# Part II, Chapter 7

# Finite elements in $\boldsymbol{H}(\mathrm{div})$

The goal of this chapter is to construct vector-valued finite elements such that the local interpolation operator can simultaneously and optimally approximate a vector field and its divergence. The contravariant Piola transformation defined by (4.18) plays an important role in this construction. The main focus of the chapter is on simplicial Raviart–Thomas finite elements. These finite elements and their generalizations have many applications in solid and fluid mechanics, porous media flows, and electromagnetism. We assume in the entire chapter that $d \geq 2$ and that the meshes are affine.

## 7.1 The polynomial space $\mathbb{RT}_{k,d}$

Let $k \in \mathbb{N}$ and let $d \geq 2$. We start by introducing the notion of homogeneous polynomials. Recall the polynomial space $\mathbb{P}_{k,d}$ from §3.2 and the multi-index set $\mathcal{A}_{k,d} = \{\alpha \in \mathbb{N}^d \mid |\alpha| \leq k\}$ where $|\alpha| = \alpha_1 + \ldots + \alpha_d$. We additionally introduce the subset $\mathcal{A}^{\mathrm{H}}_{k,d} = \{\alpha \in \mathcal{A}_{k,d} \mid |\alpha| = k\}$. For instance, $\mathcal{A}_{1,2} = \{(0,0),(1,0),(0,1)\}$ and $\mathcal{A}^{\mathrm{H}}_{1,2} = \{(1,0),(0,1)\}$.

**Definition 7.1 ($\mathbb{P}^{\mathrm{H}}_{k,d}$).** *Let $k \in \mathbb{N}$ and let $d \geq 2$. A polynomial $p \in \mathbb{P}_{k,d}$ is said to be* homogeneous *of degree $k$ if $p(\boldsymbol{x}) = \sum_{\alpha \in \mathcal{A}^{\mathrm{H}}_{k,d}} a_\alpha \boldsymbol{x}^\alpha$ with real coefficients $a_\alpha$. The real vector space composed of homogeneous polynomials is denoted $\mathbb{P}^{\mathrm{H}}_{k,d}$ or $\mathbb{P}^{\mathrm{H}}_{k}$ when the context is unambiguous.*

**Lemma 7.2 (Properties of $\mathbb{P}^{\mathrm{H}}_{k,d}$).** *$\boldsymbol{x}\cdot\nabla q = kq$ (Euler's identity) and $\nabla\cdot(\boldsymbol{x}q) = (k+d)q$ for all $q \in \mathbb{P}^{\mathrm{H}}_{k,d}$.*

*Proof.* By linearity, it suffices to verify the assertion with $q(\boldsymbol{x}) = \boldsymbol{x}^\alpha$ with $\alpha \in \mathcal{A}^{\mathrm{H}}_{k,d}$. Then, $\boldsymbol{x}\cdot\nabla q = \sum_{i=1}^d \alpha_i x_i x_1^{\alpha_1} \ldots x_i^{\alpha_i - 1} \ldots x_d^{\alpha_d} = (\sum_{i=1}^d \alpha_i) q = kq$. Moreover, the assertion for $\nabla\cdot(\boldsymbol{x}q)$ follows from the observation that $\nabla\cdot\boldsymbol{x} = d$ and $\nabla\cdot(\boldsymbol{x}q) = q\nabla\cdot\boldsymbol{x} + \boldsymbol{x}\cdot\nabla q$. $\square$

**Definition 7.3 ($\mathbb{RT}_{k,d}$).** *Let $k \in \mathbb{N}$. We define the following real vector space of $\mathbb{R}^d$-valued polynomials:*

$$\mathbb{RT}_{k,d} := \boldsymbol{\mathbb{P}}_{k,d} \oplus \boldsymbol{x}\,\mathbb{P}^{\mathrm{H}}_{k,d}, \quad \text{with} \quad \boldsymbol{\mathbb{P}}_{k,d} := [\mathbb{P}_{k,d}]^d. \tag{7.1}$$

Note that the above sum is direct since polynomials in $\boldsymbol{x}\,\mathbb{P}^{\mathrm{H}}_{k,d}$ are members of $[\mathbb{P}^{\mathrm{H}}_{k+1,d}]^d$, whereas the degree of any polynomial in $\boldsymbol{\mathbb{P}}_{k,d}$ does not exceed $k$.

**Example 7.4 ($2D$, $k = 0$ or $1$).** In two space dimensions, $\dim(\mathbb{RT}_{0,2}) = 3$ and $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}x_1\\x_2\end{pmatrix}\right\}$ is a basis of $\mathbb{RT}_{0,2}$, $\dim(\mathbb{RT}_{1,2}) = 8$ and $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}x_1\\0\end{pmatrix}, \begin{pmatrix}x_2\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\x_1\end{pmatrix}, \begin{pmatrix}0\\x_2\end{pmatrix}, \begin{pmatrix}x_1^2\\x_1x_2\end{pmatrix}, \begin{pmatrix}x_1x_2\\x_2^2\end{pmatrix}\right\}$ is a basis of $\mathbb{RT}_{1,2}$.

**Lemma 7.5 (Properties of $\mathbb{RT}_{k,d}$).** $\dim(\mathbb{RT}_{k,d}) = (k+d+1)\binom{k+d-1}{k}$, *in particular* $\dim(\mathbb{RT}_{k,2}) = (k+3)(k+1)$ *and* $\dim(\mathbb{RT}_{k,3}) = \frac{1}{2}(k+4)(k+2)(k+1)$. *Moreover, $\nabla\cdot\boldsymbol{v} \in \mathbb{P}_{k,d}$ for all $\boldsymbol{v} \in \mathbb{RT}_{k,d}$.*

*Proof.* Since $\dim(\mathbb{P}_{k,d}) = \binom{k+d}{k}$, $\dim(\mathbb{P}^{\mathrm{H}}_{k,d}) = \binom{k+d-1}{k}$, and the sum of $\boldsymbol{\mathbb{P}}_{k,d}$ and $\mathbb{P}^{\mathrm{H}}_{k,d}$ is direct, $\dim(\mathbb{RT}_{k,d}) = d\binom{k+d}{k} + \binom{k+d-1}{k} = (k+d+1)\binom{k+d-1}{k}$. Let now $\boldsymbol{v} \in \mathbb{RT}_{k,d}$; then, $v_i \in \mathbb{P}_{k+1,d}$ for all $i \in \{1{:}d\}$, so that $\partial_i v_i \in \mathbb{P}_{k,d}$. As a result, $\nabla\cdot\boldsymbol{v} \in \mathbb{P}_{k,d}$. □

**Lemma 7.6 (Normal component).** *Let $H$ be an affine hyperplane in $\mathbb{R}^d$, let $\boldsymbol{n}_H$ be a normal vector to $H$, and let $\boldsymbol{T}_H : \mathbb{R}^{d-1} \to H$ be an affine bijective map. Then, $\boldsymbol{v}_{|H}\cdot\boldsymbol{n}_H \in \mathbb{P}_{k,d-1} \circ \boldsymbol{T}_H^{-1}$ for all $\boldsymbol{v} \in \mathbb{RT}_{k,d}$, i.e., $\boldsymbol{v}_{|H}\cdot\boldsymbol{n}_H \circ \boldsymbol{T}_H \in \mathbb{P}_{k,d-1}$.*

*Proof.* Let $\boldsymbol{v} \in \mathbb{RT}_{k,d}$ with $\boldsymbol{v} = \boldsymbol{p} + \boldsymbol{x}q$, $\boldsymbol{p} \in \boldsymbol{\mathbb{P}}_{k,d}$, and $q \in \mathbb{P}^{\mathrm{H}}_{k,d}$. Let $\boldsymbol{x} \in H$ and set $\boldsymbol{x} = \boldsymbol{T}_H(\boldsymbol{y})$. Since the quantity $\boldsymbol{x}\cdot\boldsymbol{n}_H$ is constant, say $\boldsymbol{x}\cdot\boldsymbol{n}_H = c_H$, we infer that $(\boldsymbol{v}_{|H}\cdot\boldsymbol{n}_H)(\boldsymbol{x}) = (\boldsymbol{p}_{|H}\cdot\boldsymbol{n}_H)(\boldsymbol{x}) + (\boldsymbol{x}\cdot\boldsymbol{n}_H)q(\boldsymbol{x}) = ((\boldsymbol{p}\circ\boldsymbol{T}_H)\cdot\boldsymbol{n}_H)(\boldsymbol{y}) + c_H(q\circ\boldsymbol{T}_H)(\boldsymbol{y})$, and both summands are in $\mathbb{P}_{k,d-1}$. □

## 7.2 Simplicial Raviart–Thomas elements

Let $K$ be a simplex in $\mathbb{R}^d$, see Definition 3.1. Let $\{F_i\}_{i\in\{0:d\}}$ be the faces of $K$. The degrees of freedom of the simplicial Raviart–Thomas finite element are attached to the $(d+1)$ faces of $K$ and to $K$ itself (for $k \geq 1$). All the faces of $K$ are oriented by choosing a unit normal vector to $F_i$ denoted $\boldsymbol{n}_{F_i}$. Note that $\boldsymbol{n}_{F_i} = \pm\boldsymbol{n}_{K,F_i}$, where $\boldsymbol{n}_{K,F_i}$ is the outward unit normal to $K$, and that $\{\boldsymbol{n}_{F_i}\}_{i\in\{1:d\}}$ is a spanning set of $\mathbb{R}^d$ irrespective of the choice of the orientation (see Exercise 3.2(iv) for the case $\boldsymbol{n}_{F_i} = \boldsymbol{n}_{K,F_i}$).

