Fun with Zeta Functions of Graphs

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Thanks to the AWM !!!!!!!!!!!!!!!!!

from
Wikipedia

Amalie Emmy
Noether

Abstract

Introduction to zeta functions of graphs + history & comparisons with other zetas from number theory & geometry – e.g., Riemann's and Selberg's.

3 kinds of graph zetas will be defined: vertex, edge and path.

Basic properties

- the Ihara formula saying that the zeta function is the reciprocal of a polynomial.
- analogs of the Riemann hypothesis, zero (pole) spacings,
- connections with expander graphs and quantum chaos.
- graph theory prime number theorem

Graphs will be assumed to be finite undirected possibly irregular, usually connected.

References include my joint papers with Harold Stark in Advances in Math. See newbook.pdf on my website.

Introduction

The Riemann zeta function for Re(s)>1

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=prime} \left(1 - p^{-s}\right)^{-1}.$$

- Riemann extended to all complex s with pole at s=1.
- Functional equation relates value at s and 1-s
- Riemann hypothesis
- Duality between primes and complex zeros of zeta
- See Davenport, *Multiplicative Number Theory*.

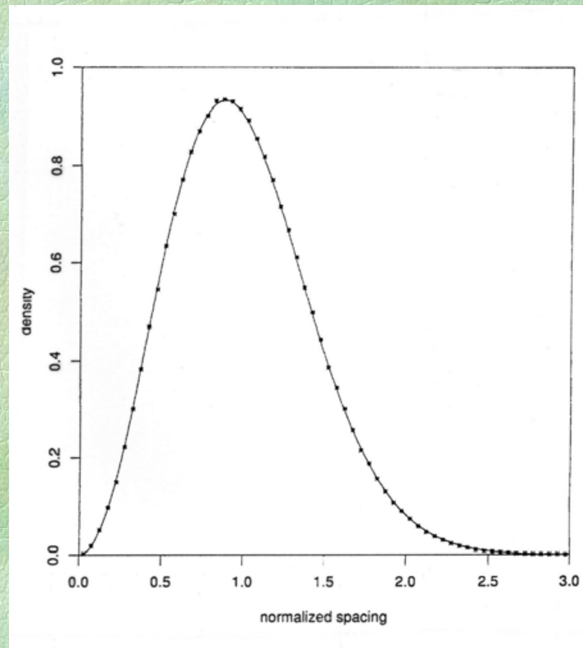
Graph of |Zeta|

Graph of z=|ζ(x+iy)| showing the pole at x+iy=1 and the first 6 zeros on the line x=1/2.

Picture made by D. Asimov and S. Wagon for their article on the evidence for the Riemann hypothesis as of 1986.

Odlyzko's Comparison of Spacings of Imaginary Parts of Zeros of Zeta and Eigenvalues of Random Hermitian Matrix.

See
B. Cipra, *What's Happening in the Mathematical Sciences, 1998-1999, A.M.S., 1999.*



Many Kinds of Zeta

Dedekind zeta of an algebraic number field F, where primes become prime ideals p and infinite product of terms $(1-Np^{-s})^{-1}$, where Np = norm of p = #(O/p), O=ring of integers in F

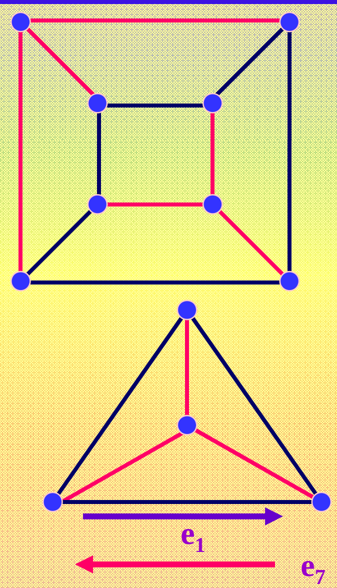
Selberg zeta associated to a compact Riemannian manifold M=Γ\H; H= upper half plane with $ds^2=(dx^2+dy^2)y^{-2}$; Γ=discrete subgroup of group of real Möbius transformations; primes = primitive closed geodesics C in M of length n(C); (primitive means only go around once)

$$Z(s)=\prod_{[C]}\prod_{j\geq 0}\left(1-e^{-(s+j)\nu(C)}\right)$$

Duality between spectrum Δ on M & lengths closed geodesics in M
Z(s+1)/Z(s) is more like Riemann zeta

References:
Lang, *Algebraic Number Theory*
my book, *Harmonic Analysis on Symmetric Spaces & Applications, I*

Labeling Edges of Graphs



X = finite connected (not-necessarily regular graph)
Orient the m edges.
Label them as follows.
Here the inverse edge has opposite orientation.

