A STROLL THROUGH THE GARDEN OF GRAPH ZETA FUNCTIONS

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Part 1. A Quick Look at Various Zeta Functions

FIGURE 1. **in the garden of zetas**

The goal of this book is to guide the reader in a stroll through the garden of zeta functions of graphs. The subject arose in the late part of the last century modelled after zetas found in the other gardens.

Number theory involves many zetas starting with Riemann's - a necessary ingredient in the study of the distribution of prime numbers. Other zetas of interest to number theorists include the Dedekind zeta function of an algebraic number field and the analog for function fields. Many Riemann hypotheses have been formulated and a few proved. The statistics of the complex zeros of zeta have been connected with the statistics of the eigenvalues of random Hermitian matrices (the GUE distribution of quantum chaos). Artin L-functions are also a kind of zeta associated to a representation of a Galois group of number or function fields. We will find graph analogs of all of these.

Differential geometry has its own zeta - the Selberg zeta function which is used to study the distribution of lengths of prime geodesics in compact or arithmetic Riemann surfaces. There is a third zeta function known as the Ruelle zeta function which is associated to dynamical systems. We will look at these zetas briefly in the introduction. The graph theory zetas are related to these zetas too.

In this part we give a brief glimpse of four sorts of zeta functions to motivate the rest of the book. Much of the first part is not necessary for the rest of the book. Feel free to skip all but Section 2 on the Ihara zeta function. Most of this book arises from joint work with Harold Stark.

Prerequisites for reading this book include linear algebra and group theory. What do groups have to say about graphs which appear to have no symmetry? The answer comes with an understanding of the fundamental group whose elements are closed paths through a vertex. This group is intrinsic to our subject. We will find that the theory Galois developed at such a young age has its applications here. Our zetas are reciprocals of polynomials, sometimes in several variables. We will have determinant formulas for these zetas. And Galois theory will lead to factorizations of the zetas of normal covering graphs, just as it leads to factorizations of Dedekind zeta functions of Galois extensions of number fields.

Thanks are due to the many people who listened to my lectures on this book and helped with the research; e.g., Matthew Horton, Derek Newland, Tom Petrillo, Adriano Garsia, Angela Hicks, Paul Horn, Yeon Kyung Kim at U.C.S.D...... I would also like to thank the people who encouraged me by attending my Ulam Seminar at the University of Colorado, Boulder; especially Lynne Walling, David Grant, Su-ion Ih, Vinod Radhakrishnan, Erika Frugoni, Jonathan Kish, and Mike Daniel.

In addition, I want to thank the Newton Institute in Cambridge, England for its hospitality while I worked on this manuscript which was prepared using Scientific Workplace.

Finally a warning: BEWARE OF TYPOS!!

1. Riemann's zeta function and other zetas from number theory

There are many popular books about the Riemann zeta and many "serious" ones as well. Serious references for this topic include H. Davenport [34], H. Edwards [37], Iwaniec and Kowalski [64], S. J. Miller and R. Takloo-Bighash [86], and S. J. Patterson [97]. I googled "zeta functions" today and got around 181,000 hits. The most extensive website was www.aimath.org.

The theory of zeta functions was developed by many people but Riemann's work in 1859 was certainly the most important. The concept was generalized for the purposes of number theorists by Dedekind, Dirichlet, Hecke, Takagi, Artin and others. Here we will concentrate on the original, namely, Riemann's zeta function. The definition is as follows.

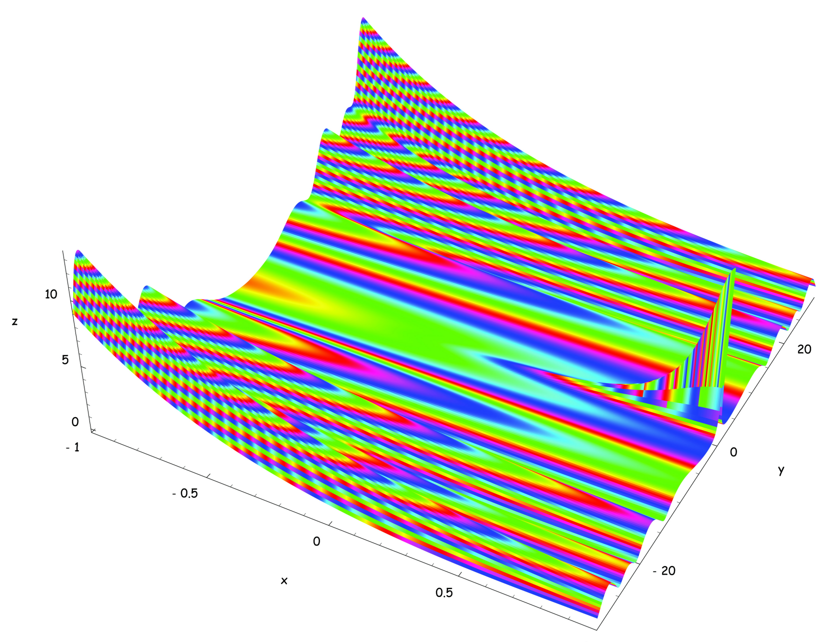


FIGURE 2. **Graph of the modulus of Rieman zeta**; i.e., $z = |\zeta(x+iy)|$ showing the pole at $x + iy = 1$ and the complex zeros nearest the real axis (all of which are on the line $\text{Re}(s) = \frac{1}{2}$, of course).

Riemann's zeta function for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ is defined to be

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=prime} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

The infinite product here is called an **Euler product**. In 1859 Riemann extended the definition of zeta to an analytic function in the whole complex plane except for a simple pole at $s = 1$. He also showed that there is an unexpected symmetry known as the **functional equation** relating the value of zeta at s and the value at $1 - s$. It says

$$\Lambda(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \Lambda(1-s). \tag{1.1}$$

The **Riemann hypothesis** (or **RH**) says that the non-real zeros of $\zeta(s)$ (equivalently those with $0 < \mathrm{Re}(s) < 1$) are on the line $\mathrm{Re}(s) = \frac{1}{2}$. It is equivalent to giving the best possible error term in the prime number theorem in formula (1.2) below. The Riemann hypothesis is now checked to 10^{13}-th zero. (October 12th 2004), by Xavier Gourdon with the help of Patrick Demichel. See Ed Pegg Jr.'s website for an article called the Ten Trillion Zeta Zeros:

$$http://www.maa.org/editorial/mathgames.$$

You win \$1 million if you have a proof of the Riemann hypothesis. See the Clay Math. Institute website:

www.claymath.org.

