
Fun with Zeta of Graphs
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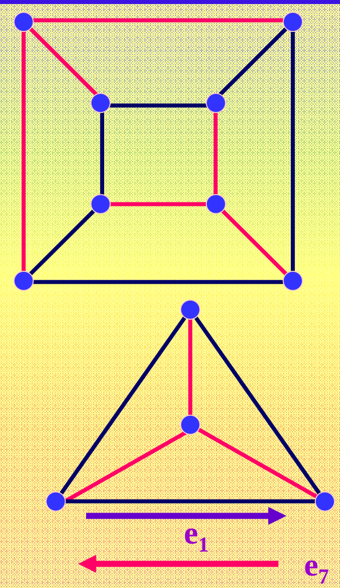
Thank You !
Joint work with H. Stark, M. Horton, etc.

Labeling Edges of Graphs



X = finite connected (not-necessarily regular graph)
Orient the m edges.
Label them as follows.
Here the inverse edge has opposite orientation.

$e_1, e_2, \ldots, e_m,$

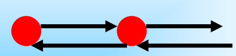
$e_{m+1}=(e_1)^{-1}, \ldots, e_{2m}=(e_m)^{-1}$

We will use this labeling in the next section on edge zetas

Primes in Graphs

are equivalence classes [C] of closed backtrackless tailless primitive paths C

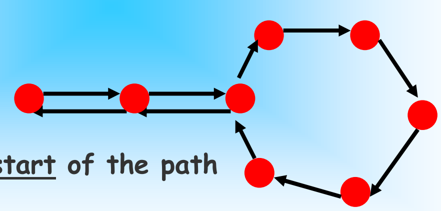
DEFINITIONS backtrack



equivalence class: change starting point

tail

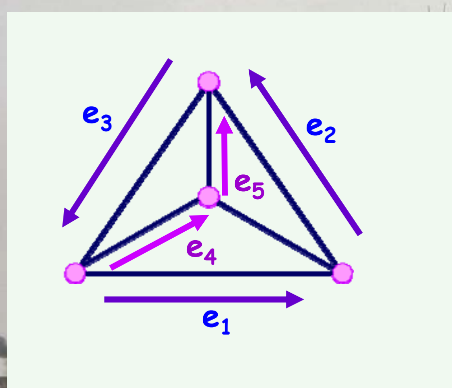
α



Here α is the start of the path

non-primitive: go around path more than once

EXAMPLES of Primes in a Graph



$[C] = [e_1 e_2 e_3]$

$[D] = [e_4 e_5 e_3]$

$[E] = [e_1 e_2 e_3 e_4 e_5 e_3]$

$\nu(C)=3$, $\nu(D)=4$, $\nu(E)=6$

E=CD
another prime $[C^n D]$, n=2,3,4, …
infinitely many primes

Ihara Zeta Function – Unweighted Possibly Irregular Graphs

$$\zeta_V(u,X) = \prod_{\substack{[C] \\ prime}} \left(1 - u^{\nu(C)}\right)^{-1}$$

|u| small enough

Ihara's Theorem (Bass, Hashimoto, etc.)

A = adjacency matrix of X
Q = diagonal matrix; jth diagonal entry
= degree jth vertex -1;
r = rank fundamental group = |E|-|V|+1

$$\zeta_V(u,X)^{-1} = (1-u^2)^{r-1} \det(I - Au + Qu^2)$$

Here V is for vertex

What happens for weighted graphs?

If each oriented edge e has weight $\nu(e)$, define length of path $C = e_1 \cdots e_s$ as

$$\nu(C) = \nu(e_1) + \cdots + \nu(e_s).$$

Just plug this ν into the definition of zeta.

Call it $\zeta(u,X,\nu)$

Question: For which weights do we get an Ihara formula?

Remarks for q+1-Regular Unweighted Graphs Mostly

- Riemann Hypothesis, (non-trivial poles on circle of radius $q^{-1/2}$ center 0), means graph is Ramanujan i.e., non-trivial spectrum of adjacency matrix is contained in the interval $(-2\sqrt{q}, 2\sqrt{q})$ = spectrum for the universal covering tree [see Lubotzky, Phillips & Sarnak, *Combinatorica*, 8 (1988)].

- Ihara zeta has functional equations relating value at u and 1/(qu), q=degree - 1
  Set $u=q^{-s}$ to get s goes to 1-s.

Alon conjecture says RH is true for “most” regular graphs but can be false. See Joel Friedman's website (www.math.ubc.ca/~jf) for a paper proving that a random regular graph is almost Ramanujan.