$e_1, e_2, \dots, e_m,$

$e_{m+1}=(e_1)^{-1}, \dots, e_{2m}=(e_m)^{-1}$

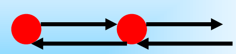
We will use this labeling in the next section on edge zetas

Primes in Graphs

are equivalence classes [C] of closed backtrackless tailless primitive paths C

DEFINITIONS

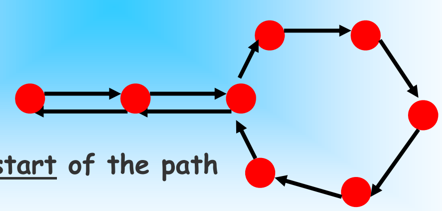
backtrack



equivalence class: change starting point

tail

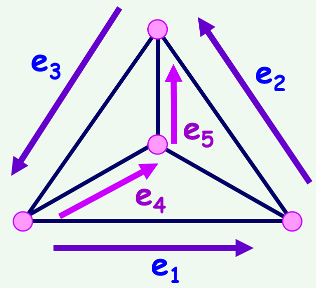
α



Here α is the start of the path

non-primitive: go around path more than once

EXAMPLES of Primes in a Graph



$[C] = [e_1 e_2 e_3]$

$[D] = [e_4 e_5 e_3]$

$[E] = [e_1 e_2 e_3 e_4 e_5 e_3]$

$\nu(C)=3,\ \nu(D)=4,\ \nu(E)=6$

E=CD
another prime $[C^n D]$, n=2,3,4, …
infinitely many primes

Ihara Zeta Function

Connected, no degree 1 vertices, possibly irregular graphs

$$\zeta_V(u,X) = \prod_{\substack{[C] \\ prime}} \left(1 - u^{\nu(C)}\right)^{-1}$$

|u| small enough

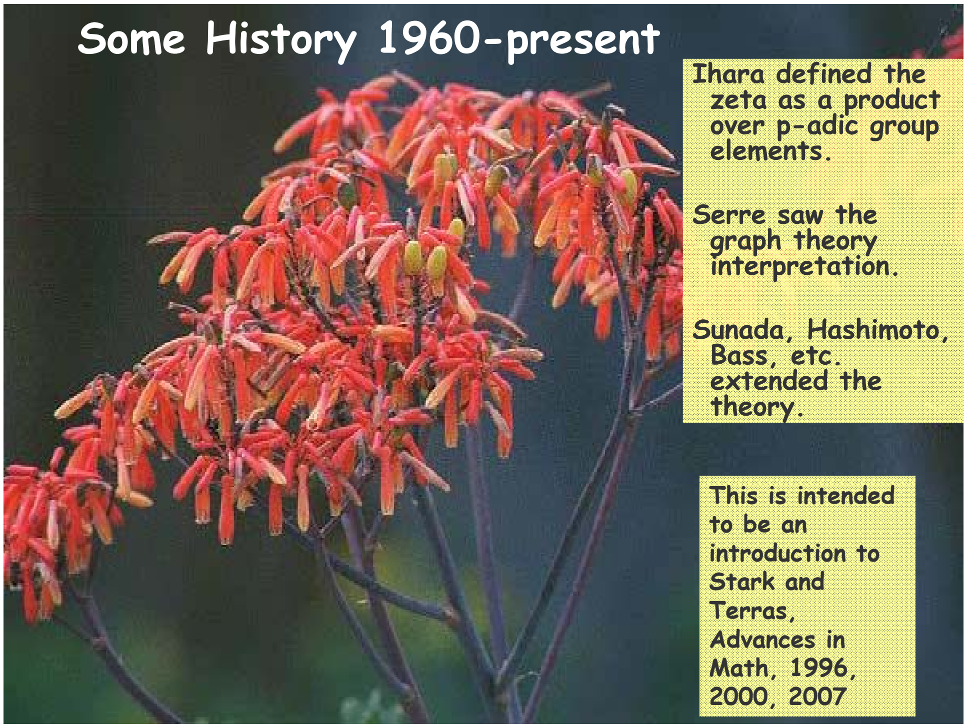
Ihara's Theorem (Bass, Hashimoto, etc.)