A. Odlyzko has studied the spacings of the zeros and found that they appear to be the spacings of eigenvalues of a random Hermitian matrix (GUE). See Figure 23 and the paper on his website

www.dtc.umn.edu/~odlyzko/doc/zeta.htm.

If one knows the Hadamard product formula for zeta (from a graduate complex analysis course) as well as the Euler product formula (1.1) above, one obtains explicit formulas displaying a relationship between primes and zeros of zeta. Such reasoning ultimately led Hadamard and de la Vallée Poussin to prove the prime number theorem about 50 years after Riemann's paper. The **prime number theorem** says

$$\#\{p = \text{prime} \mid p \leq x\} \sim \frac{x}{\log x}, \quad \text{as } x \to \infty. \tag{1.2}$$

Figure 2 is a graph of $z = |\zeta(x+iy)|$ drawn by Mathematica. The cover of *The Mathematical Intelligencer* (Vol. 8, No. 4, 1986) shows a similar graph with the pole at $x + iy = 1$ and the first 6 zeros, which are on the line $x = 1/2$, of course. The picture was made by D. Asimov and S. Wagon to accompany their article on the evidence for the Riemann hypothesis. The Mathematica people will sell you a huge poster of the Riemann zeta function.

Exercise 1. *Use Mathematica (or your favorite software) to do a contour plot of the Riemann zeta function in the same region as that of Figure 2.*

Hints. *Mathematica has a command to give you the Riemann zeta function. It is Zeta[s].*

The explicit formulas mentioned above say sums over the zeros of zeta are equal to sums over the primes. References are R. Murty [91] and S. J. Miller and R. Takloo-Bighash [86].

Many other kinds of zeta functions have been investigated since Riemann. In number theory there is the **Dedekind zeta function** of an algebraic number field K, such as $K = \mathbb{Q}(\sqrt{2})$, for example. This zeta is an infinite product over prime ideals $\mathfrak{p}$ in O_K, the ring of algebraic integers of K. For our example, $O_K = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$. The terms in the product are $\left(1 - N\mathfrak{p}^{-s}\right)^{-1}$, where $N\mathfrak{p} = \#(O_K/\mathfrak{p})$. Riemann's work can be extended to this zeta function and it can be used to prove the prime ideal theorem. The RH is unproved but conjectured to be true for the Dedekind zeta function. Surprisingly no one has yet proved (even in the case of quadratic number fields, $K = \mathbb{Q}(\sqrt{m})$), that there cannot be a real zero near 1. Such a possible zero is called a "**Siegel zero.**" A reference for this zeta is Lang [73] who shows why the non-existence of Siegel zeros would lead to many nice consequences for number theory. Figures 3, 4, 5, and 6 give summaries of the basic facts about zeta and L-functions for $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{d})$. We will find graph theory analogs of many of these things.

There are also **function field zeta functions** where the number field K is replaced by a finite algebraic extension of $\mathbb{F}_q(x)$, the rational functions of one variable over the finite field $\mathbb{F}_q$ with q elements. André Weil proved the RH for this zeta which is a rational function of $u = q^{-s}$. See Rosen [104].

Another generalization of Riemann's zeta function is the **Dirichlet L-function** associated to a multiplicative character χ defined on the group of integers $a(\mathrm{mod}\, m)$ with a relatively prime to m. This function is thought of as a function on the integers which is 0 unless a and m have no common divisors. Then one has the **Dirichlet L-function**, for $\operatorname{Re} s > 1$ defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

This L-function also has an Euler product, analytic continuation, functional equation, Riemann hypothesis (the Extended Riemann Hypothesis or ERH). This function can be used to prove the Dirichlet theorem saying that there are infinitely many primes in an arithmetic progression of the form $a, a+d, a+2d, a+3d, \ldots, a+kd, \ldots\ldots$, assuming that a and d are relatively prime. More generally there are **Artin L-functions** attached to representations of Galois groups of normal extensions of number fields. The, as yet unproved, Artin conjecture, says that if the representation is irreducible and not trivial (i.e., not identically 1), the L-function is entire. These L-functions were named for Emil Artin. A reference for Artin L-functions is Lang [73]. We will be interested in graph theory analogs of Artin L-functions.

Yet another sort of zeta is the **Epstein's zeta function** attached to a quadratic form

$$Q[x] = \sum_{i,j=1}^{n} q_{ij} x_i x_j.$$

We assume that the q_{ij} are real and that Q is positive definite, meaning that $Q[x] > 0$, if $x \neq 0$. Then **Epstein zeta function** is defined for complex s with $\operatorname{Re} s > \frac{n}{2}$ by:

$$Z(Q,s) = \sum_{a \in \mathbb{Z}^n - 0} Q[a]^{-s}.$$

As for the Riemann zeta, there is an analytic continuation to all $s \in \mathbb{C}$ with a pole at $s = \frac{n}{2}$. And there is a functional equation relating $Z(Q,s)$ and $Z(Q,n-s)$. Even when $n = 2$, the analog of the Riemann hypothesis may be false for the Epstein zeta function. See Terras [133] for more information on this zeta function.

If $Q[x] \in \mathbb{Z}$, for all $x \in \mathbb{Z}^n$, then defining $N_m(Q) = |\{x \in \mathbb{Z}^n | \, Q[x] = m\}|$, we see that $Z(Q,s) = \sum_{m \geq 1} N_m m^{-s}$, assuming $\operatorname{Re} s > \frac{n}{2}$. Similarly one can define zeta functions attached to many lists of numbers like $N_m(Q)$, in particular, to the Fourier coefficients of modular forms. Classically modular forms are holomorphic functions on the upper half plane having an invariance property under a group of fractional linear transformations like the modular group $SL(2,\mathbb{Z})$ consisting of 2×2 matrices with integer entries and determinant 1. See S. J. Miller and R. Takloo-Bighash [86], Sarnak [109], or Terras [133] for more information. Now the idea of modular forms has been vastly generalized and even plays a role in Andrew Wiles proof of Fermat's last theorem.