**Definition 7.7 (Degrees of freedom).** *Let $\{\zeta_m\}_{m\in\{1:n_{\rm sh}^{\rm s}\}}$ be a fixed basis of $\mathbb{P}_{k,d-1}$ with $n_{\rm sh}^{\rm s}=\binom{d+k-1}{k}$. Let $\{\psi_m\}_{m\in\{1:n_{\rm sh}^{\rm v}\}}$ be a fixed basis of $\mathbb{P}_{k-1,d}$, if $k\ge 1$, with $n_{\rm sh}^{\rm v}=\binom{d+k-1}{k-1}$. Let $\boldsymbol{T}_{F_i}$ be an affine bijective map from the unit simplex $\widehat{S}^{d-1}$ in $\mathbb{R}^{d-1}$ onto $F_i$, for all $i\in\{1:d\}$. We denote by $\Sigma$ the collection of the following linear forms acting on $\mathbb{RT}_{k,d}$:*

$$\sigma_{i,m}^{\rm s}(\boldsymbol{v})=\int_{F_i}(\boldsymbol{v}\cdot\boldsymbol{n}_{F_i})(\zeta_m\circ\boldsymbol{T}_{F_i}^{-1})\,{\rm d}s,\qquad i\in\{0:d\},\ m\in\{1:n_{\rm sh}^{\rm s}\},\tag{7.2a}$$

$$\sigma_{i,m}^{\rm v}(\boldsymbol{v})=\frac{|F_i|}{|K|}\int_K(\boldsymbol{v}\cdot\boldsymbol{n}_{F_i})\psi_m\,{\rm d}x,\qquad i\in\{1:d\},\ m\in\{1:n_{\rm sh}^{\rm v}\},\tag{7.2b}$$

*and we set $\Sigma_i^{\rm s}:=\{\sigma_{i,m}^{\rm s}\}_{m\in\{1:n_{\rm sh}^{\rm s}\}}$ and $\Sigma^{\rm v}:=\{\sigma_{i,m}^{\rm v}\}_{i\in\{1:d\},m\in\{1:n_{\rm sh}^{\rm v}\}}$.*

The following result is particularly important to characterize $\boldsymbol{H}(\mathrm{div})$-conformity.

**Lemma 7.8 (Face unisolvence).** *Let $\boldsymbol{v}\in\mathbb{RT}_{k,d}$; then, for all $i\in\{0:d\}$, the following holds:*

$$[\,\sigma(\boldsymbol{v})=0,\ \forall\sigma\in\Sigma_i^{\rm s}\,]\Longleftrightarrow[\,\boldsymbol{v}_{|F_i}\cdot\boldsymbol{n}_{F_i}=0\,].\tag{7.3}$$

*Proof.* The condition $\sigma(\boldsymbol{v})=0$ for all $\sigma\in\Sigma_i^{\rm s}$ means that $\boldsymbol{v}_{|F_i}\cdot\boldsymbol{n}_{F_i}$ is orthogonal to $\mathbb{P}_{k,d-1}\circ\boldsymbol{T}_{F_i}^{-1}$, which, by virtue of Lemma 7.6, is equivalent to $\boldsymbol{v}_{|F_i}\cdot\boldsymbol{n}_{F_i}=0$. □

**Proposition 7.9 (Finite element).** *The triple $(K,\mathbb{RT}_{k,d},\Sigma)$ is a finite element.*

*Proof.* Observe first that the cardinal number of $\Sigma$ can be evaluated as follows:

$$\begin{aligned}\mathrm{card}(\Sigma)&=d\,n_{\rm sh}^{\rm v}+(d+1)n_{\rm sh}^{\rm s}=d\binom{d+k-1}{k-1}+(d+1)\binom{d+k-1}{k}\\&=\frac{(d+k-1)!}{(d-1)!(k-1)!}\left(1+\frac{d+1}{k}\right)=\dim(\mathbb{RT}_{k,d}).\end{aligned}$$

As a result, the statement will be proved once it is established that zero is the only function in $\mathbb{RT}_{k,d}$ that annihilates the degrees of freedom in $\Sigma$. Let $\boldsymbol{v}\in\mathbb{RT}_{k,d}$ be such that $\sigma(\boldsymbol{v})=0$ for all $\sigma\in\Sigma$. Owing to Lemma 7.8, we infer that $\boldsymbol{v}_{|F_i}\cdot\boldsymbol{n}_{F_i}=0$ for all $i\in\{0:d\}$. This in turn implies that $\int_K\boldsymbol{v}\cdot(\nabla\nabla\cdot\boldsymbol{v})\,{\rm d}x=-\int_K(\nabla\cdot\boldsymbol{v})^2\,{\rm d}x$. Observing that $\nabla\nabla\cdot\boldsymbol{v}$ is in $\boldsymbol{\mathbb{P}}_{k-1,d}$ (recall that $\nabla\cdot\boldsymbol{v}\in\mathbb{P}_{k,d}$ from Lemma 7.5), the assumption that $\sigma(\boldsymbol{v})=0$ for all $\sigma\in\Sigma^{\rm v}$ (i.e., $\boldsymbol{v}$ is orthogonal to $\boldsymbol{\mathbb{P}}_{k-1,d}$), together with the above identity, implies that $\nabla\cdot\boldsymbol{v}=0$. By definition there are $\boldsymbol{p}\in\boldsymbol{\mathbb{P}}_{k,d}$ and $q\in\mathbb{P}_{k,d}^{\rm H}$ such that $\boldsymbol{v}=\boldsymbol{p}+\boldsymbol{x}q$. The above argument means that $\nabla\cdot\boldsymbol{p}+(k+d)q=0$, which implies that $q=0$ since $\mathbb{P}_{k,d}^{\rm H}\cap\mathbb{P}_{k-1,d}=\{0\}$ if $k\ge 1$, the argument for $k=0$ being trivial. In conclusion, $\boldsymbol{v}$ is in $\boldsymbol{\mathbb{P}}_{k,d}$ and is such that $\boldsymbol{v}_{|F_i}\cdot\boldsymbol{n}_{F_i}=0$ for all $i\in\{0:d\}$.

If $k=0$, this means that $\boldsymbol{v}=\boldsymbol{0}$, since $\{\boldsymbol{n}_{F_i}\}_{i\in\{0:d\}}$ is a spanning set of $\mathbb{R}^d$. Otherwise, we conclude that $\boldsymbol{v}(\boldsymbol{x})\cdot\boldsymbol{n}_{F_i}=\lambda_i(\boldsymbol{x})r_i(\boldsymbol{x})$ for all $\boldsymbol{x}\in\mathbb{R}^d$, where $\lambda_i$ is the barycentric coordinate of $K$ associated with the vertex opposite to $F_i$ (so that $\lambda_i$ vanishes on $F_i$) and $r_i\in\mathbb{P}_{k-1,d}$, see Exercise 3.3(iv). The condition $\sigma(\boldsymbol{v})=0$ for all $\sigma\in\Sigma^{\mathrm{v}}$ implies that $\int_K(\boldsymbol{v}\cdot\boldsymbol{n}_{F_i})r_i\,\mathrm{d}x=0$, which in turn means that $0=\int_K(\boldsymbol{v}\cdot\boldsymbol{n}_{F_i})r_i\,\mathrm{d}x=\int_K\lambda_i r_i^2\,\mathrm{d}x$, thereby proving that $r_i=0$. In conclusion $\boldsymbol{v}(\boldsymbol{x})\cdot\boldsymbol{n}_{F_i}=0$ for all $i\in\{0:d\}$ and all $\boldsymbol{x}\in K$, i.e., $\boldsymbol{v}=\boldsymbol{0}$. □

The above finite element has been introduced in Raviart and Thomas [348, 349] in two space dimensions. The extension to three space dimensions and the presentation adopted here are due to Nédélec [322]. The reading of [322] is highly recommended. The notation $\mathbb{RT}_{k,d}$ seems to be an accepted practice in the literature. Further results can be found in Brezzi and Fortin [91, p. 113], Monk [316, p. 118-126], and Quarteroni and Valli [343, p. 82].

