2}=\frac{X^{2}+Y^{2}+Z^{2}}{n}
$$

will result in the desired Equation (1) whose general solution is now expressed in a new and simpler way. Indeed, we first find

$$
\left(\xi^{\prime}-\xi\right)^{2}+\left(\eta^{\prime}-\eta\right)^{2}+\left(\zeta^{\prime}-\zeta\right)^{2}=\frac{n^{2}\left[\left(X^{\prime}-X\right)^{2}+\left(Y^{\prime}-Y\right)^{2}+\left(Z^{\prime}-Z\right)^{2}\right]}{\left(X^{2}+Y^{2}+Z^{2}\right)\left(X^{\prime 2}+Y^{\prime 2}+Z^{\prime 2}\right)}
$$

then

$$
\left(X^{\prime}-X\right)^{2}+\left(Y^{\prime}-Y\right)^{2}+\left(Z^{\prime}-Z\right)^{2}=\left(x^{\prime}-x\right)^{2}+\left(y^{\prime}-y\right)^{2}+\left(z^{\prime}-z\right)^{2}
$$

because of the condition equations for $\alpha, \beta$, etc. And from there we get

$$
\left(\xi^{\prime}-\xi\right)^{2}+\left(\eta^{\prime}-\eta\right)^{2}+\left(\zeta^{\prime}-\zeta\right)^{2}=\frac{n^{2}\left[\left(x^{\prime}-x\right)^{2}+\left(y^{\prime}-y\right)^{2}+\left(z^{\prime}-z\right)^{2}\right]}{\left(X^{2}+Y^{2}+Z^{2}\right)\left(X^{\prime 2}+Y^{\prime 2}+Z^{\prime 2}\right)},
$$

that is to say Equation (1) by taking

$$
p^{2}=\frac{X^{2}+Y^{2}+Z^{2}}{n}=\frac{\left(x-x_{1}\right)^{2}+\left(y-y_{1}\right)^{2}+\left(z-z_{1}\right)^{2}}{n} .
$$

3. We can inversely write the values of $x, y, \approx$ in $\xi, \eta, \zeta$; but it is clear without calculation, and a priori, that these values must be expressed by formulas of the same kind as those which give $\xi, \eta, \zeta$ in $x, y, z$. Indeed, as $p$ being a function of $x, y, z$, we can view this quantity as a function of $\xi, \eta, \zeta$. Let

$$
p=\frac{1}{\varpi}, \quad p^{\prime}=\frac{1}{\varpi^{\prime}},
$$

$\varpi$ being a certain function of $\xi, \eta, \zeta$ and $\varpi^{\prime}$ the same function of $\xi^{\prime}, \eta^{\prime}, \zeta^{\prime}$. Equation (1) then changes into the new equation

$$
\left(x^{\prime}-x\right)^{2}+\left(y^{\prime}-y\right)^{2}+\left(z^{\prime}-z\right)^{2}=\frac{\left(\xi^{\prime}-\xi\right)^{2}+\left(\eta^{\prime}-\eta\right)+\left(\zeta^{\prime}-\zeta\right)^{2}}{\varpi \varpi^{\prime}}
$$

which is of a form very similar to Equation (1) itself, and which, consequently, will give $x, y, z$ in $\xi, \eta, \zeta$ in the same way that Equation (1) gave $\xi, \eta, \zeta$ in $x, y, z$.
4. We see that, by the exchange of the letters $x, y, z$ and $\xi, \eta, \zeta$ into each other, a particular solution of Equation (1), meaning a solution in which the constants would have particular values, will give another one, most of the time different, although always falling, of course, into the general type indicated earlier. It is also easy to see that two given solutions provide a third. Indeed, suppose that when taking $\xi, \eta, \zeta, q$ as functions of $U, V, W$ we have

$$
\left(\xi^{\prime}-\xi\right)^{2}+\left(\eta^{\prime}-\eta\right)^{2}+\left(\zeta^{\prime}-\zeta\right)^{2}=\frac{\left(U^{\prime}-U\right)^{2}+\left(V^{\prime}-V\right)^{2}+\left(W^{\prime}-W\right)^{2}}{q^{2} q^{\prime 2}}
$$

and that, similarly, when taking $U, V, W, p$ as functions of $x, y, z$ we have

$$
\left(U^{\prime}-U\right)^{2}+\left(V^{\prime}-V\right)^{2}+\left(W^{\prime}-W\right)^{2}=\frac{\left(x^{\prime}-x\right)^{2}+\left(y^{\prime}-y\right)^{2}+\left(z^{\prime}-z\right)^{2}}{p^{2}{p^{\prime}}^{2}}
$$

it is clear that we can also $\operatorname{express} q, \xi, \eta, \zeta$ in $x, y, z$, and that will be

$$
\left(\xi^{\prime}-\xi\right)^{2}+\left(\eta^{\prime}-\eta\right)^{2}+\left(\zeta^{\prime}-\zeta\right)^{2}=\frac{\left(x^{\prime}-x\right)^{2}+\left(y^{\prime}-y\right)^{2}+\left(z^{\prime}-z\right)^{2}}{p^{2} q^{2} \cdot p^{2} q^{\prime 2}}
$$