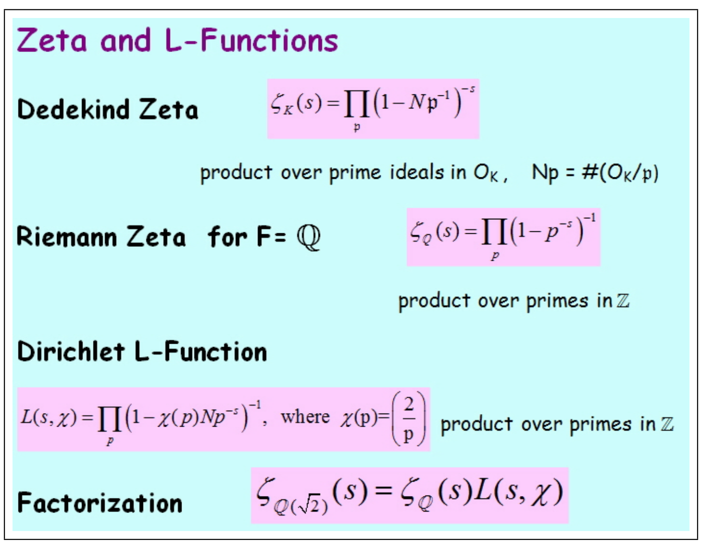


FIGURE 3. **A summary of facts about zeta and L-functions associated to the number fields** $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2})$. See Figure 6 for the definition of the Legendre symbol $\left(\frac{2}{p}\right)$.