**Example 7.10 ($\mathbb{RT}_{0,d}$).** Using $\zeta_1=1$ for the unique basis function of $\mathbb{P}_{0,d-1}$ and outward normals, the shape functions of the $\mathbb{RT}_{0,d}$ element are given by

$$\boldsymbol{\theta}_i^{\mathrm{s}}(\boldsymbol{x})=\frac{1}{d|K|}(\boldsymbol{x}-\boldsymbol{z}_i),\qquad \forall\boldsymbol{x}\in\mathbb{R}^d,\ \forall i\in\{0:d\},\tag{7.4}$$

where $\boldsymbol{z}_i$ is the vertex of $K$ opposite to $F_i$. Owing to Lemma 7.6, the normal component of $\boldsymbol{\theta}_i^{\mathrm{s}}$ is constant over the faces of $K$; it is zero on $F_j$ if $j\neq i$. A graphic representation of the degrees of freedom is shown in Figure 7.1. An arrow means that the flux of the normal component is taken over the face. See Exercise 7.1 for additional properties of the $\mathbb{RT}_{0,d}$ shape functions. □

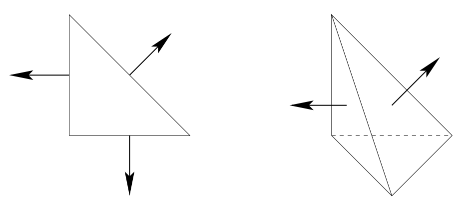

**Fig. 7.1.** $\mathbb{RT}_{0,d}$ finite element in two (left) and three (right) dimensions; only visible degrees of freedom are shown in three dimensions.

## 7.3 Generation of Raviart–Thomas elements

Let $\widehat{K}$ be the reference simplex in $\mathbb{R}^d$ with faces $\widehat{F}_i$ and unit normals $\widehat{\boldsymbol{n}}_{\widehat{F}_i}$ for all $i\in\{0:d\}$. Let $\mathcal{T}_h$ be an affine mesh. Let $K=\boldsymbol{T}_K(\widehat{K})$ be a mesh cell with faces $F_i=\boldsymbol{T}_K(\widehat{F}_i)$ and unit normals $\boldsymbol{n}^{\mathrm{s}}_{K,F_i}:=\|\mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{n}}_{\widehat{F}_i}\|^{-1}_{\ell^2(\mathbb{R}^d)}\mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{n}}_{\widehat{F}_i}$

(see Lemma 4.12(i)) for all $i \in \{0{:}d\}$, where $\mathbb{J}_K$ is the Jacobian matrix of $\boldsymbol{T}_K$. Notice that $\boldsymbol{n}^{\mathrm{s}}_{K,F_i}$ coincides with the unit normal $\boldsymbol{n}_{K,F_i}$ pointing outward $K$ when $\widehat{\boldsymbol{n}}_{\widehat{F}}$ points outward $\widehat{F}$. We now consider the contravariant Piola transformation introduced in (4.18):

$$\boldsymbol{\psi}^{\mathrm{d}}_K(\boldsymbol{v}) = \det(\mathbb{J}_K)\mathbb{J}_K^{-1}(\boldsymbol{v}\circ \boldsymbol{T}_K). \tag{7.5}$$

This definition is motivated by the following important result.

**Lemma 7.11 (Transformation of degrees of freedom).** *Assume that the geometric map $\boldsymbol{T}_K$ is affine. Let $\boldsymbol{v} \in C^0(K;\mathbb{R}^d)$ and let $q \in C^0(K)$. Set $\widehat{\boldsymbol{v}} = \boldsymbol{\psi}^{\mathrm{d}}_K(\boldsymbol{v})$ and $\widehat{q} = \psi^{\mathrm{g}}_K(q) := q\circ \boldsymbol{T}_K$. Then, the following holds:*

(i) $\int_{F_i}(\boldsymbol{v}{\cdot}\boldsymbol{n}^{\mathrm{s}}_{K,F_i})q\,\mathrm{d}s = \int_{\widehat{F}_i}(\widehat{\boldsymbol{v}}{\cdot}\widehat{\boldsymbol{n}}_{\widehat{F}_i})\widehat{q}\,\mathrm{d}\widehat{s}$, *for all* $i \in \{0{:}d\}$.

(ii) $\frac{|F_i|}{|K|}\int_K(\boldsymbol{v}{\cdot}\boldsymbol{n}^{\mathrm{s}}_{K,F_i})q\,\mathrm{d}x = \frac{|\widehat{F}_i|}{|\widehat{K}|}\int_{\widehat{K}}(\widehat{\boldsymbol{v}}{\cdot}\widehat{\boldsymbol{n}}_{\widehat{F}_i})\widehat{q}\,\mathrm{d}\widehat{x}$, *for all* $i \in \{1{:}d\}$.

*Proof.* See Exercise 7.2. □

**Proposition 7.12 (Generation).** *Let $(\widehat{K},\widehat{\boldsymbol{P}},\widehat{\Sigma})$ be a simplicial $\mathbb{RT}_{k,d}$ element. The finite element generated in $K$ using Proposition 4.16 with the map (7.5) is a simplicial $\mathbb{RT}_{k,d}$ finite element with degrees of freedom*

$$\sigma^{\mathrm{s}}_{i,m}(\boldsymbol{v}) = \int_{F_i}(\boldsymbol{v}{\cdot}\boldsymbol{n}_{K,F_i})(\zeta_m\circ \boldsymbol{T}^{-1}_{K,F_i})\,\mathrm{d}s, \qquad i\in\{0{:}d\},\ m\in\{1{:}n^{\mathrm{s}}_{\mathrm{sh}}\}, \tag{7.6a}$$

$$\sigma^{\mathrm{v}}_{i,m}(\boldsymbol{v}) = \frac{|F_i|}{|K|}\int_K(\boldsymbol{v}{\cdot}\boldsymbol{n}_{K,F_i})\psi_m\,\mathrm{d}x, \qquad i\in\{1{:}d\},\ m\in\{1{:}n^{\mathrm{v}}_{\mathrm{sh}}\}, \tag{7.6b}$$

*where $\boldsymbol{T}_{K,F_i} = \boldsymbol{T}_K\circ \boldsymbol{T}_{\widehat{F}_i}$ is an affine bijective map from $\widehat{S}^{d-1}$ onto $F_i$.*

*Proof.* Let us first prove that $\boldsymbol{P} = \mathbb{RT}_{k,d}$. Let $\boldsymbol{T}_K(\widehat{\boldsymbol{x}}) = \mathbb{J}_K\widehat{\boldsymbol{x}} + \boldsymbol{b}_K$ with $\mathbb{J}_K \in \mathbb{R}^{d\times d}$ and $\boldsymbol{b}_K\in\mathbb{R}^d$. Let $\boldsymbol{v}$ be a member of $\boldsymbol{P}$, then $\boldsymbol{\psi}^{\mathrm{d}}_K(\boldsymbol{v}) = \widehat{\boldsymbol{p}} + \widehat{\boldsymbol{x}}\widehat{q}$ with $\widehat{\boldsymbol{p}}\in\mathbb{P}_{k,d}$ and $\widehat{q}\in\mathbb{P}^{\mathrm{H}}_{k,d}$, yielding

$$\boldsymbol{v} = (\boldsymbol{\psi}^{\mathrm{d}}_K)^{-1}(\widehat{\boldsymbol{p}} + \widehat{\boldsymbol{x}}\widehat{q}) = \frac{1}{\det(\mathbb{J}_K)}\mathbb{J}_K(\widehat{\boldsymbol{p}}\circ \boldsymbol{T}_K^{-1} + (\widehat{\boldsymbol{x}}\widehat{q})\circ \boldsymbol{T}_K^{-1}).$$