The Prime Number Theorem (irregular unweighted graphs)

$\pi_X(m)$ = number of primes [C] in X of length m

Δ = g.c.d. of lengths of primes in X

R = radius of largest circle of convergence of $\zeta(u,X)$

If Δ divides m, then

R=1/q, if graph is q+1-regular

$$\pi_X(m) \sim \Delta R^{-m}/m, \text{ as } m \to \infty.$$

The proof comes from exact formula for $\pi_X(m)$ by analogous method to that of Rosen, *Number Theory in Function Fields*, page 56.

N_m=# closed paths of length m with no backtrack, no tails

$$u\frac{d\log\zeta(u,X)}{du} = \sum_{m=1}^{\infty} N_m u^m$$

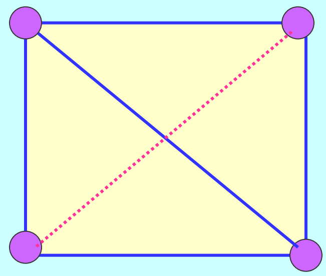
What about PNT for graph X with positive integer weights ν?

You can inflate edge e by adding $\nu(e)$-1 vertices. New graph X_ν has determinant formulas and PNT similar to previous.

Some things do change:

e.g. size of adjacency matrix, exact formula.

2 Examples
K_4 and
$X=K_4$-edge



$$\zeta_V(u,K_4)^{-1} = (1-u^2)^2(1-u)(1-2u)(1+u+2u^2)^3$$

For weighted graphs with non-integer wts, 1/zeta not a polynomial

$$\zeta_V(u,X)^{-1} = (1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3)$$

N_m for the examples

$x\ d/dx\ \log \zeta(x,K_4)$

$= 24x^3+24x^4+96x^6+168x^7+168x^8+528x^9+O(x^{10})$

$\pi(3)=8$ (orientation counts) $\pi(4)=6$ $\pi(5)=0$

$$N_6 = \sum_{d|6} d\pi(d) = \pi(1)+2\pi(2)+3\pi(3)+6\pi(6)$$

$$\pi(6)=24$$

$x\ d/dx\ \log \zeta(x,K_4\text{-}e)$

$= 12x^3+8x^4+24x^6+28x^7+8x^8+48x^9+O(x^{10})$

$\pi(3)=4$ $\pi(4)=2$ $\pi(5)=0$ $\pi(6)=2$

Poles of Zeta for K_4 are

$$\{1,1,1,-1,-1,\tfrac{1}{2},r_+,r_+,r_+,r_-,r_-,r_-\}$$

where $r_\pm=(-1\pm\sqrt{-7})/4$ and $|r|=1/\sqrt{2}$

$\frac{1}{2}$=Pole closest to 0 - governs prime number thm

Poles of zeta for K_4-e are

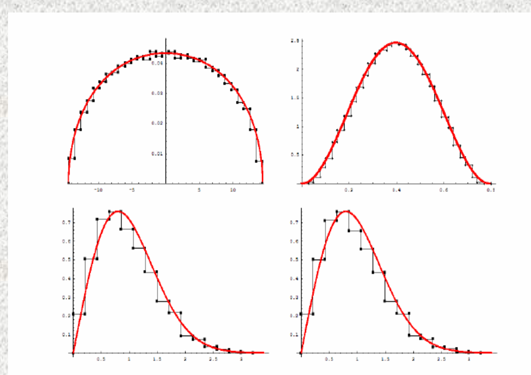
$$\{1,1,-1,i,-i,r_+,r_-,\alpha,\beta,\bar{\beta}\}$$

$R = \alpha$ real root of cubic $\cong$ .6573

β complex root of cubic

Derek Newland's Experiments

Mathematica experiment with random 53-regular graph - 2000 vertices



Spectrum adjacency matrix $\qquad$ $\zeta(52^{-s})$ as a function of s

Top row = distributions for eigenvalues of A on left and Imaginary parts of the zeta poles on right $s=\frac{1}{2}+it$.

Bottom row contains their respective normalized level spacings.

Red line on bottom: Wigner surmise, $y = (\pi x/2)\exp(-\pi x^2/4)$.

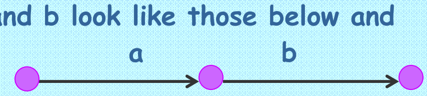
What are Edge Zetas?

Edge Zetas

Orient the edges of the graph. Recall the labeling!