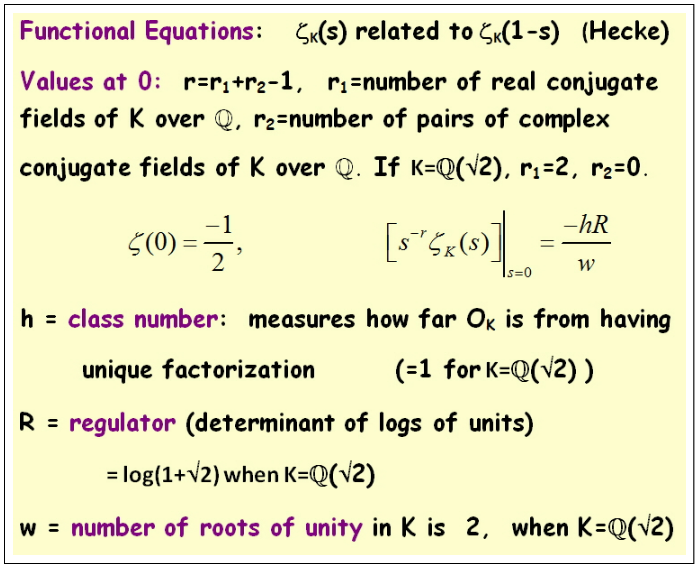


FIGURE 4. **What the zeta and L-functions say about the number fields**

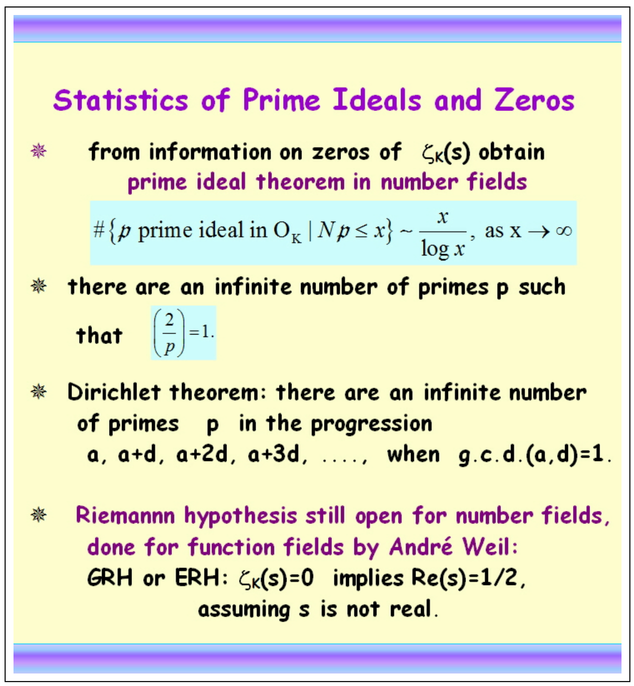


FIGURE 5. **Statistics of prime ideals**

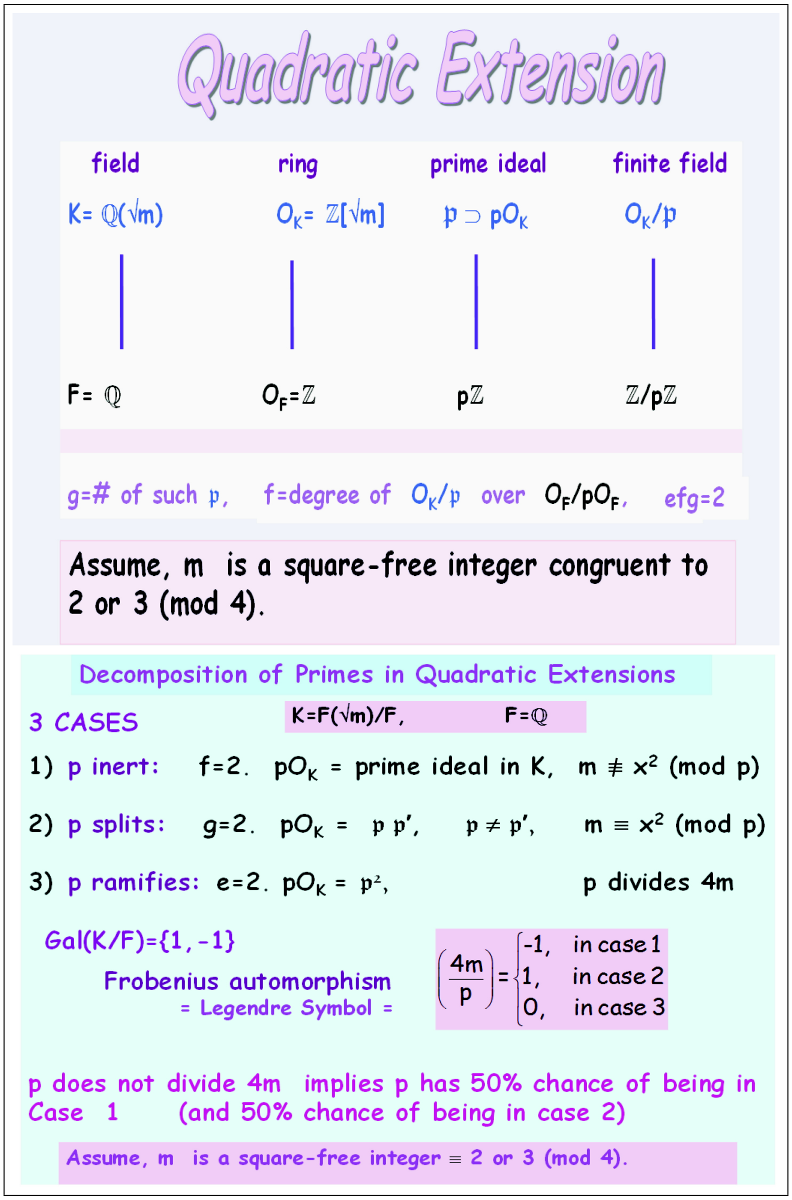


FIGURE 6. **Splitting of primes in quadratic extensions.** At the top, moving left to right, the 4 blue lines represent the number field extension $\mathbb{Q}(\sqrt{m})/\mathbb{Q}$, then the corresponding rings of integers, then prime ideals, and finally the finite residue fields. Here f is the degree of the extension of finite residue fields, g is the number of primes of O_K containing the prime p of $\mathbb{Z}$, and e is the ramification exponent. We have $2 = efg$ in the case under discussion with $K = \mathbb{Q}(\sqrt{m})$.

2. Ihara's Zeta Function

The Usual Hypotheses.

Our graphs will be finite, connected and undirected. It will usually be assumed that they contain no degree 1 vertices (called "leaves" or "hair" or "danglers"). We will also usually assume the graphs are not cycles or cycles with hair. A **cycle graph** is obtained by arranging the vertices in a circle and connecting each vertex to the 2 vertices next to it on the circle. A bad graph is pictured in Figure 7. We will allow our graphs to have loops and multiple edges.

Why do we make these assumptions? They are necessary hypotheses of many of the main theorems (for example, the graph theory prime number theorem, i.e., formula (2.4)). References for graph theory include Biggs [15], Bollobás [19], Chung [25], and Cvetković et al [32].

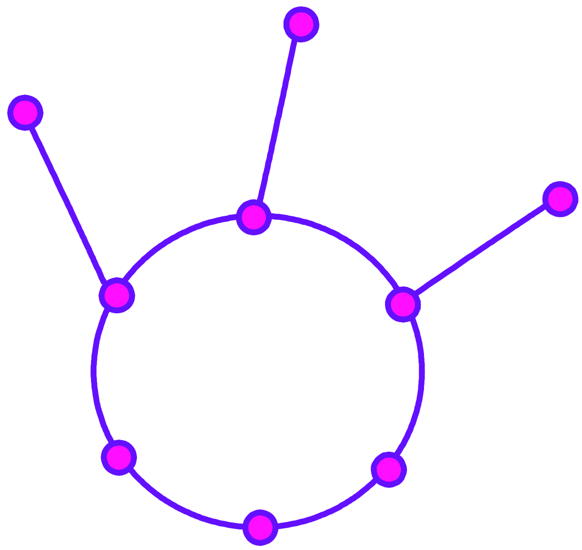


FIGURE 7. **This is an example of a bad graph for the theory of zeta functions.** For this graph, there are only finitely many primes (2 to be exact), as defined below.

A **regular graph** is a graph each of whose vertices has the same **degree** ; i.e., number of edges coming out of the vertex. A graph is **k-regular** if every vertex has degree k. Our graphs need not be regular and they may also have loops and multiple edges.

Definition 1. *Let V denote the vertex set of our graph X with n=$|V|$. The* ***adjacency matrix*** *A of X is an $n \times n$ matrix with i,j entry*

$$a_{ij} = \begin{cases} \text{number of undirected edges connecting vertex } i \text{ to vertex } j, & \text{if } i \neq j\,; \\ 2 \times \textit{number of loops at vertex } i, & \textit{if } i = j\,. \end{cases}$$

In order to define the Ihara zeta function, we need to define a prime in a graph X with edge set E having $m = |E|$ elements. To do this, we first direct or orient the edges of our graph arbitrarily and **label the edges** as follows

$$e_1, ..., e_m, e_{m+1} = e_1^{-1}, ..., e_{2m} = e_m^{-1}. \tag{2.1}$$

Here $m = |E|$ is the number of unoriented edges of X and $e_j^{-1} = e_{j+m}$ is the edge e_j with the opposite orientation. See Figure 8 for an example.

Primes in X, Some Definitions.

A path or walk $C = a_1 \cdots a_s$, where a_j is an oriented edge of X, is said to have a **backtrack** if $a_{j+1} = a_j^{-1}$, for some $j = 1, ..., s-1$. A path $C = a_1 \cdots a_s$ is said to have a **tail** if $a_s = a_1^{-1}$. The **length** of $C = a_1 \cdots a_s$ is $s = \nu(C)$. A **closed path** or **cycle** means the starting vertex is the same as the terminal vertex. The closed path $C = a_1 \cdots a_s$ is called a **primitive or prime path** if it has no backtrack or tail and $C \neq D^f$, for $f > 1$. That is, you can only go around the path once. For the closed path $C = a_1 \cdots a_s$, the **equivalence class** $[C]$ means the following

$$[C] = \{a_1 \cdots a_s,\ a_2 \cdots a_s a_1,\ \ldots\ ,\ a_s a_1 \cdots a_{s-1}\}\,. \tag{2.2}$$

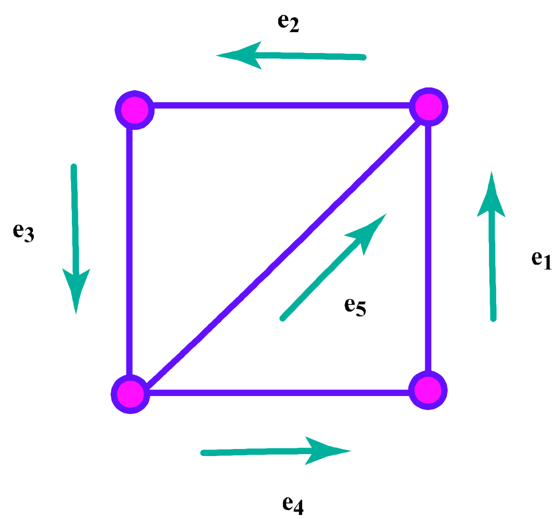


FIGURE 8. **We choose an arbitrary orientation of the edges of a graph. Then we label the inverse edges** via $e_{j+5} = e_j^{-1}$, for $j = 1, ..., 5$.