Then, using $\widehat{\boldsymbol{x}} = \mathbb{J}_K^{-1}(\boldsymbol{x}-\boldsymbol{b}_K)$, we have $\widehat{q}\circ \boldsymbol{T}_K^{-1} = \widehat{q}(\mathbb{J}_K^{-1}\boldsymbol{x} - \mathbb{J}_K^{-1}\boldsymbol{b}_K) = \widehat{q}(\mathbb{J}_K^{-1}\boldsymbol{x}) + \boldsymbol{r}$ where $\boldsymbol{r}\in\mathbb{P}_{k-1,d}$, and it can be verified that $\widehat{q}\circ\mathbb{J}_K^{-1}\in\mathbb{P}^{\mathrm{H}}_{k,d}$. Hence, $\boldsymbol{v} = \boldsymbol{s} + \frac{1}{\det(\mathbb{J}_K)}\mathbb{J}_K\mathbb{J}_K^{-1}\boldsymbol{x}(\widehat{q}\circ\mathbb{J}_K^{-1}) = \boldsymbol{s} + \boldsymbol{x}t$ where $\boldsymbol{s}\in\mathbb{P}_{k,d}$ and $t\in\mathbb{P}^{\mathrm{H}}_{k,d}$. As a result, $\boldsymbol{P}\subset\mathbb{RT}_{k,d}$; the converse follows from a dimension argument. Finally, the definition of the degrees of freedom results from Lemma 7.11. □

**Remark 7.13 (Unit).** The shape functions scale as $L^{1-d}$ where $L$ is a length unit (i.e., per unit length for $d=2$ and per unit surface for $d=3$). □

**Remark 7.14 (Non-affine meshes).** Proposition 4.16 together with the map (7.5) can still be used to generate a finite element $(K,\boldsymbol{P},\Sigma)$ if the geometric map $\boldsymbol{T}_K$ is non-affine. The function space $\boldsymbol{P}$ and the degrees of freedom in $\Sigma$ then differ from those of the $\mathbb{RT}_{k,d}$ element. □

## 7.4 Local interpolation

Let $K$ be a simplex in $\mathbb{R}^d$. We consider two options to extend the degrees of freedom of the $\mathbb{RT}_{k,d}$ element introduced in §7.2:

$$\boldsymbol{V}^{\mathrm{d}}(K) = \boldsymbol{W}^{s,p}(K), \quad p \in [1, +\infty], \quad s > \tfrac{1}{p}, \tag{7.7a}$$

$$\text{or} \quad \boldsymbol{V}^{\mathrm{d}}(K) = \{\boldsymbol{v} \in \boldsymbol{L}^p(K) \mid \nabla\cdot\boldsymbol{v} \in L^s(K)\}, \quad p > 2, \quad s \geq \tfrac{pd}{p+d}, \tag{7.7b}$$

where $\boldsymbol{W}^{s,p}(K) := W^{s,p}(K;\mathbb{R}^d)$. In the definition (7.7a) for $\boldsymbol{V}^{\mathrm{d}}(K)$, all the components of $\boldsymbol{v}$ play the same role. The definition (7.7b) is relevant when there is some information on the integrability of $\nabla\cdot\boldsymbol{v}$ (for instance, because $\boldsymbol{v}$ solves a given PDE). For both definitions in (7.7), it is shown in Exercise 7.6 that the degrees of freedom defined in (7.2) are bounded on $\boldsymbol{V}^{\mathrm{d}}(K)$, and that the contravariant Piola transformation $\boldsymbol{\psi}_K^{\mathrm{d}}$ defined by (7.5) maps boundedly from $\boldsymbol{V}^{\mathrm{d}}(K)$ onto $\boldsymbol{V}^{\mathrm{d}}(\widehat{K})$. Moreover, the identities stated in Lemma 7.11 hold for all $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{d}}(K)$ and all $q \in C^0(K)$.

**Remark 7.15 (Choice of $s$).** The lower the value of $s$, the larger the space $\boldsymbol{V}^{\mathrm{d}}(K)$. In (7.7a), the choice $s = 1$ is always legitimate, since it is possible to take $s = 1$ if $p = 1$. In (7.7b), it is always possible to take $s = 2$; notice also that $\frac{pd}{p+d} > 1$ if $p > 2$ and $d \geq 2$, so that $s > 1$ in (7.7b). □

Let us now consider a sequence of affine simplicial meshes $(\mathcal{T}_h)_{h>0}$ and let us generate a simplicial $\mathbb{RT}_{k,d}$ element in each mesh cell $K \in \mathcal{T}_h$ from a reference $\mathbb{RT}_{k,d}$ element by proceeding as described in §7.3. Let $\{\widehat{\sigma}_i\}_{i\in\mathcal{N}}$ be the reference degrees of freedom and let $\{\widehat{\boldsymbol{\theta}}_i\}_{i\in\mathcal{N}}$ be the reference shape functions, where $\mathcal{N} = \{1{:}n_{\mathrm{sh}}\}$ and $n_{\mathrm{sh}} = \mathrm{card}(\widehat{\Sigma})$. Let $\mathcal{I}_{\widehat{K}}^{\mathrm{d}} : \boldsymbol{V}^{\mathrm{d}}(\widehat{K}) \to \mathbb{RT}_{k,d}$ be the associated interpolation operator with $\boldsymbol{V}^{\mathrm{d}}(\widehat{K})$ defined by either (7.7a) or (7.7b) (with $\widehat{K}$ in lieu of $K$). The local Raviart–Thomas interpolation operator $\mathcal{I}_K^{\mathrm{d}} : \boldsymbol{V}^{\mathrm{d}}(K) \to \mathbb{RT}_{k,d}$ is defined as follows:

$$\mathcal{I}_K^{\mathrm{d}}(\boldsymbol{v})(\boldsymbol{x}) = \sum_{i\in\mathcal{N}} \sigma_i(\boldsymbol{v})\boldsymbol{\theta}_i(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in K, \tag{7.8}$$

for all $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{d}}(K)$, where $\sigma_i(\boldsymbol{v}) = \widehat{\sigma}_i(\widehat{\boldsymbol{v}})$ with $\widehat{\boldsymbol{v}} = \boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v})$ and $\boldsymbol{\theta}_i = (\boldsymbol{\psi}_K^{\mathrm{d}})^{-1}(\widehat{\boldsymbol{\theta}}_i)$. Owing to Proposition 4.17, one key consequence of this construction is that

$$(\boldsymbol{\psi}_K^{\mathrm{d}} \circ \mathcal{I}_K^{\mathrm{d}})(\boldsymbol{v}) = (\mathcal{I}_{\widehat{K}}^{\mathrm{d}} \circ \boldsymbol{\psi}_K^{\mathrm{d}})(\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \boldsymbol{V}^{\mathrm{d}}(K). \tag{7.9}$$

**Lemma 7.16 (Commuting with $\nabla\cdot$).** *Let $\mathcal{I}_K^{\mathrm{d}}$ be defined in (7.8) and let $\mathcal{I}_K^{\mathrm{b}} : V^{\mathrm{b}}(K) := L^1(K) \to \mathbb{P}_{k,d}$ be the $L^2$-orthogonal projection onto $\mathbb{P}_{k,d}$, i.e., $\int_K (\mathcal{I}_K^{\mathrm{b}}(\phi) - \phi) q \,\mathrm{d}x = 0$ for all $\phi \in L^1(K)$ and all $q \in \mathbb{P}_{k,d}$. The following diagram commutes:*

$$
\begin{tikzcd}
\check{\boldsymbol{V}}^{\mathrm{d}}(K) \arrow[r, "\nabla\cdot"] \arrow[d, "\mathcal{I}_K^{\mathrm{d}}"] & V^{\mathrm{b}}(K) \arrow[d, "\mathcal{I}_K^{\mathrm{b}}"] \\
\mathbb{RT}_{k,d} \arrow[r, "\nabla\cdot"] & \mathbb{P}_{k,d}
\end{tikzcd}
$$

*where* $\check{\boldsymbol{V}}^{\mathrm{d}}(K) = \{\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{d}}(K) \mid \nabla\cdot\boldsymbol{v} \in V^{\mathrm{b}}(K)\}$.