Define **Edge matrix** W to have a,b entry w_{ab} in **C** & set $w(a,b)=w_{ab}$

if the edges a and b look like those below and $a \neq b^{-1}$



Otherwise set $w_{ab} = 0$

W is 2|E| x 2|E| matrix

If $C = a_1 a_2 \ldots a_s$ where a_j is an edge, define **edge norm** to be

$$N_E(C) = w(a_s,a_1)w(a_1,a_2)w(a_2,a_3)\cdots w(a_{s-1},a_s)$$

Edge Zeta

$$\zeta_E(W,X) = \prod_{\substack{[C] \\ prime}} \left(1 - N_E(C)\right)^{-1}$$

$|w_{ab}|$ small

Properties of Edge Zeta

- Set all non-0 variables, $w_{ab}=u$ in the edge zeta & get Ihara zeta.
- Cut an edge, compute the new edge zeta by setting all variables equal to 0 if the cut edge or its inverse appear in subscripts.
- Edge zeta is the reciprocal of a polynomial given by a much simpler determinant formula than the Ihara zeta
- Better yet, the proof is simpler (compare Bowen & Lanford proof for dynamical zetas) and Bass deduces Ihara from this

$$\zeta_E(W,X)=\det(I-W)^{-1}$$

Determinant Formula for Zeta of Weighted Graph

Given weights $\nu(e)$ on edges. For non-0, variables set $w_{ab}=u^{\nu(a)}$ in W matrix & get weighted graph zeta. Call matrix W_ν.

So obtain $\zeta(u,X,\nu)^{-1} = \det(I-W_\nu)$.

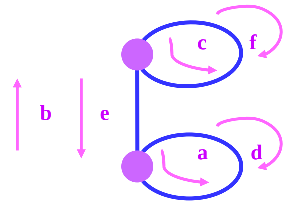
If we make added assumption $\nu(e^{-1}) = 2- \nu(e)$, then Bass proof (as in Snowbird volume paper) gives an Ihara-type formula with a new A.

$$(A_\nu)_{a,b}=\sum_{\substack{e\\ a\to b}} u^{\nu(e)-1}$$

It's old if $\nu=1$.

$$\zeta(u,X,\nu)^{-1}=\left(1-u^2\right)^{r-1}\det\left(1-A_\nu u+Qu^2\right)$$

Example. Dumbbell Graph



$$\zeta_E(W,D)^{-1} = \det\begin{pmatrix} w_{aa}-1 & w_{ab} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{bc} & 0 & 0 & w_{bf} \\ 0 & 0 & w_{cc}-1 & 0 & w_{ce} & 0 \\ 0 & w_{db} & 0 & w_{dd}-1 & 0 & 0 \\ w_{ea} & 0 & 0 & w_{ed} & -1 & 0 \\ 0 & 0 & 0 & 0 & w_{fe} & w_{ff}-1 \end{pmatrix}$$

Here b & e are vertical edges.
Specialize all variables with b & e to be 0
get zeta fn of subgraph with vertical edge removed
Fission.

Graph Galois Theory

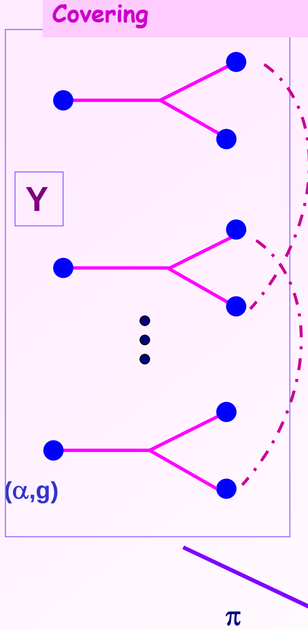
Graph Y an unramified covering of Graph X means
(assuming no loops or multiple edges)

$\pi: Y \to X$ is an onto graph map such that
for every $x \in X$ & for every $y \in \pi^{-1}(x)$,
π maps the points $z \in Y$ adjacent to y
1-1, onto the points $w \in X$ adjacent to x.

Normal d-sheeted Covering means:
$\exists$ d graph isomorphisms $g_1, \ldots, g_d$ mapping $Y \to Y$
such that $\pi\, g_j(y) = \pi(y)$, $\forall\, y \in Y$
Galois group $G(Y/X) = \{ g_1, \ldots, g_d \}$.