That is, we call two closed paths **equivalent** if we get one from the other by changing the starting vertex. A **prime** in the graph X is an equivalence class $[C]$ of prime paths. The **length of the path** C is $\nu(C) = s$, the number of edges in C.

Examples of Primes in a Graph.

For the graph in Figure 8, we have primes $[C] = [e_2e_3e_5]$, $[D] = [e_1e_2e_3e_4]$, $E = [e_1e_2e_3e_4e_1e_{10}e_4]$. Here $e_{10} = e_5^{-1}$ and the lengths of these primes are: $\nu(C) = 3$, $\nu(D) = 4$, $\nu(E) = 7$. We have infinitely many primes since $E_n = [(e_1e_2e_3e_4)^n e_1e_{10}e_4]$ is prime for all $n \geq 1$. But we don't have unique factorization into primes. The only non-primes are powers of primes. In particular, for the graph theory version of things, one does not have unique factorization into primes.

Definition 2. *The* ***Ihara zeta function*** *for a finite connected graph (without degree 1 vertices) is defined to be the following function of the complex number u, with $|u|$ sufficiently small:*

$$\zeta_X(u) = \zeta(u, X) = \prod_{[P]} \left(1 - u^{\nu(P)}\right)^{-1},$$

where the product is over all primes $[P]$ in X. Recall that $\nu(P)$ denotes the length of P.

In the product defining the Ihara zeta function, we distinguish the prime $[P]$ from $[P^{-1}]$, which is the path traversed in the opposite direction. Generally the product is infinite. We will see later how small $|u|$ must be for the product to converge. There is one case, however, when the product is finite. Normally we will exclude this case - the cycle graph.

Example 1. ***Cycle Graph.*** *Let X be a cycle graph with n vertices. Then, since there are only two primes:*

$$\zeta_X(u) = (1 - u^n)^{-2}.$$

As a power series in the complex variable u, the Ihara zeta function has non-negative coefficients. Thus, by a classic theorem of Landau, both the series and the product defining $\zeta_X(u)$ will converge absolutely in a circle $|u| < R_X$ with a singularity (pole of order 1 for connected X) at $u = R_X$. See Apostol [3], p. 237 for Landau's theorem.

Definition 3. *R_X is the* ***radius of the largest circle of convergence*** *of the Ihara zeta function.*

In fact, R_X is rather small. When X is a $(q+1)$-regular graph, $R_X = 1/q$. We will say more about the size of R_X for irregular graphs later. Amazingly the Ihara zeta function is the reciprocal of a polynomial by Theorem 1 below.

The Fundamental Group of a Graph and its Connection with Primes.

One of the favorite determinant formulas for the Ihara zeta function (in Theorem 1 below) involves the **fundamental group** $\Gamma = \pi_1(X, v)$ of the graph X. Later we will even define the path zeta which is more clearly attached to this group.

The fundamental group of a topological space such as our graph X has elements which are closed directed paths starting and ending at a fixed basepoint $v \in X$. Two paths are **equivalent** iff one can be continuously deformed into the other (i.e., one is

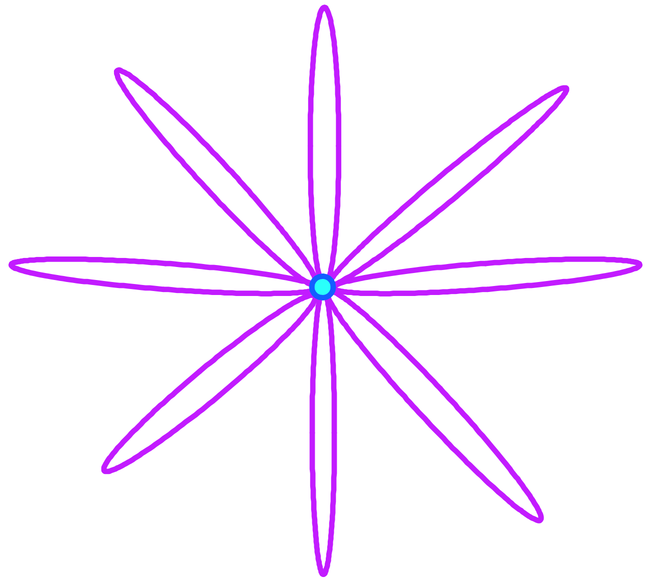


FIGURE 9. **A bouquet of loops.**

"homotopic" to the other within X, while still starting and ending at v). The **product** of 2 paths a, b means first go around a then b.

It turns out (by the Seifert-Van Kampen theorem, for example) that the fundamental group of graph X is a free group on r generators, where r is the number of edges left out of a spanning tree for X. Let us try to explain this a bit. More information can be found on the web; e.g., the algebraic topology book of Allen Hatcher, Chapter 1: www.math.cornell.edu/~hatcher. You could also look at Massey [83], p. 198, or Gross and Tucker [47].

What is a **free group** G on a set S of r generators? Here r is the **rank** of G. The group G is the set of words obtained by forming finite strings or words $a_1 \cdots a_t$ of symbols $a_j \in S$ modulo an equivalence relation. Two words $a_1 a_2 \cdots a_t$ and $a_1 a_2 \cdots b b^{-1} \cdots a_t$, $b \in S$ are called equivalent. The product of words $a_1 \cdots a_t$ and $b_1 \cdots b_s$ is $a_1 \cdots a_t b_1 \cdots b_s$. The result is a group.

What is a spanning tree T in a graph X? First we say that a graph T is a **tree** if it is a connected graph without any closed backtrackless paths of length ≥ 1. A **spanning tree** T for graph X means a tree which is a subgraph of X containing all the vertices of X. Every graph has a spanning tree.

From the graph X we construct a new graph $X^\#$ by shrinking a spanning tree T of X to a point. The new graph will be a bouquet of r loops as in Figure 9. It turns out that the fundamental group of X is the same as that of $X^\#$. Why? The quotient map $X \to X/T$ is what algebraic topologists call a "homotopy equivalence." This means that intuitively you can continuously deform one graph into the other. For more information, see Allen Hatcher, Chapter 0: www.math.cornell.edu/~hatcher.

So what is the fundamental group of the bouquet of r loops in Figure 9? We claim it is clearly the free group on r generators. The generators are the loops! The elements are the words in these loops modulo the equivalence relation defined above for words in a free group. The rank r of the fundamental group of the original graph X is thus the number of edges left out of X to get a spanning tree.