*Proof.* Let $\boldsymbol{v} \in \check{\boldsymbol{V}}^{\mathrm{d}}(K)$ and let $q \in \mathbb{P}_{k,d}$. We observe that

$$
\int_K q\mathcal{I}_K^{\mathrm{b}}(\nabla\cdot\boldsymbol{v})\,\mathrm{d}x = \int_K q\nabla\cdot\boldsymbol{v}\,\mathrm{d}x = -\int_K \boldsymbol{v}\cdot\nabla q\,\mathrm{d}x + \sum_{i\in\{0:d\}} \int_{F_i} (\boldsymbol{v}\cdot\boldsymbol{n}_K)q\,\mathrm{d}s,
$$

where $\boldsymbol{n}_K$ is the unit outward normal to $K$. Since $q \circ \boldsymbol{T}_{K,F_i} \in \mathbb{P}_{k,d-1}$, we infer that $\int_{F_i}(\boldsymbol{v}\cdot\boldsymbol{n}_{K,F_i})q\,\mathrm{d}s = \int_{F_i}(\mathcal{I}_K^{\mathrm{d}}(\boldsymbol{v})\cdot\boldsymbol{n}_{K,F_i})q\,\mathrm{d}s$ owing to (7.2a) (recall that $\boldsymbol{n}_{K,F_i} = \pm\boldsymbol{n}_K$ on $F_i$). In addition, since, for $k \geq 1$, $\nabla q \in \boldsymbol{\mathbb{P}}_{k-1,d}$, we infer that $\int_K \boldsymbol{v}\cdot\nabla q\,\mathrm{d}x = \int_K \mathcal{I}_K^{\mathrm{d}}(\boldsymbol{v})\cdot\nabla q\,\mathrm{d}x$ owing to (7.2b), (the statement is evident for $k = 0$). This, in turn, implies that

$$
\begin{aligned}
\int_K q\mathcal{I}_K^{\mathrm{b}}(\nabla\cdot\boldsymbol{v})\,\mathrm{d}x &= -\int_K (\mathcal{I}_K^{\mathrm{d}}\boldsymbol{v})\cdot\nabla q\,\mathrm{d}x + \int_{\partial K} (\mathcal{I}_K^{\mathrm{d}}(\boldsymbol{v})\cdot\boldsymbol{n}_K)q\,\mathrm{d}s \\
&= \int_K q\nabla\cdot(\mathcal{I}_K^{\mathrm{d}}\boldsymbol{v})\,\mathrm{d}x. \qquad \square
\end{aligned}
$$

**Example 7.17 (Laplacian projection).** Let $\phi \in W^{2,1}(K)$ and set $\boldsymbol{v} = \mathcal{I}_K^{\mathrm{d}}(\nabla\phi) \in \mathbb{RT}_{k,d}$, $k \geq 0$. Then, Lemma 7.16 implies that $\nabla\cdot\boldsymbol{v} = \nabla\cdot\mathcal{I}_K^{\mathrm{d}}(\nabla\phi) = \mathcal{I}_K^{\mathrm{b}}(\nabla\cdot\nabla\phi) = \mathcal{I}_K^{\mathrm{b}}\Delta\phi$. $\square$

**Theorem 7.18 (Local interpolation).** *Assume that the mesh sequence is shape-regular. Let* $\mathcal{I}_K^{\mathrm{d}}$ *be the local* $\mathbb{RT}_{k,d}$ *interpolation operator.*

(i) *There is c such that the following estimate holds:*

$$
\|\boldsymbol{v} - \mathcal{I}_K^{\mathrm{d}}\boldsymbol{v}\|_{\boldsymbol{W}^{m,p}(K)} \leq c\,h_K^{r-m}|\boldsymbol{v}|_{\boldsymbol{W}^{r,p}(K)}, \tag{7.10}
$$

*for all* $r \in \{1{:}k+1\}$, *all* $m \in \{0{:}r\}$, *all* $p \in [1,\infty]$, *all* $\boldsymbol{v} \in \boldsymbol{W}^{r,p}(K)$, *and all* $K \in \mathcal{T}_h$.

(ii) *There is c such that the following estimate holds:*

$$
\|\nabla\cdot(\boldsymbol{v} - \mathcal{I}_K^{\mathrm{d}}\boldsymbol{v})\|_{W^{m,p}(K)} \leq c\,h_K^{r-m}|\nabla\cdot\boldsymbol{v}|_{W^{r,p}(K)}, \tag{7.11}
$$

*for all* $r \in \{0{:}k+1\}$, *all* $m \in \{0{:}r\}$, *all* $p \in [1,\infty]$, *all* $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{d}}(K)$ *such that* $\nabla\cdot\boldsymbol{v} \in W^{r,p}(K)$, *and all* $K \in \mathcal{T}_h$.

*Proof.* The contravariant Piola transformation $\boldsymbol{\psi}_K^{\mathrm{d}}$ is of the form (5.1) with $\mathbb{A}_K = \det(\mathbb{J}_K)\mathbb{J}_K^{-1}$, which satisfies the bound (5.12) with $c = 1$. Then, the bound (7.10) is a consequence of Theorem 5.12 with $\psi_K = \boldsymbol{\psi}_K^{\mathrm{d}}$ (and $q = d$), $l = 1$ (since $\boldsymbol{W}^{1,p}(\widehat{K}) \hookrightarrow \boldsymbol{V}^{\mathrm{d}}(\widehat{K})$ using (7.7a), see Remark 7.15), and $r \in$

$\{1{:}k+1\}$ (so that $s = \max(l,r) = r$). To prove the estimate (7.11) on the divergence, we use Lemma 7.16 to infer that

$$\|\nabla{\cdot}(\boldsymbol{v} - \mathcal{I}^{\mathrm{d}}_K \boldsymbol{v})\|_{L^p(K)} = \|\nabla{\cdot}\boldsymbol{v} - \mathcal{I}^{\mathrm{b}}_K(\nabla{\cdot}\boldsymbol{v})\|_{L^p(K)},$$

and conclude using Lemma 7.20 below (note that $P_K = \mathbb{P}_{k,d}$ since $\widehat{P} = \mathbb{P}_{k,d}$ and the meshes are affine). $\square$

**Remark 7.19 (Error on divergence).** It is remarkable that the error $\nabla{\cdot}(\boldsymbol{v} - \mathcal{I}^{\mathrm{d}}_K \boldsymbol{v})$ only depends on the regularity of $\nabla{\cdot}\boldsymbol{v}$. $\square$

**Lemma 7.20 ($L^2$-orthogonal projection).** *Let $\widehat{P}$ be a finite-dimensional space such that $\mathbb{P}_{k,d} \subset \widehat{P} \subset W^{1,\infty}(\widehat{K})$. Let $(\mathcal{T}_h)_{h>0}$ be a shape-regular family of affine meshes. Let $\psi_K$ be the pullback by the geometric map $\boldsymbol{T}_K$ for all $K \in \mathcal{T}_h$. Set $P_K := \psi_K^{-1}(\widehat{P})$. Let $\mathcal{I}^{\mathrm{b}}_K : L^1(K) \to P_K$ be the $L^2$-orthogonal projection onto $P_K$, i.e., $\int_K (\mathcal{I}^{\mathrm{b}}_K(\phi) - \phi) q \,\mathrm{d}x = 0$ for all $\phi \in L^1(K)$ and all $q \in P_K$. Then, there is $c$ such that the following estimate holds:*

$$|\phi - \mathcal{I}^{\mathrm{b}}_K \phi|_{W^{m,p}(K)} \le c\, h_K^{r-m} |\phi|_{W^{r,p}(K)}, \tag{7.12}$$

*for all $r \in \{0{:}k+1\}$, all $m \in \{0{:}r\}$, all $\phi \in W^{r,p}(K)$, and all $K \in \mathcal{T}_h$.*

*Proof.* Let $\mathcal{I}^{\mathrm{b}}_{\widehat{K}} : V(\widehat{K}) = L^1(\widehat{K}) \to \widehat{P}$ be the $L^2$-orthogonal projection onto $\widehat{P}$, i.e., $\int_{\widehat{K}} (\mathcal{I}^{\mathrm{b}}_{\widehat{K}}(\widehat{\phi}) - \widehat{\phi})\widehat{q} \,\mathrm{d}\widehat{x} = 0$ for all $\widehat{\phi} \in L^1(\widehat{K})$ and all $\widehat{q} \in \widehat{P}$. Note also that $\mathcal{I}^{\mathrm{b}}_{\widehat{K}}$ leaves $\widehat{P}$ pointwise invariant. Moreover, the mesh sequence being affine, we infer that

$$\begin{aligned}
\int_{\widehat{K}} \mathcal{I}^{\mathrm{b}}_{\widehat{K}}(v \circ \boldsymbol{T}_K) q \circ \boldsymbol{T}_K \,\mathrm{d}\widehat{x} &= \int_{\widehat{K}} (v \circ \boldsymbol{T}_K) q \circ \boldsymbol{T}_K \,\mathrm{d}\widehat{x} = \frac{|\widehat{K}|}{|K|} \int_K vq \,\mathrm{d}x \\
&= \frac{|\widehat{K}|}{|K|} \int_K \mathcal{I}^{\mathrm{b}}_K(v) q \,\mathrm{d}x = \int_{\widehat{K}} (\mathcal{I}^{\mathrm{b}}_K(v) \circ \boldsymbol{T}_K) q \circ \boldsymbol{T}_K \,\mathrm{d}\widehat{x},
\end{aligned}$$

for all $q \in P_K$. Hence, $\mathcal{I}^{\mathrm{b}}_K = \psi_K^{-1} \circ \mathcal{I}^{\mathrm{b}}_{\widehat{K}} \circ \psi_K$ where $\psi_K$ is the pullback by $\boldsymbol{T}_K$. In conclusion, the assumptions of Theorem 5.12 hold with with $l = 0$ and $s = \max(l,r) = r$ since $V(\widehat{K}) = L^1(\widehat{K})$. Then (7.12) follows readily. $\square$