a new solution of our problem.
We can say, in other words, that with various transformations solving this problem being operated successively, the single transformation composed of this ensemble also solves it. And from the way we verified our general solution above, it is obvious that this solution is only the result of a series of particular solutions thereby composed together so to speak.
5. There is a particular solution of Equation (1) which we must study especially because it constitutes, strictly speaking, the essential element of our general formulas, and that it will also help us to show the geometric meaning. It was employed by Mr. Thomson, and it is given by

$$
\xi=\frac{n x}{x^{2}+y^{2}+z^{2}}, \quad \eta=\frac{n y}{x^{2}+y^{2}+z^{2}}, \quad \zeta=\frac{n z}{x^{2}+y^{2}+z^{2}},
$$

which indeed results in the equation

$$
\left(\xi^{\prime}-\xi\right)^{2}+\left(\eta^{\prime}-\eta\right)^{2}+\left(\zeta^{\prime}-\zeta\right)^{2}=n^{2} \frac{\left(x^{\prime}-x\right)^{2}+\left(y^{\prime}-y\right)^{2}+\left(z^{\prime}-z\right)^{2}}{\left(x^{2}+y^{2}+z^{2}\right)\left(x^{\prime 2}+y^{2}+z^{\prime 2}\right)}
$$

that is, Equation (1) with

$$
p^{2}=\frac{x^{2}+y^{2}+z^{2}}{n}
$$

We then have

$$
\xi^{2}+\eta^{2}+\zeta^{2}=\frac{n^{2}}{x^{2}+y^{2}+z^{2}}
$$

and therefore,

$$
x=\frac{n \xi}{\xi^{2}+\eta^{2}+\zeta^{2}}, \quad y=\frac{n \eta}{\xi^{2}+\eta^{2}+\zeta^{2}}, \quad z=\frac{n \zeta}{\xi^{2}+\eta^{2}+\zeta^{2}},
$$

which are values of the same composition in $\xi, \eta, \zeta$ as the previous ones in $x, y, z$.

We can interpret these formulas geometrically by viewing $x, y, z$, for example, as rectangular coordinates, and $\xi, \eta, \zeta$ as parameters. The surfaces $(\xi),(\eta),(\zeta)$, for which one of these parameters remains the same value, are spheres which intersect pairwise orthogonally, and by the intersection of three of which Mr. Thomson determines the position of each point $(x, y, z)$ or $(\xi, \eta, \zeta)$. From this point of view, $\xi, \eta, \zeta$ are the curvilinear coordinates which relate to the same figure as the rectilinear coordinates $x, y, z$. But it is more convenient, I believe, to introduce into our research one of those transmutations of figures so familiar to geometers, and which have contributed so much to the progress of science in recent times. The transformation in question is well known, and moreover very simple; it is the one Mr. Thomson himself used under the name of the principle of images*. Consider $x, y, z$ as the coordinates of any point $m$ from a figure relative to three rectangular axes $O x, O y, O z, \xi, \eta, \zeta$ as those of a point $\mu$ from another figure relative to three axes $O \xi, O \eta, O \zeta$, also rectangular, and to which we give the same origin $O$ and respectively the same directions, each of these figures deriving from the other, and the point $\mu$, in particular, corresponding to the point $m$ by virtue of the relations by which $\xi, \eta, \zeta$ are expressed in $x, y, z$, or $x, y, z$ in $\xi, \eta, \zeta$. It is obvious that the two corresponding points $m, \mu$ are in a straight line with the origin $O$, and that the product $O m . O \mu$ of the radius vectors $O m, O \mu$ is constant and $=n$. One of the figures is therefore deduced from the other by taking, on each of the radius vectors led from point $O$ to any point in the first figure, other radius vectors in inverse ratio to the first; the ends of these new radius vectors determine the second figure. We will give this transformation the name of transformation by reciprocal radius vectors relative to the origin $O$. If, for a point $m$, we have $O m=\sqrt{n}$, we will also have $O \mu=\sqrt{n}$, and thus the corresponding points $m, \mu$ in the two figures will coincide. By arranging $n$, we can ensure that a given point $m$ remains fixed under the transformation; it suffices to take $n=O \bar{m}^{2}$, so all the points located on the sphere of which

[^1]$O$ is the center and $O m$ the radius, will also remain fixed, but all the others will be moved.
6. Using this transformation by reciprocal radius vectors, we will derive from a given figure infinitely many other figures, either by changing the origin $O$ from which the radius vectors depart, or by taking various values of $n$ with the same origin $O$, which only, in addition, gives rise to transformed figures all similar to each other, at least as long as $n$ keeps the same sign; because the figures responding to two equal values of $n$ and opposite signs are symmetrical. We can also perform, one after another, transformations relative to different origins. But I say that our general formulas of § 2 can always be interpreted using a single transformation of this sort, so we would not really get anything new by adding other transformations to this one.