Exercise 2. *a) Find the fundamental groups for the graphs in Figures 7, 9, and 11.*
b) Show that if r is the rank of the fundamental group, then $r - 1 = |E| - |V|$.

We have a 1-1 correspondence between **conjugacy classes** $\{C\} = \left\{ xCx^{-1} \middle| x \in \Gamma = \pi_1(X, v) \right\}$ and equivalence classes of backtrackless, tailless cycles $[C^*]$ in X defined in formula (2.2). If a closed path C starting and ending at point v gives rise to a conjugacy class $\{C\}$ in Γ, we may take C in its homotopy class so that C has no backtracking. We then remove the tail from C so as to get a tailless cycle C^*. The 1-1 correspondence referred to comes from the fact that the conjugacy class of C in Γ corresponds to the equivalence class of the backtrackless, tailless cycle C^* It can be shown (**Exercise**) that the change of C in its conjugacy class corresponds to a change of C^* in its equivalence class. In the other direction of the correspondence, given C^*, we grow a tail so as to reach v, thus getting a path C which determines an element of Γ. A different tail simply conjugates C. Another way of thinking about the correspondence is that the elements of the equivalence class of C^* are precisely the closed cycles of minimal length which are freely homotopic to C. **Freely homotopic** means the base point v is not fixed.

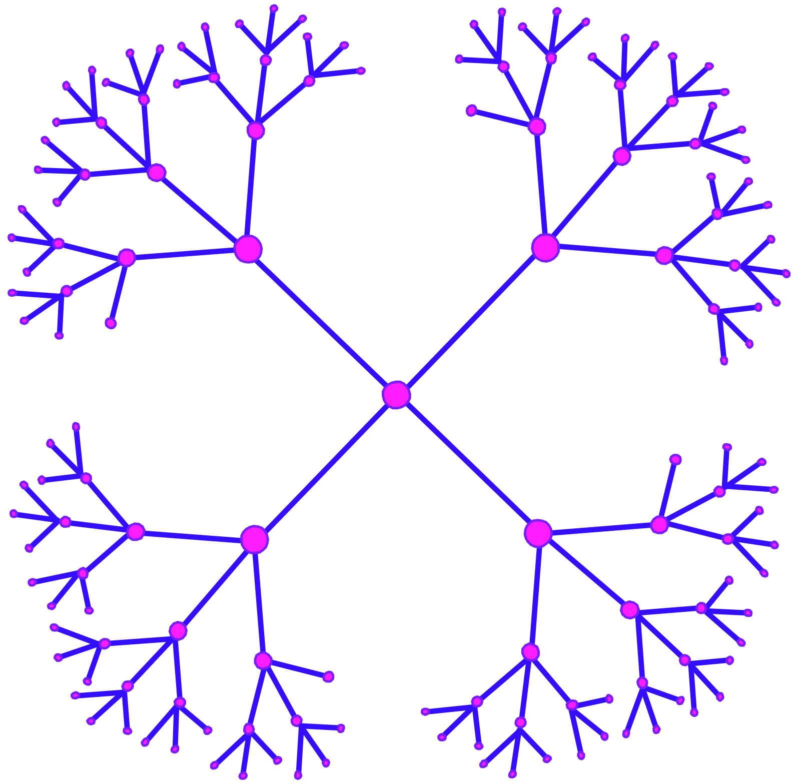


FIGURE 10. **Part of the 4-regular tree.**

The fundamental group Γ is a free group of rank r. Thus, for $\gamma \neq 1$ in Γ, the **centralizer** $C_\Gamma(\gamma) = \{\delta \in \Gamma | \gamma\delta = \delta\gamma\}$ is a cyclic subgroup of Γ. Under the 1-1 correspondence between classes $[C]$ of backtrackless, tailless cycles in Γ, prime cycles P correspond to conjugacy classes $\{P\}$ in Γ such that the centralizer $C_\Gamma(P)$ is generated by P. Such conjugacy classes $\{P\}$ are called **primitive.** Thus primes $[P]$ in X are in 1-1 correspondence with conjugacy classes of primitive elements in the fundamental group of X. We say more about this correspondence in the section on path zeta functions of graphs.

Note that although an irregular graph may not appear at first glance to have any "symmetry" in the sense of group of symmetries, there is always the fundamental group lurking around.

We remark here that we will not always be consistent in our notation. We will want to use capital Latin letters for paths in a graph. We will want to use small Greek letters for elements of the fundamental group (or a Galois group acting on a graph covering), but sometimes conflicts will arise and consistency will seem impossible.

Exercise 3. *Prove that the centralizer of $\gamma \neq 1$ in the fundamental group Γ is cyclic.*

Algebraic topology (see the references above) tells us that there is a "universal covering tree" T (meaning that it is without cycles and is a covering of the original graph X as in Definition 5) below. See Figure 10 for a picture of the 4-regular tree which is the universal cover of any 4-regular graph.

Exercise 4. *Draw part of the universal covering tree for the irregular graph $K_4 - e$, obtained by removing one edge from the tetrahedron K_4, and pictured on the right in Figure 11 below.*

There is an action of the fundamental group Γ on T such that we can identify T/Γ with X. You can also view the tree in the $(p+1)$-regular case as coming from p-adic matrix groups. See Serre [113], Trimble [137], and the last chapter of Terras [132] as well as [59] and [136].

If V is the set of vertices of graph X and E the set of edges, then the **Euler characteristic** of X is $\chi(X) = |V| - |E| = 1 - r$, where r is the rank of the fundamental group Γ of X. See Exercise 2.

One moral of the preceding considerations is that we can rewrite the product in Definition 2 in the language of the fundamental group of X as

$$\zeta_X(u) = \prod_{\{C\}} \left(1 - u^{\nu(C)}\right)^{-1}, \tag{2.3}$$

where the product is over primitive conjugacy classes in the fundamental group Γ of X. A primitive conjugacy class $\{C\}$ in Γ means that C generates the centralizer of C in Γ. Here $\nu(C)$ means the length of the element C^* in the equivalence class of tailless, backtrackless paths corresponding to $\{C\}$. Alternatively $\nu(C)$ is the minimal length of all cycles freely homotopic to C. As above, we distinguish between C and C^{-1} in this product.

Mercifully the zeta function can be computed from the Ihara determinant formula in the theorem which follows.