A = adjacency matrix of X
Q = diagonal matrix; jth diagonal entry
= degree jth vertex -1;
r = rank fundamental group = |E|-|V|+1

$$\zeta_V(u,X)^{-1} = (1-u^2)^{r-1} \det(I - Au + Qu^2)$$

Here V is for vertex

Some History 1960-present



Ihara defined the zeta as a product over p-adic group elements.

Serre saw the graph theory interpretation.

Sunada, Hashimoto, Bass, etc. extended the theory.

This is intended to be an introduction to Stark and Terras, Advances in Math, 1996, 2000, 2007

Remarks for q+1-Regular Unweighted Graphs Mostly

- Riemann Hypothesis, (non-trivial poles on circle of radius $q^{-1/2}$ center 0), means graph is <u>Ramanujan</u> i.e., non-trivial spectrum of adjacency matrix is contained in the interval $(-2\sqrt{q}, 2\sqrt{q})$ = spectrum for the universal covering tree

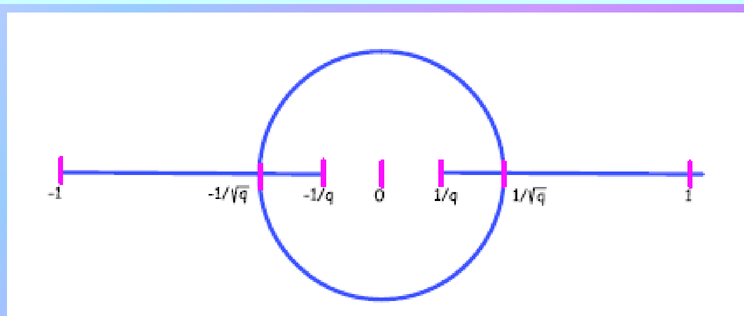
  for the proof, use the Ihara theorem

  [see Lubotzky, Phillips & Sarnak, *Combinatorica*, 8 (1988)].

  Ramanujan graph ⇒ good expander, efficient communication network

- For (q+1)-regular graphs, Ihara zeta has functional equations relating value at u and 1/(qu), q=degree - 1. Set $u=q^{-s}$ to get s goes to 1-s.

Poles of Zeta for q+1 Regular Graph



Possible Locations of Poles of zeta for a regular graph:
1/q is always the closest to the origin in absolute value

Circle of radius $1/\sqrt{q}$ corresponds to the part of the spectrum of the adjacency matrix satisfying the Ramanujan inequality

Real poles correspond to the non-Ramanujan eigenvalues of A; except the two poles on the circle itself.

Alon conjecture for regular graphs says RH is true for "most" regular graphs but can be false. See Steven J. Miller, Tim Novikoff & Anthony Sabelli for definition of "most." (> 51%) See Joel Friedman's website (www.math.ubc.ca/~jf) for a paper proving that a random regular graph is almost Ramanujan.

RH for Irregular Graphs

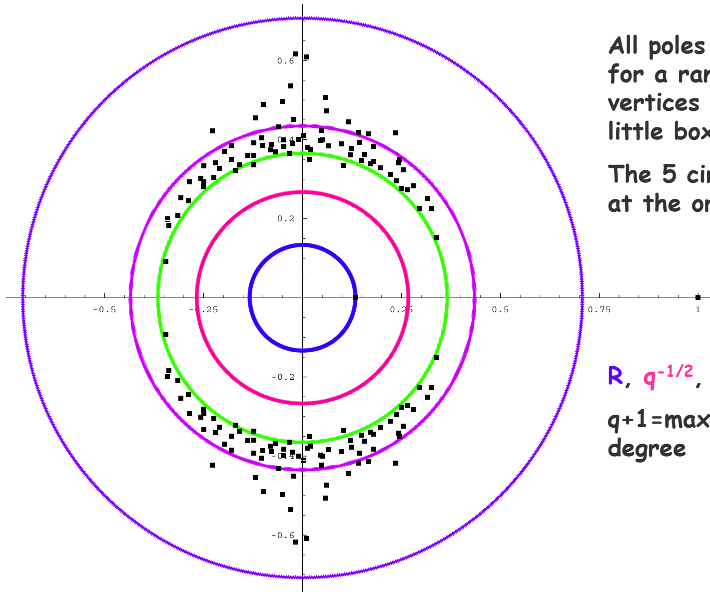
For irregular graphs, replace 1/q by

R=closest pole of Ihara zeta to 0. (necessarily R>0)

The RH: zeta is pole free when $R<|u|<\sqrt{R}$.