## 7.5 Cartesian Raviart–Thomas elements

In this section, we briefly review the Cartesian Raviart–Thomas finite element; we refer the reader to Exercise 7.3 for the proofs. For a multi-index $\alpha \in \mathbb{N}^d$, we define the (anisotropic) polynomial space $\mathbb{Q}_{\alpha_1,\ldots,\alpha_d}$ composed of $d$-variate polynomials whose degree with respect to $x_i$ is at most $\alpha_i$, for all $i \in \{1{:}d\}$. Let $k \in \mathbb{N}$ and define

$$\mathbb{RT}^{\square}_{k,d} = \mathbb{Q}_{k+1,k,\ldots,k} \times \ldots \times \mathbb{Q}_{k,\ldots,k,k+1}. \tag{7.13}$$

One can verify that $\dim(\mathbb{RT}^{\square}_{k,d}) = d(k+2)(k+1)^{d-1}$ and the following holds:

$$\nabla{\cdot}\boldsymbol{v} \in \mathbb{Q}_{k,d}, \qquad \boldsymbol{v}_{|H}{\cdot}\boldsymbol{n}_H \in \mathbb{Q}_{k,d-1} \circ \boldsymbol{T}_H^{-1}, \tag{7.14}$$

for all $\boldsymbol{v} \in \mathbb{RT}^{\square}_{k,d}$, where $H$ is any affine hyperplane in $\mathbb{R}^d$ with normal vector $\boldsymbol{n}_H$ and $\boldsymbol{T}_H : \mathbb{R}^{d-1} \to H$ is an affine bijective map.

Let $K = [0,1]^d$ be the unit cube in $\mathbb{R}^d$ with faces $\{F_i\}_{i\in\{1:2d\}}$ outer-oriented by unit normal vectors $\{\boldsymbol{n}_{F_i}\}_{i\in\{1:2d\}}$. Assume that the enumeration is done so that $F_i = \{\boldsymbol{x} = (x_1,\ldots,x_d) \in K \mid x_i = 0\}$, for all $i \in \{1{:}d\}$. Let $\boldsymbol{T}_{F_i}$ be an affine bijective map from $[0,1]^{d-1}$ onto $F_i$. Let $\{\zeta_m\}_{m\in\{1:n^{\mathrm{s}}_{\mathrm{sh}}\}}$ be a fixed basis of $\mathbb{Q}_{k,d-1}$ with $n^{\mathrm{s}}_{\mathrm{sh}} = (k+1)^{d-1}$. Let $\{\psi_{i,m}\}_{m\in\{1:n^{\mathrm{v}}_{\mathrm{sh}}\}}$ be a fixed basis of $\mathbb{Q}_{k,\ldots,k,k-1,k,\ldots,k}$ (for $k \geq 1$) with $n^{\mathrm{v}}_{\mathrm{sh}} = k(k+1)^{d-1}$, with the index $(k-1)$ at the $i$-th position for all $i \in \{1{:}d\}$. Let $\Sigma$ be the set composed of the following linear forms:

$$\sigma^{\mathrm{s}}_{i,m}(\boldsymbol{v}) = \int_{F_i} (\boldsymbol{v}{\cdot}\boldsymbol{n}_{F_i})(\zeta_m \circ \boldsymbol{T}_{F_i}^{-1})\,\mathrm{d}s, \quad i \in \{1{:}2d\},\ m \in \{1{:}n^{\mathrm{s}}_{\mathrm{sh}}\}, \tag{7.15a}$$

$$\sigma^{\mathrm{v}}_{i,m}(\boldsymbol{v}) = \frac{|F_i|}{|K|} \int_K (\boldsymbol{v}{\cdot}\boldsymbol{n}_{F_i})\psi_{i,m}\,\mathrm{d}x, \qquad i \in \{1{:}d\},\ m \in \{1{:}n^{\mathrm{v}}_{\mathrm{sh}}\}. \tag{7.15b}$$

**Proposition 7.21.** *The triple* $\{K, \mathbb{RT}^{\square}_{k,d}, \Sigma\}$ *is a finite element.*

Using affine geometric maps and the contravariant Piola transformation defined by (7.5), Cartesian Raviart–Thomas elements can be generated on all the mesh cells of an affine mesh composed of cuboids (or parallelotopes).

**Example 7.22 (Shape functions for $\mathbb{RT}^{\square}_{0,d}$).** Let $F_i$ and $F_{d+i}$ be the faces defined by $x_i = 0$ and $x_i = 1$, respectively, for all $i \in \{1{:}d\}$. Using $\zeta_1 = 1$ for the basis function of $\mathbb{Q}_{0,d-1}$, the $2d$ degrees of freedom are the mean-value of the normal component over each face of $K$, and the shape functions are $\boldsymbol{\theta}^{\mathrm{s}}_i(\boldsymbol{x}) = (1-x_i)\boldsymbol{n}_{F_i}$ and $\boldsymbol{\theta}^{\mathrm{s}}_{d+i}(\boldsymbol{x}) = x_i\boldsymbol{n}_{F_i}$ for all $i \in \{1{:}d\}$. □

## Exercises

**Exercise 7.1 ($\mathbb{RT}_{0,d}$).**

(i) Prove (7.4). (*Hint*: $\int_{F_j} \boldsymbol{x}\,\mathrm{d}s = |F_j|\boldsymbol{c}_{F_j}$, where $\boldsymbol{c}_{F_j} = \frac{1}{d}\sum_{j'\in\{0:d\}\setminus\{j\}} \boldsymbol{z}_{j'}$ is the center of gravity of $F_j$; recall also that $|K| = \frac{1}{d}|F_i|(\boldsymbol{z}_{i'} - \boldsymbol{z}_i){\cdot}\boldsymbol{n}_{K,F_i}$ for all $i' \neq i$, $i \in \{0{:}d\}$.)

(ii) Prove that $\int_K \boldsymbol{\theta}^{\mathrm{s}}_i\,\mathrm{d}x = \boldsymbol{c}_{F_i} - \boldsymbol{c}_K$ where $\boldsymbol{c}_K$ is the center of gravity of $K$. (*Hint*: use (7.4) and verify that $\boldsymbol{c}_K - \boldsymbol{z}_i = d(\boldsymbol{c}_{F_i} - \boldsymbol{c}_K)$.) Provide a second proof without using (7.4). (*Hint*: Fix $\boldsymbol{e} \in \mathbb{R}^d$, define $\phi(\boldsymbol{x}) = (\boldsymbol{x} - \boldsymbol{c}_{F_i}){\cdot}\boldsymbol{e}$, observe that $\nabla\phi = \boldsymbol{e}$, and compute $\boldsymbol{e}{\cdot}\int_K \boldsymbol{\theta}^{\mathrm{s}}_i\,\mathrm{d}x$.)

(iii) Prove that $\sum_{i\in\{0:d\}} |F_i|\boldsymbol{\theta}_i^{\mathrm{s}}(\boldsymbol{x}) \otimes \boldsymbol{n}_{K,F_i} = \mathbb{I}_d$ for all $\boldsymbol{x} \in K$. (*Hint*: use (3.6).)

(iv) Prove that $\boldsymbol{v}(\boldsymbol{x}) = \langle \boldsymbol{v}\rangle_K + \frac{1}{d}(\nabla{\cdot}\boldsymbol{v})(\boldsymbol{x} - \boldsymbol{c}_K)$ for all $\boldsymbol{v} \in \mathbb{RT}_{0,d}$, where $\langle \boldsymbol{v}\rangle_K = \frac{1}{|K|}\int_K \boldsymbol{v}\,\mathrm{d}x$ is the mean-value of $\boldsymbol{v}$ in $K$.

(v) Let $d = 3$. Let $F_i$, $i \in \{0:3\}$, be a face of $K$ with vertices $\{\boldsymbol{a}_p, \boldsymbol{a}_q, \boldsymbol{a}_r\}$ so that $[(\boldsymbol{a}_p - \boldsymbol{a}_q)\times(\boldsymbol{a}_q - \boldsymbol{a}_r)]{\cdot}\boldsymbol{n}_{K,F_i} > 0$. Prove that $\boldsymbol{\theta}_i^{\mathrm{s}} = 2(\lambda_p\nabla\lambda_q{\times}\nabla\lambda_r + \lambda_q\nabla\lambda_r{\times}\nabla\lambda_p + \lambda_r\nabla\lambda_p{\times}\nabla\lambda_q)$. Find the counterpart of this formula for $d = 2$.