Theorem 1. ***(Ihara theorem generalized by Bass, Hashimoto, etc.).*** *Let A be the adjacency matrix of X and Q the diagonal matrix with jth diagonal entry q_j such that $q_j + 1$ is the degree of the jth vertex of X. Suppose that r is the rank of the fundamental group of X; $r - 1 = |E| - |V|$. Then we have the* ***Ihara determinant formula***

$$\zeta_X(u)^{-1} = (1-u^2)^{r-1} \det\left(I - Au + Qu^2\right).$$

Exercise 5. *Show that $r - 1 = \frac{1}{2} Tr(Q - I)$.*

There is an elementary proof of the preceding theorem using the method of Bass [12]. We will present it in Part 3. In [132] we presented another proof for k-regular graphs using the Selberg trace formula on the k-regular tree. For k-regular graphs, the Ihara zeta function has much in common with the Riemann zeta function.

Suppose X is a $q+1$ regular graph. Then the Ihara zeta function has functional equations relating the value at u with the value at $1/(qu)$. Setting $u = q^{-s}$, one finds that there is a functional equation relating the value at s with that at $1-s$, just as for Riemann's zeta function. See Proposition 3 below.

When the graph X is $(q+1)$-regular, there is also an analog of the Riemann hypothesis. It turns out to hold if and only if the graph is Ramanujan as defined by Lubotzky, Phillips and Sarnak in [79].

Definition 4. *A connected $(q+1)$-regular graph X is* ***Ramanujan,*** *iff, when*

$$\mu = \max\{|\lambda| \,|\lambda \in Spectrum(A), |\lambda| \neq q+1\}$$

then $\mu \leq 2\sqrt{q}$.

Some graphs are Ramanujan and some are not. In the 1980s, Margulies and independently Lubotzky, Phillips and Sarnak [79] found a construction of infinite families of Ramanujan graphs of fixed degree equal to $1 + p^e$, where p is a prime. They used the Ramanujan conjecture (now proved by Deligne) to show that the graphs were Ramanujan. Such graphs are of interest to computer scientists because they provide efficient communication as they have good expansion properties. See Guiliana Davidoff et al [35], Lubotzky [77], Sarnak [109], or Terras [132] for more information. Friedman [42] proves that a random regular graph is almost Ramanujan (the **Alon conjecture**). See Steven J. Miller et al [87] for experiments leading to the conjecture that the percent of regular graphs exactly satisfying the RH approaches 27% as the number of vertices approaches infinity. The argument involves the Tracy-Widom distribution from random matrix theory. A survey on expander graphs and their applications is that of Hoory, Linial and Wigderson [55].

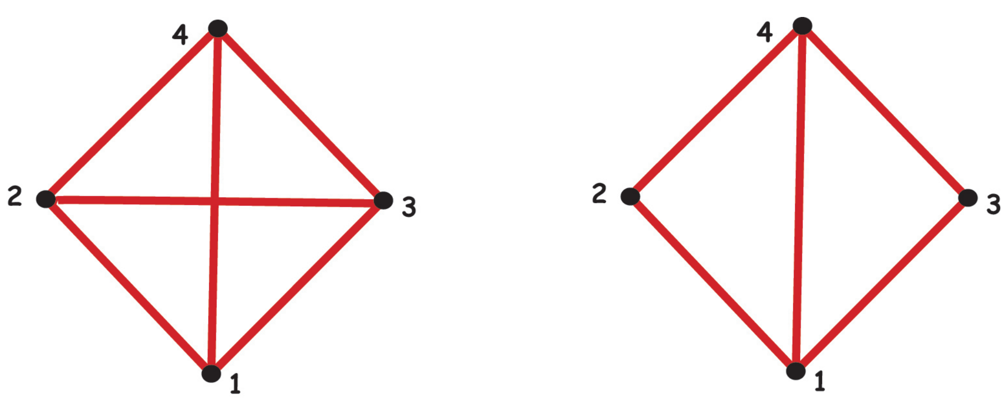


FIGURE 11. **On the left is the tetrahedron graph** also known as K_4, the complete graph on 4 vertices. **On the right is the graph $K_4 - e$ obtained from the tetrahedron by deleting an edge.** The vertices are numbered.

Example 2. *The* ***tetrahedron graph*** K_4 *in Figure 11 has Ihara zeta function*

$$\zeta_{K_4}(u)^{-1} = \left(1-u^2\right)^2 (1-u)(1-2u)\left(1+u+2u^2\right)^3.$$

The 5 poles of this zeta function are located at the points $-1, \frac{1}{2}, 1, \frac{-1\pm\sqrt{-7}}{4}$. *The absolute value of the complex pole is* $\frac{1}{\sqrt{2}} \cong 0.707\,11$. *The closest pole to the origin is* $\frac{1}{2} = \frac{1}{q} = R_{K_4}$.

Of course, K_4 *is a Ramanujan graph and thus the Riemann hypothesis holds for this graph.*

Next we illustrate what the Ihara zeta counts. The nth coefficient of the generating function $u\frac{d}{du}\log\zeta_{K_4}(u)$ *is the number* N_m *of length n closed paths in* K_4 *(without backtracking or tails). So there are 8 primes of length 3 in* K_4*, for example. See Definition 9 and formula (4.5) proved in the section on Ruelle's zeta function below. We find that*

$$u\frac{d}{du}\log\zeta_{K_4}(u) = 24u^3 + 24u^4 + 96u^6 + 168u^7 + 168u^8 + 528u^9 + 1200u^{10} + 1848u^{11} + O(u^{12}).$$

Figure 12 is a contour map of $z = \left|\zeta_{K_4}(x+iy)\right|^{-1}$, while Figure 13 is a contour map of $z = \left|\zeta_{K_4}(2^{-(x+iy)})\right|^{-1}$. The second graph is more like that for the Riemann zeta function which was Figure 2.