Research Problems:

1) Connect this with spectrum of universal covering tree.
   See preprint of Friedman, Hoory, and Angel.
2) Connect with expansion properties of the graph.



All poles except -1 of $\zeta_X(u)$ for a random graph with 80 vertices are denoted by little boxes.

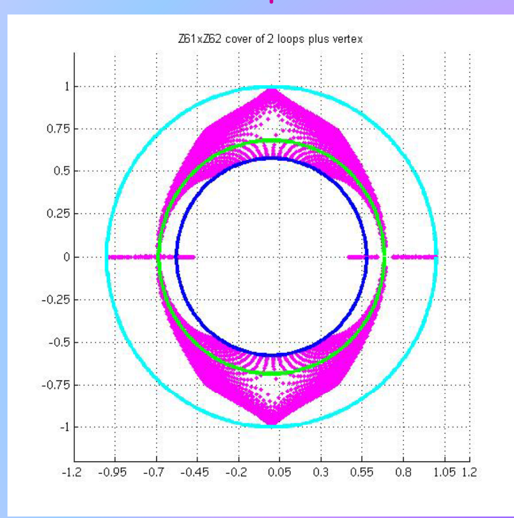
The 5 circles are centered at the origin and have radii

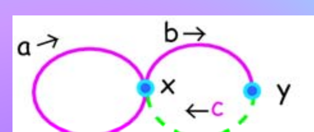
R, $q^{-1/2}$, $R^{1/2}$, $(pq)^{-1/4}$, $p^{-1/4}$

$q+1$=max degree, $p+1$=min degree

Experiments on Locations of Zeros of Ihara Zeta of Irregular Graphs - joint work with Matthew Horton

Poles of Ihara Zeta for a $\mathbb{Z}_{61}\times\mathbb{Z}_{62}$-Cover of 2 Loops + Extra Vertex are pink dots

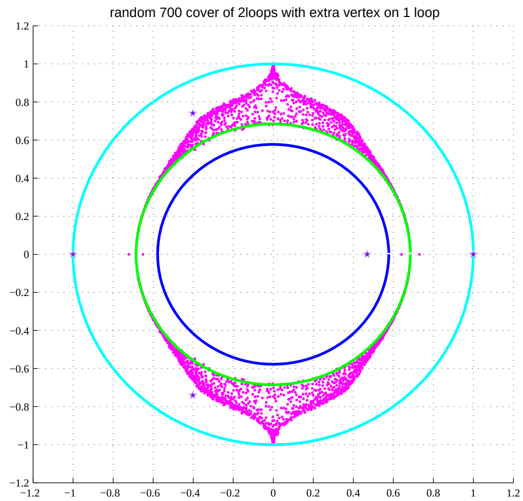


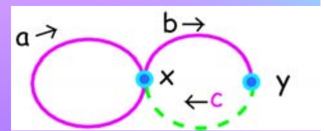


joint work with H. Stark and M. Horton

Circles Centers (0,0); Radii: $3^{-1/2}$, $R^{1/2}$, 1; $R \cong .47$

RH False





joint work with H. Stark and M. Horton

Z is random 700 cover of 2 loops plus vertex graph in picture.

The pink dots are at poles of ζ_Z. Circles have radii $q^{-1/2}$, $R^{1/2}$, $p^{-1/2}$, with $q=3$, $p=1$, $R \cong .4694$. RH approximately True.

Prime Number Theorem for irregular unweighted graphs

Assume graph connected, no degree 1 vertices, not a cycle

$\pi_X(m)$ = number of primes [C] in X of length m

Δ = g.c.d. of lengths of primes in X

R = radius of largest circle of convergence of $\zeta(u,X)$

If Δ divides m, then

R=1/q, if graph is q+1-regular

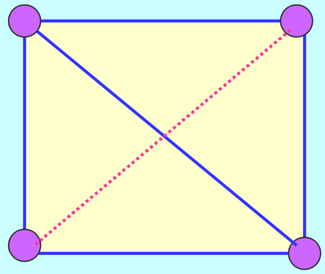
$$\pi_X(m) \sim \Delta R^{-m}/m, \quad \text{as } m \to \infty.$$

A proof comes from exact formula for $\pi_X(m)$ by analogous method to that of Rosen, *Number Theory in Function Fields*, page 56.