**Exercise 7.2 (Transformation of degrees of freedom).**

(i) Prove Lemma 7.11. (*Hint*: use Lemma 4.12 and Lemma 4.13.)

(ii) Prove that $\int_K q\,\nabla{\cdot}\boldsymbol{v}\,\mathrm{d}x = \int_{\widehat{K}} \widehat{q}\,\nabla{\cdot}\widehat{\boldsymbol{v}}\,\mathrm{d}\widehat{x}$ for all $\boldsymbol{v} \in C^1(K;\mathbb{R}^d)$ and all $q \in C^0(K)$, with $\widehat{\boldsymbol{v}} = \boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v})$ and $\widehat{q} = q \circ \boldsymbol{T}_K$.

**Exercise 7.3 (Cartesian Raviart–Thomas element).**

(i) Propose a basis for $\mathbb{RT}_{0,2}^{\square}$ and for $\mathbb{RT}_{0,3}^{\square}$.

(ii) Prove that $\dim(\mathbb{RT}_{k,d}^{\square}) = d(k+2)(k+1)^{d-1}$ and prove (7.14).

(iii) Prove Proposition 7.21.

**Exercise 7.4 (Brezzi–Douglas–Marini element).** Let $K$ be a triangle and set $P = [\mathbb{P}_1]^2$. On each face $F$ of $K$ with unit normal $\boldsymbol{n}_F$, consider the two linear forms $\sigma_{F,1} : P \ni p \mapsto \int_F p(s){\cdot}\boldsymbol{n}_F\,\mathrm{d}s$ and $\sigma_{F,2} : P \ni p \mapsto \int_F p(s){\cdot}\boldsymbol{n}_F\, s\,\mathrm{d}s$. Set $\Sigma = \{\sigma_{F,1}, \sigma_{F,2}\}_{F\subset\partial K}$. Prove that $(K, P, \Sigma)$ is a finite element; see Brezzi et al. [92, 93].

**Exercise 7.5 (Divergence-free $\mathbb{RT}_{k,d}$).** Show that if $\boldsymbol{v} \in \mathbb{RT}_{k,d}$ is divergence-free, then $\boldsymbol{v} \in \mathbb{P}_{k,d}$. (need $\mathrm{BDM}_k$)

**Exercise 7.6 (Definition of $\boldsymbol{V}^{\mathrm{d}}(K)$).**

(i) Prove that (7.7a) is a suitable definition for $\boldsymbol{V}^{\mathrm{d}}(K)$. (*Hint*: use Theorem B.107.) Prove that $\boldsymbol{\psi}_K^{\mathrm{d}} \in \mathcal{L}(\boldsymbol{V}^{\mathrm{d}}(K); \boldsymbol{V}^{\mathrm{d}}(\widehat{K}))$ and that $\boldsymbol{\psi}_K^{\mathrm{d}}$ is bijective. (*Hint*: use (4.5).)

(ii) Do the same exercise for definition (7.7b). (*Hint*: Set $\frac{1}{p} + \frac{1}{p'} = 1$ and use the fact that for any face $F \subset \partial K$, there exists $w \in W^{1,p'}(K)$ such that $w_{|F} = 1$ and $w_{|\partial K\setminus F} = 0$. Conclude using (B.68) and Theorem B.99.)

# Solution to exercises

**Exercise 7.1 ($\mathbb{RT}_{0,d}$).**

(i) Let $i, j \in \{0{:}d\}$. Using the first hint, we infer that

$$\int_{F_i} \boldsymbol{\theta}_j^{\mathrm{s}} \cdot \boldsymbol{n}_{K,F_i} \, \mathrm{d}s = \frac{|F_i|}{d|K|} (\boldsymbol{c}_{F_i} - \boldsymbol{z}_j) \cdot \boldsymbol{n}_{K,F_i}.$$

If $i \neq j$, then $\boldsymbol{c}_{F_i} - \boldsymbol{z}_j = \frac{1}{d} \sum_{j' \in \{0:d\} \setminus \{i,j\}} (\boldsymbol{z}_{j'} - \boldsymbol{z}_j)$ is a linear combination of vectors tangent to $F_i$, so that $(\boldsymbol{c}_{F_i} - \boldsymbol{z}_j) \cdot \boldsymbol{n}_{K,F_i} = 0$ yielding $\int_{F_i} \boldsymbol{\theta}_j^{\mathrm{s}} \cdot \boldsymbol{n}_{K,F_i} \, \mathrm{d}s = 0$. If $i = j$, then $(\boldsymbol{c}_{F_i} - \boldsymbol{z}_i) \cdot \boldsymbol{n}_{K,F_i} = (\boldsymbol{z}_{i'} - \boldsymbol{z}_i) \cdot \boldsymbol{n}_{K,F_i}$ for all $i' \neq i$, so that the assertion $\int_{F_i} \boldsymbol{\theta}_i^{\mathrm{s}} \cdot \boldsymbol{n}_{K,F_i} \, \mathrm{d}s = 1$ follows from the second hint.

(ii) To prove that $\boldsymbol{c}_K - \boldsymbol{z}_i = d(\boldsymbol{c}_{F_i} - \boldsymbol{c}_K)$, we use $\boldsymbol{c}_K = \frac{1}{d+1} \sum_{j \in \{0:d\}} \boldsymbol{z}_j$ to infer that

$$\begin{aligned} d(\boldsymbol{c}_{F_i} - \boldsymbol{c}_K) &= \left( \sum_{j \in \{0:d\} \setminus \{i\}} \boldsymbol{z}_j \right) - d\boldsymbol{c}_K = \left( \sum_{j \in \{0:d\}} \boldsymbol{z}_j \right) - \boldsymbol{z}_i - d\boldsymbol{c}_K \\ &= (d+1)\boldsymbol{c}_K - \boldsymbol{z}_i - d\boldsymbol{c}_K = \boldsymbol{c}_K - \boldsymbol{z}_i. \end{aligned}$$

Hence, since $\int_K \boldsymbol{x} \, \mathrm{d}x = |K| \boldsymbol{c}_K$, we infer that

$$\int_K \boldsymbol{\theta}_i^{\mathrm{s}} \, \mathrm{d}x = \frac{1}{d} (\boldsymbol{c}_K - \boldsymbol{z}_i) = \boldsymbol{c}_{F_i} - \boldsymbol{c}_K.$$

For the second proof, let $\boldsymbol{e} \in \mathbb{R}^d$. Let $\phi(\boldsymbol{x}) = (\boldsymbol{x} - \boldsymbol{x}_{F_i}) \cdot \boldsymbol{e}$ and observe that $\nabla \phi = \boldsymbol{e}$. Then,

$$\boldsymbol{e} \cdot \int_K \boldsymbol{\theta}_i^{\mathrm{s}} \, \mathrm{d}x = \int_K \boldsymbol{\theta}_i^{\mathrm{s}} \cdot \nabla \phi \, \mathrm{d}x = -\int_K \phi \nabla \cdot \boldsymbol{\theta}_i^{\mathrm{s}} \, \mathrm{d}x + \sum_{j=0}^{d} \int_{F_j} (\boldsymbol{\theta}_i^{\mathrm{s}} \cdot \boldsymbol{n}_{K,F_j}) \phi \, \mathrm{d}s$$

Owing to Lemma 7.6, $\boldsymbol{\theta}_i^{\mathrm{s}} \cdot \boldsymbol{n}_{K,F_j}$ is piecewise constant and equal to $\boldsymbol{\theta}_i^{\mathrm{s}} \cdot \boldsymbol{n}_{K,F_j} = \frac{\delta_{ij}}{|F_i|}$. Moreover $|K| \nabla \cdot \boldsymbol{\theta}_i^{\mathrm{s}} = \int_K \nabla \cdot \boldsymbol{\theta}_i^{\mathrm{s}} \, \mathrm{d}x = \int_{F_i} \boldsymbol{\theta}_i^{\mathrm{s}} \cdot \boldsymbol{n}_{K,F_i} \, \mathrm{d}s = 1$. In conclusion,

$$\boldsymbol{e} \cdot \int_K \boldsymbol{\theta}_i^{\mathrm{s}} \, \mathrm{d}x = -\frac{1}{|K|} \int_K \phi \, \mathrm{d}x + \frac{1}{|F_i|} \int_{F_i} \phi \, \mathrm{d}s = -(\boldsymbol{x}_K - \boldsymbol{x}_{F_i}) \cdot \boldsymbol{e},$$

since $\int_K \phi \, \mathrm{d}x = \phi(\boldsymbol{c}_K)|K|$ and $\int_{F_i} \phi \, \mathrm{d}s = 0$. This implies that $\int_K \boldsymbol{\theta}_i^{\mathrm{s}} \, \mathrm{d}x = \boldsymbol{x}_{F_i} - \boldsymbol{x}_K$, since the above equality holds for all $\boldsymbol{e} \in \mathbb{R}^d$.