Gives generalization of Cayley & Schreier graphs

How to Label the Sheets of a Covering

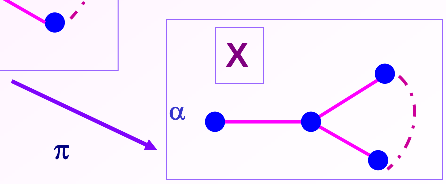


First pick a spanning tree in X (no cycles, connected, includes all vertices of X).

Second make n=|G| copies of the tree T in X. These are the sheets of Y. Label the sheets with $g \in G$. Then

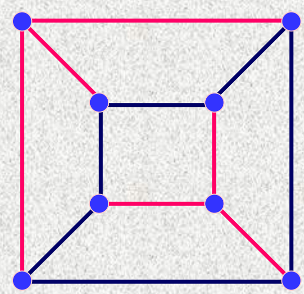
g(sheet h)=sheet(gh)
$g(\alpha,h)=(\alpha,gh)$

g(path from (α,h) to (β,j))
= path from (α,gh) to (β,gj)

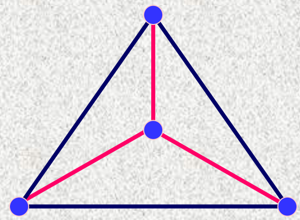


Given G, get examples Y by giving permutation representation of generators of G to lift edges of X left out of T.

Example 1. Quadratic Cover



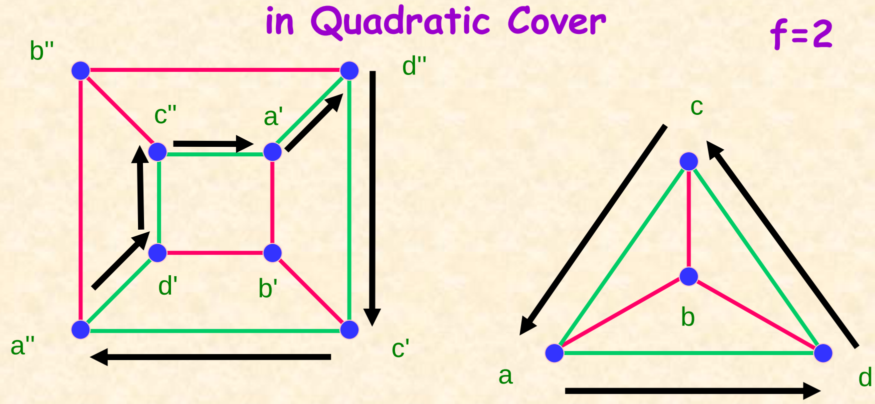
Cube covers Tetrahedron



Spanning Tree in X is red.
Corresponding sheets of Y are also red

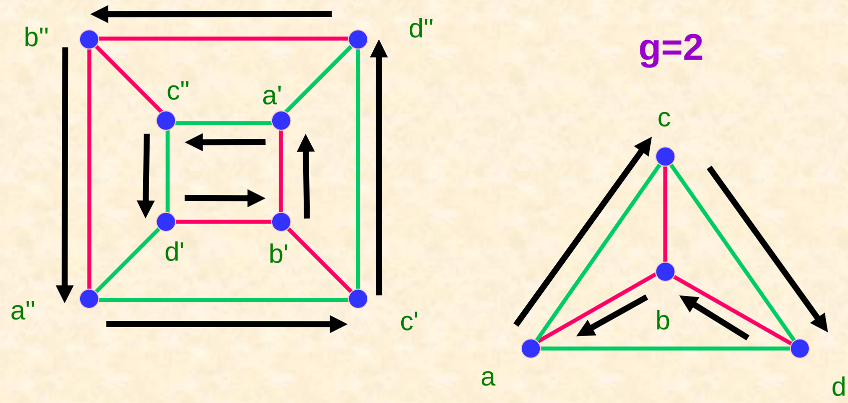
Example of Splitting of Primes in Quadratic Cover

f=2



Picture of Splitting of Prime which is inert;
i.e., f=2, g=1, e=1
1 prime cycle D above, & D is lift of C^2.