N_m=# closed paths of length m with no backtrack, no tails

$$u\frac{d\log\zeta(u,X)}{du} = \sum_{m=1}^{\infty} N_m u^m$$

2 Examples
K_4 and
$X=K_4$-edge



$$\zeta_V\left(u,K_4\right)^{-1} =$$
$$(1-u^2)^2(1-u)(1-2u)(1+u+2u^2)^3$$

$$\zeta_V\left(u,X\right)^{-1} =$$
$$(1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3)$$

N_m for the examples

x d/dx log ζ (x,K_4)

$=24x^3+24x^4+96x^6+168x^7+168x^8+528x^9+O(x^{10})$

$\pi(3)=8$ (orientation counts) $\pi(4)=6$ $\pi(5)=0$

$$N_6=\sum_{d|6} d\pi(d)=\pi(1)+2\pi(2)+3\pi(3)+6\pi(6)$$
$$\pi(6)=24$$

x d/dx log ζ $(x,K_4\text{-}e)$

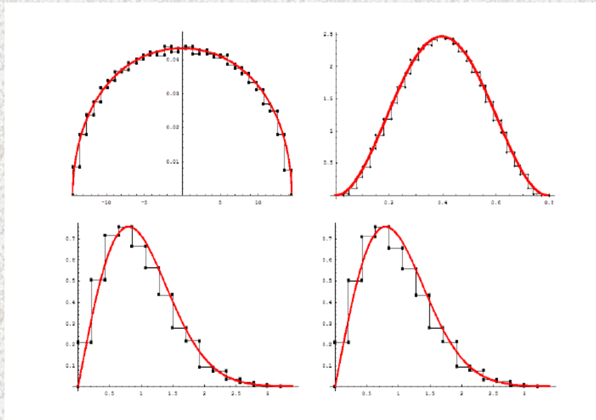
$= 12x^3+8x^4+24x^6+28x^7+8x^8+48x^9+O(x^{10})$

$\pi(3)=4$ $\pi(4)=2$ $\pi(5)=0$ $\pi(6)=2$

Derek Newland's Experiments

Compare with Odlyzko experiments for Riemann zeta

Mathematica experiment with random 53-regular graph - 2000 vertices



Spectrum adjacency matrix $\zeta(52^{-s})$ as a function of s

Top row = distributions for eigenvalues of A on left and imaginary parts of the zeta poles on right $s=\frac{1}{2}+it$.

Bottom row contains their respective normalized level spacings.

Red line on bottom: Wigner surmise GOE, $y = (\pi x/2)\exp(-\pi x^2/4)$.

Compare Katz & Sarnak work on Zeta Functions of Curves over $\mathbb{F}_q$ – almost all GUE as q and genus $\to\infty$. But no examples exist.

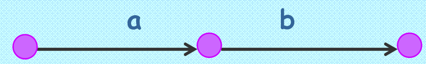
Edge Zetas

Orient the edges of the graph. Recall the labeling!

Define **Edge matrix** W to have a,b entry w_{ab} in $\mathbb{C}$ & set

$w(a,b)=w_{ab}$

if the edges a and b look like those below and $a \neq b^{-1}$



Otherwise set $w_{ab} = 0$

W is 2|E| x 2|E| matrix

If $C = a_1 a_2 \ldots a_s$ where a_j is an edge, define **edge norm** to be

$$N_E(C) = w(a_s, a_1) w(a_1, a_2) w(a_2, a_3) \cdots w(a_{s-1}, a_s)$$

Edge Zeta

$$\zeta_E(W, X) = \prod_{\substack{[C] \\ prime}} \left(1 - N_E(C)\right)^{-1}$$

$|w_{ab}|$ small

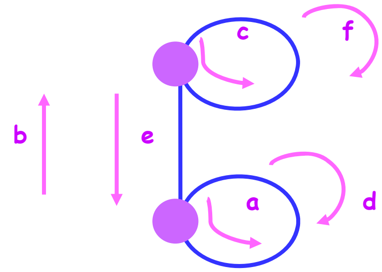
Properties of Edge Zeta

- Set all non-0 variables, $w_{ab}=u$ in the edge zeta & get Ihara zeta.
- Cut (remove) an edge, compute the new edge zeta by setting all variables equal to 0 if the cut edge or its inverse appear in subscripts.
- Edge zeta is the reciprocal of a polynomial given by a much simpler determinant formula than the Ihara zeta