Indeed, in the most general case, we can still consider $x, y, z$ and $\xi, \eta, \zeta$ as the coordinates of two points $m, \mu$ belonging to two different figures and relative to two systems of rectangular axes $x, y, z$ and $\xi, \eta, \zeta$. Here is how the transformation of one of the figures into the other works.

First we go from $x, y, z$ to $X, Y, Z$ by

$$
\begin{aligned}
& X=\alpha\left(x-x_{0}\right)+\beta\left(y-y_{0}\right)+\gamma\left(z-z_{0}\right) \\
& Y=\alpha^{\prime}\left(x-x_{0}\right)+\beta^{\prime}\left(y-y_{0}\right)+\gamma^{\prime}\left(z-z_{0}\right), \\
& Z=\alpha^{\prime \prime}\left(x-x_{0}\right)+\beta^{\prime \prime}\left(y-y_{0}\right)+\gamma^{\prime \prime}\left(z-z_{0}\right) .
\end{aligned}
$$

Now, because of the condition equations between $\alpha, \beta$, etc., this step is only a change from rectangular coordinates to other rectangular coordinates, which does not in any way alter the first figure to which it is applied; we can assume it operated in advance, and then confuse $X, Y, Z$ with $x, y, z$.

From there we go to formulas

$$
\xi-\xi^{0}=\frac{n X}{X^{2}+Y^{2}+Z^{2}}, \quad \eta-\eta^{0}=\frac{n Y}{X^{2}+Y^{2}+Z^{2}}, \quad \zeta-\zeta^{0}=\frac{n Z}{X^{2}+Y^{2}+Z^{2}}
$$

and so we have a transformation from $X, Y, Z$ into $\xi-\xi^{0}, \eta-\eta^{0}, \zeta-\zeta^{0}$, which we regard as rectangular coordinates with respect to the same axes. This transformation is with reciprocal radius vectors, as we have seen in $\S 5$. It operates by taking, on the radius vectors led from the current origin,
lengths inversely proportional to these radius vectors; the old figure is thus changed to what results from the ends of all these lengths. Next, pass from $\xi-\xi^{0}, \eta-\eta^{0}, \zeta-\zeta^{0}$ to $\xi, \eta, \zeta$, which is only a simple displacement of the origin with the axes remaining parallel to themselves; this does not produce within the transformed figure any alternation.

Our formulas of § 2 therefore result from a transformation by reciprocal radius vectors, combined with ordinary changes of coordinates. Such transformations in any number always give rise to an equation of the form (1), and the geometric interpretation of the formulas by which we had first linked (§ 1) $x, y, z$ and $\xi, \eta, \zeta$, seemed to ask for two, relative to two different origins, one for the step from $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ to $u, v, w$, the other for the step from $u, v, w$ to $\xi, \eta, \zeta$; now we see, by the above, and thanks to the simpler formula of $\S 2$, that a single transformation suffices to lead to the most general result; it was important to demonstrate it.
7. The geometric considerations which we have just used, to interpret the formulas which lead to Equation (1), give rise to remarkable consequences about which we will say a few words. In the two figures described respectively by the coordinates $x, y, z$ and the coordinates $\xi, \eta, \zeta$, let us consider in one figure any two points $m, m^{\prime}$, and in the other figure the corresponding points $\mu, \mu^{\prime}$. Let $D$ be the distance between the first two, $\Delta$ that of the other two, so that

$$
\begin{aligned}
& D^{2}=\left(x^{\prime}-x\right)^{2}+\left(y^{\prime}-y\right)^{2}+\left(z^{\prime}-z\right)^{2} \\
& \Delta^{2}=\left(\xi^{\prime}-\xi\right)^{2}+\left(\eta^{\prime}-\eta\right)^{2}+\left(\zeta^{\prime}-\zeta\right)^{2}
\end{aligned}
$$

Equation (1), which can be written as

$$
\Delta^{2}=\frac{D^{2}}{p^{2}{p^{\prime}}^{2}}, \quad \Delta=\frac{D}{p p^{\prime}}, \quad \frac{1}{\Delta}=\frac{p p^{\prime}}{D}
$$

provides a relationship between the distance of two points $\mu, \mu^{\prime}$ in one of the figures and the quantities $D, p, p^{\prime}$. We just said that $D$ is the distance of the corresponding two points $m, m^{\prime}$ in the other figure; as for $p$ and $p^{\prime}$, these are, up to a constant factor, the distances of points $m, m^{\prime}$ to a certain fixed point. Any metric relationship between two or more distances $\Delta$ in one of the figures will therefore immediately provide an analogous relationship in the other
figure. But one should not think of that the various points corresponding to those of the straight segment $\Delta$ are on the straight segment $D$; this applies to the end points by the very definition of these straight segments, but does not hold, in general, for the intermediate points. In general, the series of points corresponding to those of a straight line in the first figure forms in the second figure a circle circumference, which is reduced to a straight line only in one particular case, that when its radius is infinite.