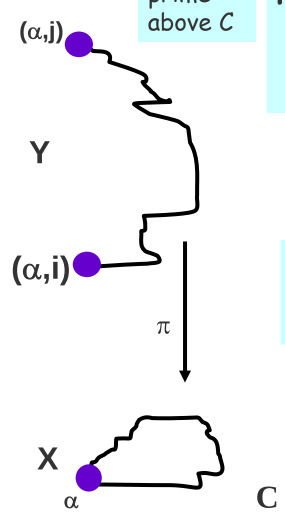
Example of Splitting of Primes in Quadratic Cover



Picture of Splitting of Prime which splits completely; i.e., **f=1, g=2, e=1**
2 primes cycles above

Frobenius Automorphism

D a prime above C



$$\mathrm{Frob}(D) = \left(\frac{Y/X}{D}\right) = ji^{-1} \in G=\mathrm{Gal}(Y/X)$$

where ji^{-1} maps sheet i to sheet j

The unique lift of C in Y starts at (α,i) ends at (α,j)

Exercise: Compute Frob(D) on preceding pages, G={1,g}.

Properties of Frobenius

1) Replace (α,i) with (α,hi). Then Frob(D) = ji^{-1} is replaced with $hji^{-1}h^{-1}$. Or replace D with different prime above C and see that
   Conjugacy class of Frob(D) $\in$ Gal(Y/X) unchanged.
2) Varying α=start of C does not change Frob(D).
3) $\text{Frob}(D)^j = \text{Frob}(D^j)$.

Artin L-Function

ρ = representation of G=Gal(Y/X), u complex, |u| small

$$L(u,\rho,Y/X)=\prod_{[C]}\det\left(1-\rho\left(\frac{Y/X}{D}\right)u^{\nu(C)}\right)^{-1}$$

[C]=primes of X
$\nu(C)$=length C, D a prime in Y over C

Properties of Artin L-Functions

1) L(u,1,Y/X) = $\zeta(u,X)$ = **Ihara zeta function** of X (our analogue of the **Dedekind zeta** function, also **Selberg zeta**)

2)
$$\zeta(u,Y)=\prod_{\rho}L(u,\rho,Y/X)^{d_\rho}$$

product over all irreducible reps of G, d_ρ=degree ρ

Edge Artin L-Function

Defined as before with edge norm and representation ρ

$$L_E(W,\rho,Y/X) = \prod_{[C]} \det(I-\rho([Y/X,D])\, N_E(C))^{-1}$$

Let m=|E|. Define W_ρ to be a 2dm x 2dm matrix with e,f block given by $w_{ef}\ \rho(\sigma(e))$. Then

$$L_E(W,\rho,Y/X) = \det(I-W\rho)^{-1}.$$

Ihara Theorem for L-Functions

$$L(u,\chi_\rho,Y/X)^{-1}$$

$$=(1-u^2)^{(r-1)d_\rho}\det(I'-A'_\rho u+Q'u^2)$$

r=rank fundamental group of X = |E|-|V|+1

ρ= representation of G = Gal(Y/X), d = d_ρ = degree ρ

Definitions. nd×nd matrices A', Q', I', n=|X|

nxn matrix A(g), g ∈ Gal(Y/X), has entry for α,β∈X given by

$(A(g))_{\alpha,\beta}$ = # { edges in Y from (α,e) to (β,g) }, e=identity ∈ G.

$$A'_\rho = \sum_{g\in G} A(g)\otimes\rho(g)$$

Q = diagonal matrix, jth diagonal entry = q_j = (degree of jth vertex in X)-1,

Q' = $Q\otimes I_d$, I' = I_{nd} = identity matrix.

EXAMPLE

Y=cube, X=tetrahedron: G = {e,g}

representations of G are 1 and ρ: ρ(e) = 1, ρ(g) = -1

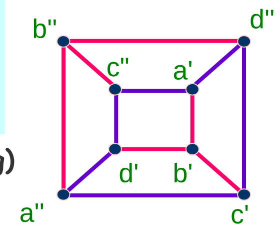
$A(e)_{u,v}$ = #{ length 1 paths u' to v' in Y}

$A(g)_{u,v}$ = #{ length 1 paths u' to v'' in Y}

(u,e)=u'

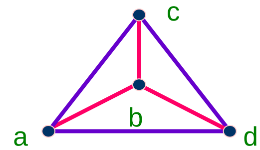
(u,g)=u''

$$A(e)=\begin{pmatrix}0&1&0&0\\1&0&1&1\\0&1&0&0\\0&1&0&0\end{pmatrix} \qquad A(g)=\begin{pmatrix}0&0&1&1\\0&0&0&0\\1&0&0&1\\1&0&1&0\end{pmatrix}$$