(iii) Let $\boldsymbol{x} \in K$. We observe that

$$\begin{aligned}\sum_{i\in\{0:d\}} |F_i|\boldsymbol{\theta}_i^{\mathrm{s}}(\boldsymbol{x})\otimes\boldsymbol{n}_{K,F_i} &= \sum_{i\in\{0:d\}} \frac{|F_i|}{d|K|}(\boldsymbol{x}-\boldsymbol{z}_i)\otimes\boldsymbol{n}_{K,F_i} \\ &= \sum_{i\in\{0:d\}} \frac{|F_i|}{d|K|}(\boldsymbol{c}_K-\boldsymbol{z}_i)\otimes\boldsymbol{n}_{K,F_i} \\ &= \sum_{i\in\{0:d\}} \frac{|F_i|}{|K|}(\boldsymbol{c}_{F_i}-\boldsymbol{c}_K)\otimes\boldsymbol{n}_{K,F_i} = \mathbb{I}_d,\end{aligned}$$

where we have used the definition of $\boldsymbol{\theta}_i^{\mathrm{s}}$, the first geometric identity in (3.6) to replace $\boldsymbol{x}$ by $\boldsymbol{c}_K$, the fact that $\boldsymbol{c}_K-\boldsymbol{z}_i = d(\boldsymbol{c}_{F_i}-\boldsymbol{c}_K)$, and the second geometric identity in (3.6) to conclude.

(iv) Let $\boldsymbol{v}\in\mathbb{RT}_{0,d}$. Then we can write $\boldsymbol{v}=\boldsymbol{a}+b(\boldsymbol{x}-\boldsymbol{c}_K)$, where $\boldsymbol{a}\in\mathbb{R}^d$, $b\in\mathbb{R}$, whence we infer that $\nabla{\cdot}\boldsymbol{v}=bd$, i.e., $b=\frac{1}{d}\nabla{\cdot}\boldsymbol{v}$. Moreover, since $(\boldsymbol{x}-\boldsymbol{c}_K)$ has zero mean-value in $K$, we infer that $\boldsymbol{a}=\langle\boldsymbol{v}\rangle_K$. In conclusion, $\boldsymbol{v}=\langle\boldsymbol{v}\rangle_K+\frac{1}{d}(\nabla{\cdot}\boldsymbol{v})(\boldsymbol{x}-\boldsymbol{c}_K)$.

(v)

**Exercise 7.2 (Transformation of degrees of freedom).**

(i) Items (i) and (ii) are a consequence of the definitions of $\widehat{\boldsymbol{v}}$ and $\widehat{q}$ and of item (i) in Lemma 4.12 and item (i) in Lemma 4.13.

(ii) This is a consequence of (4.5), i.e., $\nabla{\cdot}\boldsymbol{v}(x)=\frac{1}{\det(\mathbb{J}_K(\widehat{\boldsymbol{x}}))}\nabla{\cdot}\widehat{v}(\widehat{\boldsymbol{x}})$.

**Exercise 7.3 (Cartesian Raviart–Thomas element).**

(i) A basis for $\mathbb{RT}^{\square}_{0,2}$ is $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}$, $\begin{pmatrix}x_1\\0\end{pmatrix}$, $\begin{pmatrix}0\\x_2\end{pmatrix}$, while a basis for $\mathbb{RT}^{\square}_{0,3}$ is $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\\1\end{pmatrix}$, $\begin{pmatrix}x_1\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\x_2\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\\x_3\end{pmatrix}$.

(ii) TO DO

(iii) Observe first that $\mathrm{card}(\Sigma)=dk(k+1)^{d-1}+2d(k+1)^{d-1}=d(k+1)^{d-1}(k+2)=\dim(\mathbb{RT}^{\square}_k)$. Let $\boldsymbol{v}\in\mathbb{RT}^{\square}_k$ be such that $\sigma(\boldsymbol{v})=0$ for all $\sigma\in\Sigma$. The assumption $\sigma(\boldsymbol{v})=0$ for all $\sigma\in\Sigma_{F_i}$, $1\le i\le 2d$, together with item (**??**) from Lemma **??** implies that $\boldsymbol{v}{\cdot}\boldsymbol{n}_{F_i}=0$. This in turns implies that $\boldsymbol{v}$ can be rewritten as follows: $\boldsymbol{v}=(x_1(1-x_1)r_1,\ldots,x_d(1-x_d)r_d)^{\mathsf{T}}$ where $\boldsymbol{r}:=(r_1,\ldots,r_d)^{\mathsf{T}}$ is a member of $\mathbb{Q}_{k-1,k,\ldots,k}\times\ldots\times\mathbb{Q}_{k,\ldots,k,k-1}$. Then the assumption $\sigma(\boldsymbol{v})=0$ for all $\sigma\in\Sigma_K$ implies that $\int_K \boldsymbol{v}{\cdot}\boldsymbol{r}\,\mathrm{d}x=0$, which in turns leads to $\boldsymbol{r}=0$, thereby proving that $\boldsymbol{v}=0$.

**Exercise 7.4 (Brezzi–Douglas–Marini element).**

**Exercise 7.5 (Divergence-free $\mathbb{RT}_{k,d}$).**

**Exercise 7.6 (Definition of $\boldsymbol{V}^{\mathrm{d}}(K)$).**

(i) Let $\boldsymbol{V}^{\mathrm{d}}(K) = \boldsymbol{W}^{s,p}(K)$ with $s > \frac{1}{p}$ if $p > 1$ and $s \geq 1$ if $p = 1$. The degrees of freedom $\sigma^{\mathrm{s}}_{i,m}$ are bounded in $\boldsymbol{V}^{\mathrm{d}}(K)$ owing to Theorem B.107 (with $r = d-1$). The degrees of freedom $\sigma^{\mathrm{v}}_{i,m}$ are bounded in $\boldsymbol{V}^{\mathrm{d}}(K)$ as soon as $\boldsymbol{v} \in \boldsymbol{L}^1(K)$. Let $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{d}}(K)$ and let us bound $\|\boldsymbol{\psi}^{\mathrm{d}}_K(\boldsymbol{v})\|_{\boldsymbol{W}^{s,p}(\widehat{K})}$ by $\|\boldsymbol{v}\|_{\boldsymbol{W}^{s,p}(K)}$. Since the mesh is affine, we obtain

$$\|\boldsymbol{\psi}^{\mathrm{d}}_K(\boldsymbol{v})\|_{\boldsymbol{L}^p(\widehat{K})} \leq |\det(\mathbb{J}_K)|^{1-\frac{1}{p}} \|\mathbb{J}_K^{-1}\|_{\ell^2} \|\boldsymbol{v}\|_{\boldsymbol{L}^p(K)},$$

and, for $s \in (0,1)$,

$$\left( \int_{\widehat{K}} \int_{\widehat{K}} \frac{\|\boldsymbol{\psi}^{\mathrm{d}}_K(\boldsymbol{v})(\widehat{\boldsymbol{x}}) - \boldsymbol{\psi}^{\mathrm{d}}_K(\boldsymbol{v})(\widehat{\boldsymbol{y}})\|^p_{\ell^2}}{\|\widehat{\boldsymbol{x}} - \widehat{\boldsymbol{y}}\|^{sp+d}_{\ell^2}} \,\mathrm{d}\widehat{x}\,\mathrm{d}\widehat{y} \right)^{\frac{1}{p}} \leq |\det(\mathbb{J}_K)|^{1-\frac{2}{p}} \|\mathbb{J}_K^{-1}\|_{\ell^2}$$
$$\times \|\mathbb{J}_K\|^{-s+\frac{d}{p}}_{\ell^2} \left( \int_K \int_K \frac{\|\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{v}(\boldsymbol{y})\|^p_{\ell^2}}{\|\boldsymbol{x} - \boldsymbol{y}\|^{sp+d}_{\ell^2}} \,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{p}},$$

since $\|\widehat{\boldsymbol{x}} - \widehat{\boldsymbol{y}}\|_{\ell^2} = \|\mathbb{J}_K^{-1}(\boldsymbol{x} - \boldsymbol{y})\|_{\ell^2} \geq \|\mathbb{J}_K\|^{-1}_{\ell^2} \|\boldsymbol{x} - \boldsymbol{y}\|_{\ell^2}$.

(ii) TO DO