Having in $\xi, \eta, \zeta$ the equation of a surface or the equation of a line belonging to the first figure, it suffices to substitute $\xi, \eta, \zeta$ by their values to form the equation in $x, y, z$ of the surface or equations of the corresponding line. We can easily find, in this manner, that planes and spheres are transformed into spheres which can be reduced to planes when the radius becomes infinite; that, likewise, lines and circumferences of circles are transformed into circumferences of circles, etc. Now, to follow the mechanism of these transformations, it suffices to consider the transformation by reciprocal radius vectors, which combined with changes of coordinates gives, as we have seen, the most general transformation. Therefore, let

$$
\begin{aligned}
& \xi=\frac{n x}{x^{2}+y^{2}+z^{2}}=\frac{n x}{r^{2}}, \quad x=\frac{n \xi}{\xi^{2}+\eta^{2}+\zeta^{2}}=\frac{n \xi}{\rho^{2}}, \\
& \eta=\frac{n y}{x^{2}+y^{2}+z^{2}}=\frac{n y}{r^{2}}, \quad y=\frac{n \eta}{\xi^{2}+\eta^{2}+\zeta^{2}}=\frac{n \eta}{\rho^{2}}, \\
& \zeta=\frac{n z}{x^{2}+y^{2}+z^{2}}=\frac{n z}{r^{2}}, \quad z=\frac{n \zeta}{\xi^{2}+\eta^{2}+\zeta^{2}}=\frac{n \zeta}{\rho^{2}}, \\
& \Delta=\frac{D}{p p^{\prime}}=\frac{n D}{\sqrt{\left(x^{2}+y^{2}+z^{2}\right)\left(x^{\prime 2}+y^{\prime 2}+z^{\prime 2}\right)}}=\frac{n D}{r r^{\prime}},
\end{aligned}
$$

the set of formulas relative to the transformation by reciprocal radius vectors. We immediately conclude what we have just stated in advance, concerning planes and spheres, lines and circumferences of circles. Now we see, moreover, and even without calculation, that the planes which pass through the point $O$, origin of the radius vectors, are the only ones which remain planes in the transformation; their position is the same before and after, although their various points, of course, have moved to replace one another: those which were far from the origin having now become neighbors, and vice versa. Any other plane is transformed into a sphere passing through the point $O$ (where the transformation brings all the points located at infinity) and having
its center on the perpendicular to the plane drawn from point $O$; the perpendicular and the diameter of the sphere have a product equal to the constant $n$, and are therefore easily deduced from each other. It is needless to say that two spheres which correspond to two parallel planes touch at point $O$. Likewise, two spheres thus posed would transform into two parallel planes. But a sphere which does not pass through the point $O$ must remain a sphere, since it cannot acquire any point at infinity. The lines passing through the point $O$ remain straight lines, and keep their position invariant. Any other line gives rise to a circle circumference whose plane is determined by the line and by the point $O$, and whose center is located on the perpendicular casted from point $O$ to the line; the diameter is the quotient of the constant $n$ by this perpendicular. The circumferences coming from parallel lines are all tangent to a parallel drawn from point $O$ to these lines. We can see, finally, that the transform of a circumference is a straight line when the circumference passes through the point $O$, and, in any other case, remains a circumference.

A remarkable property of this kind of transformation consists of the fact that the two triangles formed by any three infinitely neighboring points of the original figure and the three corresponding points of its transform are similar to each other, so that if two lines intersect in one of the two figures they will intersect at the same angle*. The demonstration of this property is based on Equation (1), to which we have given the form

$$
\Delta=\frac{D}{p p^{\prime}}
$$

Suppose that two points $m, m^{\prime}$, or $(x, y, z),\left(x^{\prime}, y^{\prime}, z^{\prime}\right)$, are indeed infinitely neighboring, and that their distance $D$ is represented by $d s$. Represent by $d \sigma$ that of the two corresponding points $\mu, \mu^{\prime}$. Since $p$ and $p^{\prime}$ have no significant difference, what we get is

$$
d \sigma=\frac{d s}{p^{2}}
$$

[^2]The elements $d \sigma, d s$ therefore have at each location a constant ratio which depends on $p$ and changes, in general, from one location to another. Consider a third point $m^{\prime \prime}$ infinitely close to the first two, and denote by $d s^{\prime}$ and $d s^{\prime \prime}$ its distances to $m$ and to $m^{\prime}$; with $d \sigma^{\prime}, d \sigma^{\prime \prime}$ being the corresponding distances in the second figure, we will still have

$$
\begin{aligned}
d \sigma^{\prime} & =\frac{d s^{\prime}}{p^{2}} \\
d \sigma^{\prime \prime} & =\frac{d s^{\prime \prime}}{p^{2}}
\end{aligned}
$$

Therefore

$$
d \sigma: d \sigma^{\prime}: \sigma^{\prime \prime}:: d s: d s^{\prime}: d s^{\prime \prime}
$$

Thus, the infinitesimal triangle $m m^{\prime} m^{\prime \prime}$ is similar to the corresponding triangle $\mu \mu^{\prime} \mu^{\prime \prime}$. The angle of $d s$ with $d s^{\prime}$ is consequently the same as that of $d \sigma$ with $d \sigma^{\prime}$. This demonstration, as we can see, does not even require that Equation (1) holds for two points located with a finite distance; it only demands that this equation always holds for two infinitely neighboring points. It should be said as much of a theorem which I will establish, and which is only a corollary of the previous proposition.