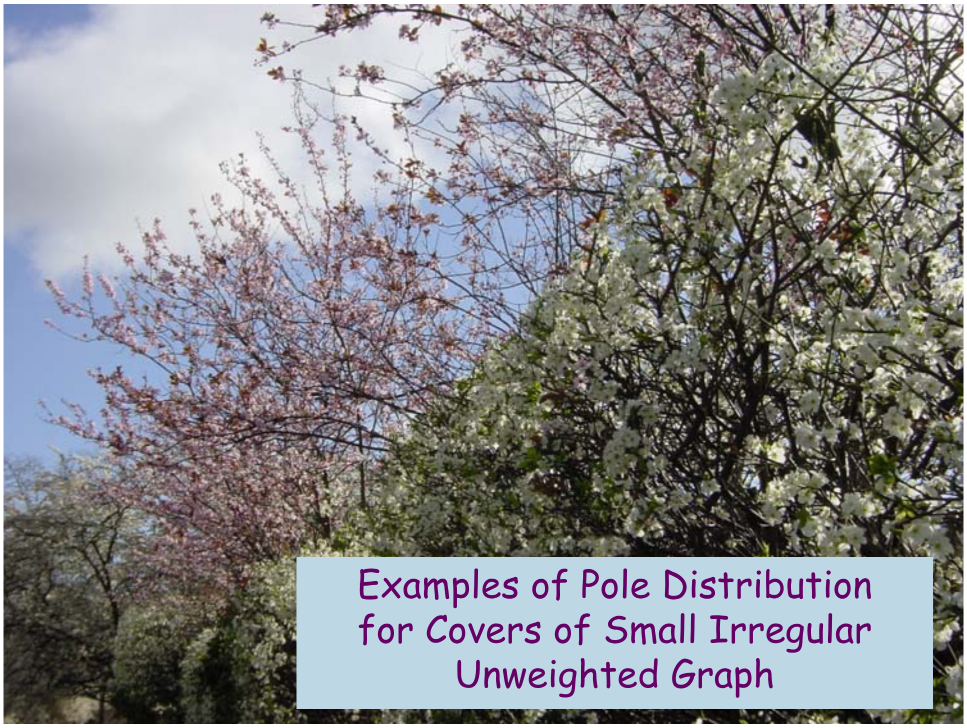
A'_1 = A = adjacency matrix of X = A(e)+A(g)

$$A'_\rho = A(e)-A(g)=\begin{pmatrix}0&1&-1&-1\\1&0&1&1\\-1&1&0&-1\\-1&1&-1&0\end{pmatrix}$$



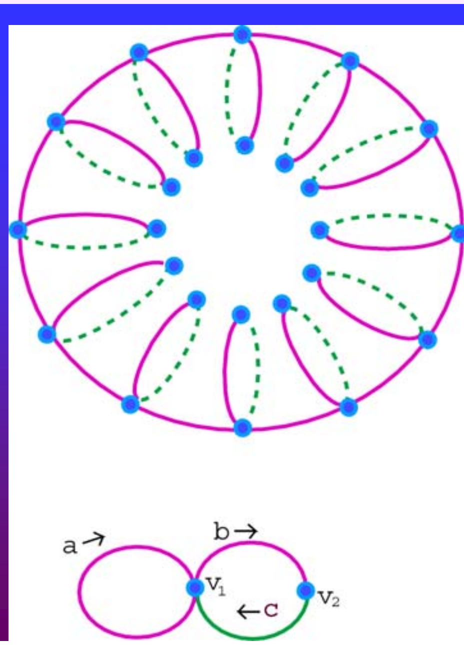
Zeta and L-Functions of Cube & Tetrahedron

- $\zeta(u,Y)^{-1} = L(u,\rho,Y/X)^{-1}\,\zeta(u,X)^{-1}$
- $L(u,\rho,Y/X)^{-1} = (1-u^2)(1+u)(1+2u)(1-u+2u^2)^3$
- $\zeta(u,X)^{-1} = (1-u^2)^2(1-u)(1-2u)(1+u+2u^2)^3$