$$\zeta_E(W, X) = \det(I - W)^{-1}$$

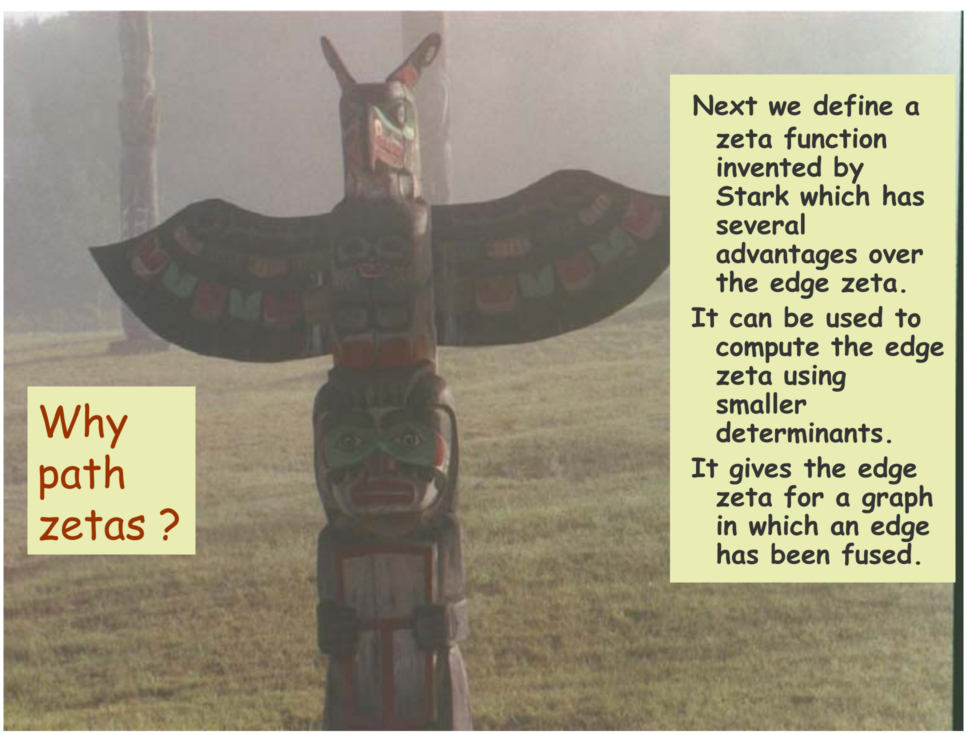
- Better yet, proof is simpler (compare Bowen & Lanford proof for dynamical zetas). Bass deduces Ihara formula from this. See 2nd joint paper with Stark in Advances for the linear algebra version of the proof or my book (newbook.pdf on my website)
- edge zeta application: R. Koetter, W.-C. W. Li, P. O. Vontobel, J. L. Walker, Pseudo-codewords of cycle codes via zeta functions, preprint, 2005

Example.
Dumbbell Graph



$$\zeta_E(W,D)^{-1} = \det\begin{pmatrix} w_{aa}-1 & w_{ab} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{bc} & 0 & 0 & w_{bf} \\ 0 & 0 & w_{cc}-1 & 0 & w_{ce} & 0 \\ 0 & w_{db} & 0 & w_{dd}-1 & 0 & 0 \\ w_{ea} & 0 & 0 & w_{ed} & -1 & 0 \\ 0 & 0 & 0 & 0 & w_{fe} & w_{ff}-1 \end{pmatrix}$$

Here b & e are vertical edges.
Specialize all variables with b & e to be 0
get zeta fn of subgraph with vertical edge removed   Fission.



Path Zeta Function

Fundamental Group of X can be identified with group generated by edges left out of a spanning tree

$$e_1, \ldots e_r, e_1^{-1}, \ldots, e_r^{-1}$$

Define 2r ×2r **path matrix** Z - ij entry

z_{ij} in ℂ if $e_j \neq e_i^{-1}$ and z_{ij} =0, otherwise.