Given a surface belonging to one of the two figures, imagine the lines of curvature of this surface, and the two families of developable surfaces, orthogonal to each other and to the given surface, which are formed by the successive normals. In the second figure, the families of corresponding surfaces will remain orthogonal to each other and to the transform of the given surface; therefore, by virtue of the beautiful theorem of Mr. Ch. Dupin, they will still trace on this transform lines of curvature. These lines of curvature will therefore result from the lines of curvature of the first given surface, and will be immediately known if the others are. It will be easy to apply this theorem to surfaces of degree two, as also to the triple systems of orthogonal surfaces that Mr. Serret indicated in a recent Note*, which by our transformation will give others that are no less curious, etc.

Let us propose, for example, to find the lines of curvature of the surface enveloped by spheres that touch three given spheres, a problem that

[^3]Mr. Ch. Dupin once solved in the Correspondance sur l'École Polytechnique, volume I, page 22. Let $O$ and $P$ be the points of intersection of these three spheres; take the point $O$ for origin, and operate a transformation by reciprocal radius vectors, which will give us a second figure from which we will easily return to the first. In the second figure, the three given spheres will be replaced by three planes which will intersect at a point $\Pi$ corresponding to the second point $P$ of intersection of our three spheres.

The envelope surface of the spheres tangent to these three planes will be (limited to one of the solid angles and to its opposite) a straight cone with circular base having its apex at point $\Pi$, and circumscribed to any of the spheres tangent to the three planes. The lines of curvature of this conical surface are: $1^{0}$ the straight generators which all pass through point $\Pi$ : returning to the first figure, these straight lines will become circles all passing through point $P$, whose tangents at $P$ will all make the same angle with the tangent of the circle that the axis of the cone is transformed into, from which a new straight cone will result, and which all pass also with similar circumstances through point $O ; 2^{\circ}$ circles, the planes of which are all parallel to each other and perpendicular to the axis of the cone, and which, after returning to the first figure, will become circles intersecting at a right angle to those resulting from rectilinear generatrices. The curvature lines of the envelope surface of the spheres tangent to three given spheres are therefore circumferences of a circle.

We demonstrate with the same ease the theorem of Mr. Dupin concerning the curve that traces on each of the three given spheres the variable sphere that touches them. Indeed, when the three given spheres are replaced by three planes, it is clear that the series of points along which the variable sphere touches any of the planes is a straight line passing through the point of intersection $\Pi$. So, returning to the three given spheres, the desired curve is a circle circumference which passes through the points $O$ and $P$. It can happen, of course, that the points $O$ and $P$ are imaginary; but then there is no essential change to be made in what we have just argued, and our conclusions remain.

The circumstance of an imaginary origin $O$ would have more disadvantage if it were a question of solving the problem of a sphere tangent to four others, by bringing it back to the very simple problem of finding a sphere tangent to a given sphere and to three given planes; but we would remedy it by increasing
by the same quantity the radii of the four given spheres, which does not change the position of the center of the tangent sphere. Likewise, by limiting ourselves to considering points all located in a plane passing through the origin $O$, we will reduce to the determination of the circle tangent to three others to that of a circle which touches a given circle and two given lines.

In general, systems of spheres or circles, and especially of spheres or circles passing through a given point, enjoy curious properties, many of which become intuitive through the transformation which we have just dealt with. We can apply this remark in particular to the theorems that Mr. Miquel gave in his Memoir on curvilinear angles*. To limit ourselves to the simplest case, it is obvious that, in a triangle $A B C$ formed by three arcs of circles all passing through the same point $O$, the sum of the angles worths 2 rights, since our transformation makes this triangle rectilinear without altering its angles.
8. The passage of metric relations from one figure to another, in the transformation by reciprocal radius vectors, going from the coordinates $\xi, \eta, \zeta$ to the coordinates $x, y, z$, is done using the formula

$$
\Delta=\frac{n D}{r r^{\prime}},
$$

or simply

$$
\Delta=\frac{D}{r r^{\prime}},
$$

by setting $n=1$, which has no drawbacks. Now by denoting by $O$ the origin, in the second figure only, and by using the other letters $A, B$, etc., to represent both the points of the first figure and the corresponding points of the second figure, this formula amounts to saying that in any relation between distances $A B, B D$, etc., it is necessary to replace each distance such as $A B$ by $\frac{A B}{O A . O B}$. That is therefore a very handy practical rule; this rule is suitable as well in the case of the plane as that of the space. Two examples will suffice.