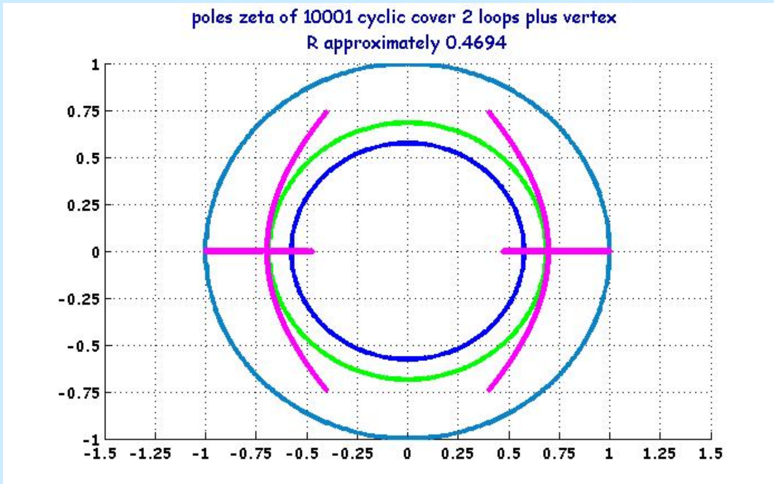


Examples of Pole Distribution for Covers of Small Irregular Unweighted Graph

Cyclic Cover of 2 Loops + Vertex



Poles of Ihara Zeta of Z_{10001} Cover of 2 Loops + Extra Vertex are pink dots

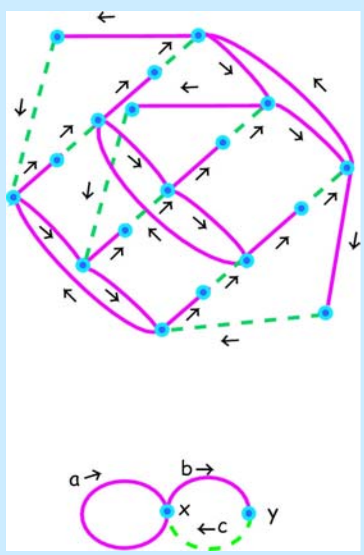


Circles Centers (0,0); Radii: $3^{-1/2}$, $R^{1/2}$, 1; $R \cong .4694$

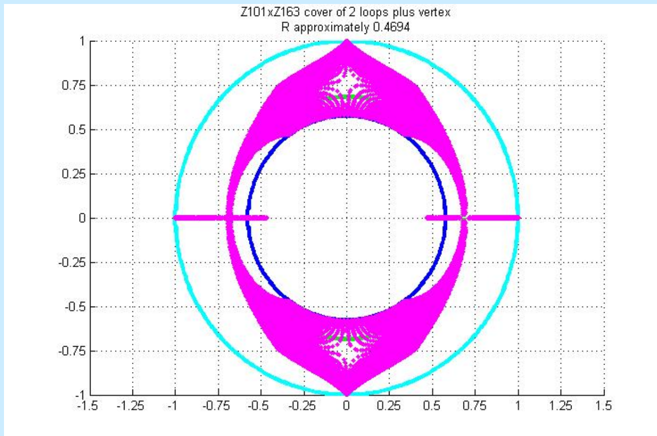
$Z_m \times Z_n$ cover of 2-Loops Plus Vertex

Sheets of Cover indexed by
(x,y) in $Z_m \times Z_n$
The edge L-fns for Characters
$\chi_{r,s}(x,y)=\exp[2\pi i\{(rx/m)+(sy/n)\}]$
Normalized Frobenius (a)=(1,0)
Normalized Frobenius (b)=(0,1)

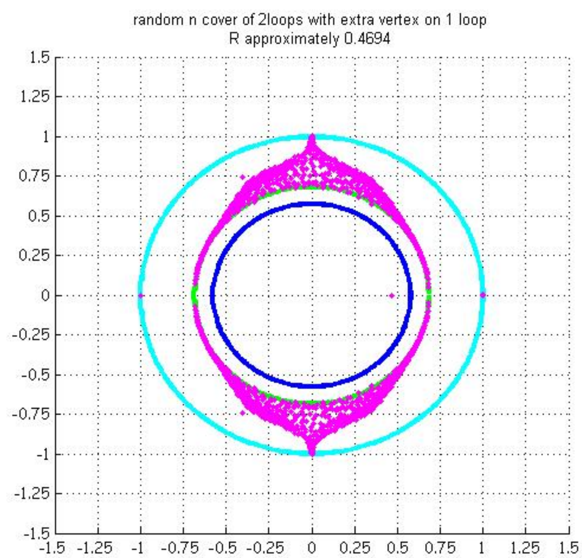
The picture shows m=n=3.