Imitate definition of edge zeta function. Define for prime path

$C = a_1 \cdots a_s,\ where\ a_j \in \{e_1^{\pm 1}, \ldots, e_r^{\pm 1}\}$ **path norm**

$$N_P(C) = z(a_s, a_1) \prod_{i=1}^{s-1} z(a_i, a_{i+1})$$

Define **path zeta**

$$\zeta_P(Z, X) = \prod_{\substack{[C] \\ prime}} \left(1 - N_P(C)\right)^{-1}$$

Specialize Path Zeta to Edge Zeta

edges left out of a spanning tree T of X are $e_1, \ldots e_r$
inverse edges are $e_{r+1} = e_1^{-1}, \ldots, e_{2r} = e_r^{-1}$

edges of the spanning tree T are $t_1, \ldots, t_{|X|-1}$
with inverse edges $t_{|X|}, \ldots t_{2|X|-2}$

A prime cycle C is first written as a product of the generators of the fundamental group e_j and then a product of actual edges e_j and t_k.

Do this by inserting $t_{k_1} \cdots t_{k_n}$ which is unique path on T joining end vertex of e_i & start vertex of e_j if e_i and e_j are adjacent in C.

Now **specialize the path matrix Z to Z(W)** with entries

$$z_{ij} = w(e_i, t_{k_1}) w(t_{k_n}, e_j) \prod_{\nu=1}^{n-1} w(t_{k_\nu}, t_{k_{\nu+1}})$$

Then

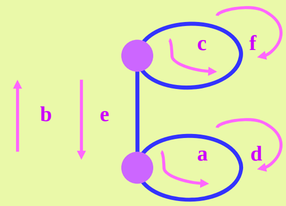
$$\zeta_P(Z(W), X) = \zeta_E(W, X)$$

Example – Again the Dumbbell

Recall that the edge zeta involved a 6x6 determinant.

The path zeta is only 4x4.

Maple computes it much faster than the 6x6.



e.g., specialize z_{ac} to $w_{ab}w_{bc}$

Fusion:

shrink edge b to a point.

edge zeta of new graph obtained by setting

$w_{xb}w_{by}=w_{xy}$

in specialized path zeta &

same for e instead of b.

$$\zeta_E(W,X)^{-1} = \det\begin{pmatrix} w_{aa}-1 & w_{ab}w_{bc} & 0 & w_{ab}w_{bf} \\ w_{ce}w_{ea} & w_{cc}-1 & w_{ce}w_{ed} & 0 \\ 0 & w_{db}w_{bc} & w_{dd}-1 & w_{db}w_{bf} \\ w_{fe}w_{ea} & 0 & w_{fe}w_{ed} & w_{ff}-1 \end{pmatrix}$$

Homework Problems

$$\left.\frac{d^r y}{dx^r}\left(\zeta^{(r)}(u)\right)^{-1}\right|_{u=1} = r!(-1)^{r+1}2^r\kappa_X$$

κ_X is the complexity = # spanning trees of graph X

Is there a graph analog of regulator, Stark Conjectures, class field theory for abelian graph coverings? Or more simply a quadratic reciprocity law, fundamental units? The ideal class group is the Jacobian of a graph and has order = number of spanning trees (paper of Roland Bacher, Pierre de la Harpe and Tatiana Nagnibeda). See also Riemann-Roch & Abel-Jacobi theory on a finite graph, by Matthew Baker & S. Norine. There is an analog of Brauer-Siegel theory (see H.S. and A.T., Part III, Advances in Math., 2007). Lorenzini has another kind of graph zeta.

- Paper with Matthew Horton & Harold Stark in Snowbird Proceedings

  Quantum Graphs and Their Applications, *Contemporary Mathematics*, v. 415, AMS, Providence, RI 2006.

- See my draft of a book:

  www.math.ucsd.edu/~aterras/newbook.pdf

- Draft of new paper: also on my website

  www.math.ucsd.edu/~aterras/cambridge.pdf

- There was a graph zetas special session of this AMS meeting – many interesting papers referred to there.

- For work on directed graphs, see Matthew Horton, Ihara zeta functions of digraphs, *Linear Algebra and its Applications,* 425 (2007) 130-142.

The End