[^4]- For straight lines starting from a fixed point $A$ each of which intersect a circle at two points $B$ and $C, B^{\prime}$ and $C^{\prime}$, etc., we will have

$$
A B \times A C=A B^{\prime} \times A C^{\prime}=\text { constant }
$$

So, in the transformed figure

$$
\frac{A B}{O A \cdot O B} \times \frac{A C}{O A \cdot O C}=\frac{A B^{\prime}}{O A \cdot O B^{\prime}} \times \frac{A C^{\prime}}{O A \cdot O C^{\prime}}
$$

and therefore

$$
\frac{A B}{O B} \times \frac{A C}{O C}=\text { constant }
$$

Moreover, points $A, B, C$, which were in a straight line, are now on a circle circumference passing through point $A$. There we see that the circles passing through two fixed points $A, O$ intersect a given circle at two points $B, C$ such that the ratio of the products of distances $A B \times A C$ and $O B \times O C$ has a constant value for all these circles.

- From the sides $B C, A C, A B$ of a rectilinear triangle $A B C$ cut at three points $A^{\prime}, B^{\prime}, C^{\prime}$ by a transversal, we find

$$
A C^{\prime} \times B A^{\prime} \times C B^{\prime}=B C^{\prime} \times C A^{\prime} \times A B^{\prime}
$$

So, in the transformed figure

$$
\frac{A C^{\prime}}{O A \cdot O C^{\prime}} \times \frac{B A^{\prime}}{O B \cdot O A^{\prime}} \times \frac{C B^{\prime}}{O C \cdot O B^{\prime}}=\frac{B C^{\prime}}{O B \cdot O C^{\prime}} \times \frac{C A^{\prime}}{O C \cdot O A^{\prime}} \times \frac{A B^{\prime}}{O A \cdot O B^{\prime}},
$$

which gives back

$$
A C^{\prime} \times B A^{\prime} \times C B^{\prime}=B C^{\prime} \times C A^{\prime} \times A B^{\prime}
$$

But now this relationship applies to a curvilinear triangle $A B C$ formed by three circles which all pass through point $O$ and whose sides are cut in $A^{\prime}, B^{\prime}, C^{\prime}$ by a fourth circle also passing through point $O$. Moreover, it is needless to say that $A C^{\prime}, B A^{\prime}$, etc., are the shortest distances from points $A$ and $C^{\prime}, B$ and $A^{\prime}$, etc., and not the segments measured on the sides of the curvilinear triangle.

We would easily generalize in the same way the theorem about a skew polygon cut by a plane. But that is enough on this subject.
9. Given two spheres that do not intersect, we can always place the origin $O$ at a real point on the straight line joining their centers, such that after the transformation by reciprocal radius vectors these two spheres become concentric. Take the line joining the centers as the $x$-axis; denote by $h$ the unknown distance from point $O$ to the center of the first sphere, and by $h+l$ its distance to the center of the second sphere; let $k, k^{\prime}$ be the radii. Before the transformation, the equations of the two spheres are

$$
\begin{array}{r}
(x-h)^{2}+y^{2}+z^{2}=k^{2}, \\
(x-h-l)^{2}+y^{2}+z^{2}=k^{\prime 2},
\end{array}
$$

and after the transformation, which replaces $x, y, z$ by

$$
\frac{x}{x^{2}+y^{2}+z^{2}}, \quad \frac{y}{x^{2}+y^{2}+z^{2}}, \quad \frac{z}{x^{2}+y^{2}+z^{2}}
$$

they become

$$
\begin{gathered}
\left(x-\frac{h}{h^{2}-k^{2}}\right)^{2}+y^{2}+z^{2}=\frac{k^{2}}{\left(h^{2}-k^{2}\right)^{2}}, \\
{\left[x-\frac{h+l}{(h+l)^{2}-k^{\prime 2}}\right]^{2}+y^{2}+z^{2}=\frac{k^{\prime 2}}{\left[(h+l)^{2}-k^{\prime 2}\right]^{2}} .}
\end{gathered}
$$

For the center to be the same now, it is necessary and sufficient that

$$
\frac{h}{h^{2}-k^{2}}=\frac{h+l}{(h+l)^{2}-k^{\prime 2}},
$$

and hence

$$
l h^{2}+\left(l^{2}+k^{2}-k^{2}\right) h+l k^{2}=0
$$

which is a quadratic equation that gives two values of $h$

$$
h=-\frac{l^{2}+k^{2}-k^{\prime 2}}{2 l} \pm \frac{1}{2 l} \sqrt{G},
$$

where

$$
G=\left(l-k-k^{\prime}\right)\left(l-k+k^{\prime}\right)\left(l+k-k^{\prime}\right)\left(l+k+k^{\prime}\right) ;
$$

it is easy to see that $G$ will be positive if the two spheres we have first given do not intersect.
10. This theorem could be useful in geometry; but above all it will have an important application to questions in mathematical physics. Here, let us try to quickly indicate the usage, in this kind of questions, of the general transformation which gives Equation (1). The letter from Mr. Thomson will serve as our guide, and we will add some developments to it. The more or less generality of the solution to Equation (1) does not change anything in the procedure; it remains the same in all cases.