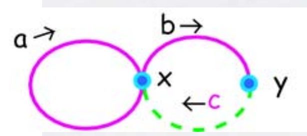


Poles of Ihara Zeta for a Z_{101}xZ_{163}-Cover of 2 Loops + Extra Vertex are pink dots



Circles Centers (0,0); Radii: $3^{-1/2}$, $R^{1/2}$,1; R ≅.47

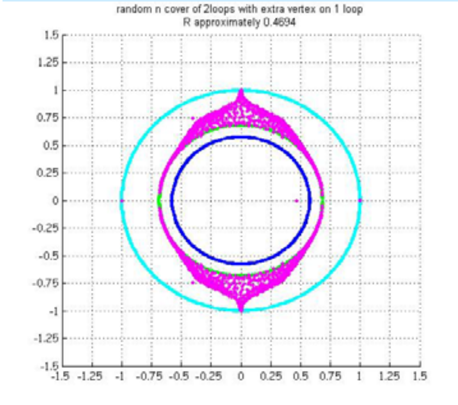


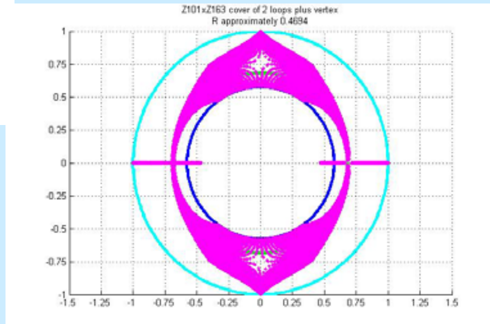


Z is random 407 cover of 2 loops plus vertex graph in picture.

The pink dots are at poles of ζ_Z.

Circles have radii $q^{-1/2}$, $R^{1/2}$, $p^{-1/2}$, with $q=3$, $p=1$, $R \cong .4694$



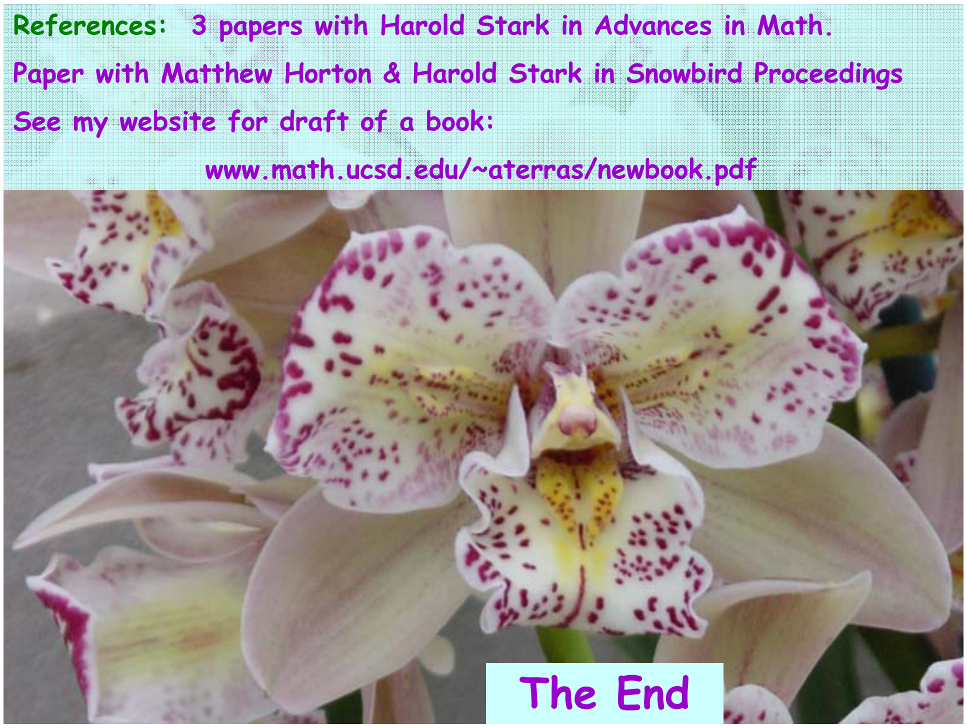


Homework Problems

1) Find the meaning of the Riemann hypothesis for irregular graphs. Are there functional equations? How does it compare with Lubotzky's definition of Ramanujan irregular graph?

2) For regular graphs, can you put define a W-matrix to make the spacings of poles of zetas that look Poisson become GOE?

3) For a large Galois cover of a fixed base graph, can you produce a distribution of poles that looks like that of a random cover?


References: 3 papers with Harold Stark in Advances in Math.
Paper with Matthew Horton & Harold Stark in Snowbird Proceedings
See my website for draft of a book:
www.math.ucsd.edu/~aterras/newbook.pdf
The End