First from the equation

$$
\frac{1}{\Delta}=\frac{p p^{\prime}}{D}
$$

we can conclude, with Mr. Thomson, that, if a function $U$ of $\xi, \eta, \zeta$ satisfies the equation

$$
\frac{d^{2} U}{d \xi^{2}}+\frac{d^{2} U}{d \eta^{2}}+\frac{d^{2} U}{d \zeta^{2}}=0
$$

this same function, divided by $p$ and expressed in $x, y, z$, will validate the equation of the same form

$$
\frac{d^{2} \cdot p^{-1} U}{d x^{2}}+\frac{d^{2} \cdot p^{-1} U}{d y^{2}}+\frac{d^{2} \cdot p^{-1} U}{d z^{2}}=0
$$

This is hence a link between two distinct problems both concerning the equilibrium of temperature in homogeneous bodies, but relating to two systems, one of which results form the other by the transformation linking $\xi, \eta, \zeta$ to $x, y, z$.

Let the first system be formed from two spheres that do not intersect, let the temperature be given at each point of their surfaces, and ask what is the state of permanent temperatures in the space between them, if one is interior to the other, or with the infinite space exterior to both if one is apart from the other, adding in this latter case the condition that the temperature is zero at infinity. We will reduce this question to the very easy case of two concentric spheres. This follows from the theorem established above and shows its importance. In addition to indicating this application to the theory of heat, Mr. Thomson adds with reason that it extends by itself to the theory of electricity.

In the theory of electricity or magnetism, and in the theory of attraction in general, the quantity that G. Green and Mr. Gauss call potential, namely the quantity that one obtains by summing the attractive or repulsive elements of a mass divided by their distances to a point, plays a crucial role. We learn the problem of Mr. Gauss: "Distribute an attractive or repellent mass on a given surface, so that the potential has a given value at each point on the surface." We have solved this problem for various surfaces, in particular for the ellipsoid. Now the solution relative to any surface gives the solution for all the surfaces which are deduced from that one by a transformation for which Equation (1) holds. Having, indeed, the equation

$$
\iint \frac{\lambda^{\prime} d \omega^{\prime}}{\Delta}=Q
$$

for the first surface, we will have for the second surface an equation of the same kind by replacing $\Delta$ and $d \omega^{\prime}$ with their new values. We have

$$
\Delta=\frac{D}{p p^{\prime}}
$$

As for $d \omega^{\prime}$, observe that the corresponding linear elements $d \sigma$ and $d s$ are linked by the formula

$$
d \sigma=\frac{d s}{p^{2}}
$$

So between two corresponding superficial elements $d \omega, d a$, we will have

$$
d \omega=\frac{d a}{p^{4}}, \quad d \omega^{\prime}=\frac{d a^{\prime}}{p^{\prime}}
$$

therefore,

$$
\iint \frac{\lambda^{\prime}}{p^{\prime 3}} \frac{d a^{\prime}}{D}=\frac{Q}{p}
$$

which solves the problem of Mr. Gauss for the transformed surface.
We can also see that the equations labeled by $(A),(B),(C)$ in my Letters to Mr. Blanchet*, and which are of such great use in most physico-mathematical

[^5]questions concerning the ellipsoid, have their analogues, which one deduces immediately for the transformed surfaces of the ellipsoid*.

We can even consider the equation

$$
\frac{d^{2} U}{d t^{2}}=\frac{d^{2} U}{d \xi^{2}}+\frac{d^{2} U}{d \eta^{2}}+\frac{d^{2} U}{d \zeta^{2}}
$$

and make it subject to the transformation from $\xi, \eta, \zeta$ to $x, y, z$.
Because of the equation

$$
d \sigma=\frac{d s}{p^{2}}
$$

which can be written as

$$
d \xi^{2}+d \eta^{2}+p \zeta^{2}=\frac{d x^{2}+d y^{2}+d z^{2}}{p^{4}}
$$

we find, by known formulas, that the quantity

$$
\frac{d^{2} U}{d \xi^{2}}+\frac{d^{2} U}{d \eta^{2}}+\frac{d^{2} U}{d \zeta^{2}}
$$

is equal to

$$
p^{6}\left(\frac{d \cdot \frac{1}{p^{2}} \frac{d U}{d x}}{d x}+\frac{d \cdot \frac{1}{p^{2}} \frac{d U}{d y}}{d y}+\frac{d \cdot \frac{1}{p^{2}} \frac{d U}{d z}}{d z}\right)
$$

that is, to

$$
p^{4}\left(\frac{d^{2} U}{d x^{2}}+\frac{d^{2} U}{d y^{2}}+\frac{d^{2} U}{d z^{2}}\right)-2 p^{2}\left(\frac{d U}{d x} \frac{d p}{d x}+\frac{d U}{d y} \frac{d p}{d y}+\frac{d U}{d z} \frac{d p}{d z}\right)
$$

[^6]or finally to
$$
p^{5}\left(\frac{d^{2} \cdot p^{-1} U}{d x^{2}}+\frac{d^{2} \cdot p^{-1} U}{d y^{2}}+\frac{d^{2} \cdot p^{-1} U}{d z^{2}}\right)
$$
recalling that
$$
\frac{d^{2} \frac{1}{p}}{d x^{2}}+\frac{d^{2} \frac{1}{p}}{d y^{2}}+\frac{d^{2} \frac{1}{p}}{d z^{2}}=0
$$

By this, we see first that the equation

$$
\frac{d^{2} U}{d \xi^{2}}+\frac{d^{2} U}{d \eta^{2}}+\frac{d^{2} U}{d \zeta^{2}}=0
$$

returns to this one:

$$
\frac{d^{2} \cdot p^{-1} U}{d x^{2}}+\frac{d^{2} \cdot p^{-1} U}{d y^{2}}+\frac{d^{2} \cdot p^{-1} U}{d z^{2}}=0
$$

as we already knew. We then see that the equation

$$
\frac{d^{2} U}{d t^{2}}=\frac{d^{2} U}{d \xi^{2}}+\frac{d^{2} U}{d \eta^{2}}+\frac{d^{2} U}{d \zeta^{2}}
$$

transforms into

$$
\frac{d^{2} U}{d t^{2}}=p^{5}\left(\frac{d^{2} \cdot p^{-1} U}{d x^{2}}+\frac{d^{2} \cdot p^{-1} U}{d y^{2}}+\frac{d^{2} \cdot p^{-1} U}{d z^{2}}\right)
$$

or, even better, into

$$
\frac{d^{2} \cdot p^{-1} U}{d t^{2}}=p^{4}\left(\frac{d^{2} \cdot p^{-1} U}{d x^{2}}+\frac{d^{2} \cdot p^{-1} U}{d y^{2}}+\frac{d^{2} \cdot p^{-1} U}{d z^{2}}\right)
$$

Conversely, this last equation, where the coefficient $p$ varies in proportion to the distance from point $(x, y, z)$ to a fixed point, is reduced to the equation

$$
\frac{d^{2} U}{d t^{2}}=\frac{d^{2} U}{d \xi^{2}}+\frac{d^{2} U}{d \eta^{2}}+\frac{d^{2} U}{d \zeta^{2}}
$$

which has constant coefficients, a result which finds a useful application in the theory of sound.

We can finally add that the partial differential equations

$$
\left(\frac{d U}{d \xi}\right)^{2}+\left(\frac{d U}{d \eta}\right)^{2}+\left(\frac{d U}{d \zeta}\right)^{2}=Q
$$

and

$$
\left(\frac{d U}{d x}\right)^{2}+\left(\frac{d U}{d y}\right)^{2}+\left(\frac{d U}{d z}\right)^{2}=\frac{Q}{p^{4}}
$$

are transformed from each other, which can be used in questions of dynamics, where Mr. Hamilton and Mr. Jacobi introduced such partial differential equations.

I will be forgiven, I hope, for these developments which I thought I could give, following the two letters so interesting from Mr. Thomson, without hindering him in his research. My purpose will be fulfilled, I repeat, if they can help to make clear the high importance of the work of this young geometer, and if Mr. Thomson himself would see in it a new proof of the friendship I endure with him and the esteem that I have for his talent.


[^0]:    * $\langle$ J. Liouville "Note au sujet de l'article précédent." Journal de mathématiques pures et appliquées, Volume XII—July 1847, page 265-290. Also in W. Thomson Reprint of papers on electrostatics and magnetism, Macmillan \& Company, 1872, page 154-177, § 221. Footnotes added in Thomson's 1872 edition are indicated by square brackets [...].)
    ${ }^{\dagger}$ 〈W. Thomson "Extraits de deux lettres addressées à M. Liouville," in the same volumeJune 1847, page 256-264. Also in W. Thomson's reprint (1872) § 211.)

[^1]:    *Volume X. (1845) of the same Journal, page 364. [W. Thomson's reprint (1872) § 207].

[^2]:    *From the similarity of the corresponding infinitely small triangles, it also follows that the transformed figure is similar to the original figure, or to its symmetrical image, in its infinitely small elements. By sticking to the first case, which is suitably that of our formulas where we naturally take the positive constant $n$, we will have, in three dimensions, a kind of representations of bodies analogous to the layout of geographic maps [those according to the "stereographic projection"], for which the similarity ratio of the corresponding elements is also variable from one place to another.

[^3]:    *Page 241 of the present volume [Liouville's Journal, 1847].

[^4]:    *Volume IX. of the Journal page 20 [Liouville's Journal, 1844].

[^5]:    *See Volume XI of this Journal.

[^6]:    *Among these surfaces, it is necessary to distinguish the one given by the transformation by reciprocal radius vectors, when putting the origin at the very center of the ellipsoid. We know that it is also the locus of the feet of the perpendiculars casted from the center onto the planes tangent to another ellipsoid whose axes have values the inverses of those of the the given ellipsoid. An analogous property takes place in the plane, for the lemniscate for example, which can thereby be generated in two different ways by mean of an equilateral hyperbola, a circumstance of which Mr. Chasles took advantage in his research sur les arcs égaux de la lemniscate (Comptes Rendus de l'Académie des Sciences, Volume XXI, session of July 21, 1845).