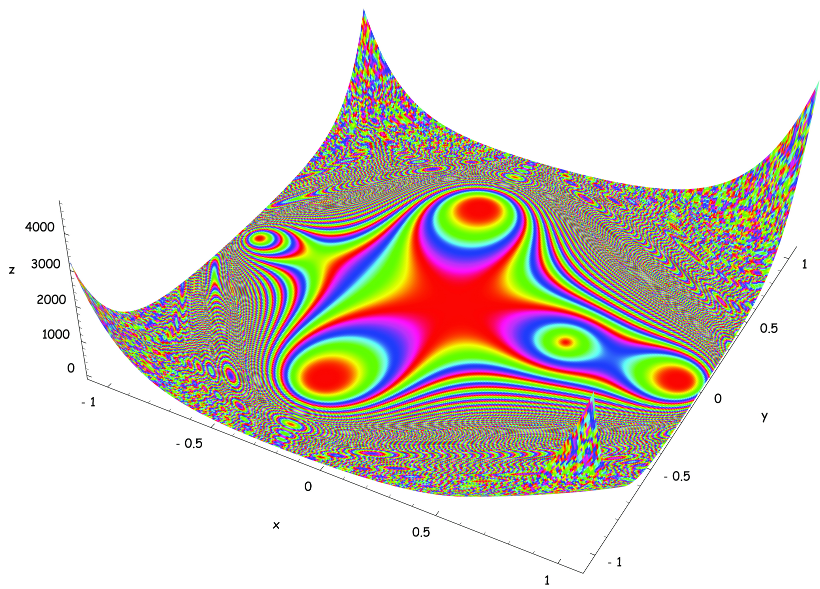


FIGURE 12. **A contour map of the modulus of the reciprocal of the Ihara zeta** for K_4 $z = 1/|\zeta_{K_4}(x+iy)|$ drawn by Mathematica. You can see the 5 roots (not counting multiplicity).

Example 3. *Let* $X = K_4 - e$ *be the* ***graph obtained from the tetrahedron*** K_4 ***by deleting an edge*** e*. See Figure 11. Then*

$$\zeta_X(u)^{-1} = \left(1-u^2\right)(1-u)\left(1+u^2\right)\left(1+u+2u^2\right)\left(1-u^2-2u^3\right).$$

From this, we have

$$u\frac{d}{du}\log\zeta_X(u) = 12x^3 + 8x^4 + 24x^6 + 28x^7 + 8x^8 + 48x^9 + \cdots.$$

So there are 4 primes of length 3 in X*. There are 9 roots:* $\pm1, \pm i, \frac{-1\pm\sqrt{-7}}{4}$*, 3 roots of the cubic,* s_1, s_2, s_3 *with* $|s_1| \cong 0.657\,30$, $|s_2| = |s_3| \cong 0.872\,18$*. So for this example,* $R_X \cong 0.657\,30$*. Later we will define what the Riemann hypothesis means for an irregular graph. See Section 8.*

The relation between zeta for the graph with an edge deleted and zeta for the original graph is not obvious. We will have more to say about this topic when we have discussed the edge zeta functions.

Exercise 6. *Compute the Ihara zeta functions of your favorite graphs; e.g., the cube, the dodecahedron, the buckyball.*

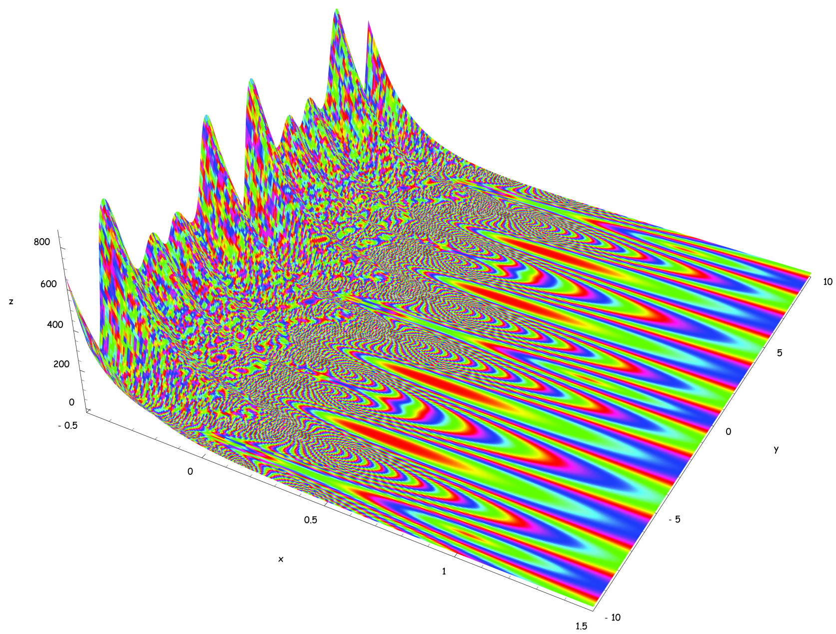


FIGURE 13. **A contour map for the modulus of the reciprocal of zeta** for K_4; i.e., $z = 1/|\zeta_{K_4}(2^{-(x+iy)})|$, as drawn by Mathematica.

Next we consider an unramified finite covering graph Y of our finite graph X. This is analogous to an extension of algebraic number fields. We assume that both X and Y are connected. A discussion of covering graphs can be found in Massey [83]. The idea is that locally at each vertex the two graphs look alike, though globally they may be very different. In the same way, the sphere and the plane are locally alike.

Definition 5. *If the graph has no multiple edges and loops we can say that the graph Y is an* ***unramified covering*** *of the graph X if we have a covering map $\pi : Y \to X$ which is an onto graph mapping (i.e., taking adjacent vertices to adjacent vertices) such that for every $x \in X$ and for every $y \in \pi^{-1}(x)$, the collection of points adjacent to $y \in Y$ is mapped 1-1 onto the collection of points adjacent to $x \in X$.*

The factorization of the Ihara zeta function of the quadratic covering in the example below is analogous to what happens for Dedekind zeta function of quadratic extensions of number fields. In a later section, we will show that the entire theory of Dedekind zeta functions (and Artin L-functions) has a graph theory analog.

Example 4. ***Unramified Quadratic Covering of K_4*** *Consider Figure 14. The cube Y is obtained by drawing two copies of a spanning tree (with red edges) for the tetrahedron $X = K_4$ and then drawing the rest of the edges of the cover to go between sheets of the cover. We find that $\zeta_Y(u)^{-1} = L(u, \rho, Y/X)^{-1}\zeta_X(u)^{-1}$, where*

$$L(u, \rho, Y/X)^{-1} = (1-u^2)(1+u)(1+2u)(1-u+2u^2)^3$$

Exercise 7. *Draw the analogs of Figure 12 and 13 for the cube.*

Exercise 8. *Find a second quadratic cover Y' of the tetrahedron by drawing two copies of a spanning tree of $X = K_4$ and then connecting the rest of the edges of Y so that only two edges in Y' go between sheets of the cover. Compute the Ihara zeta function of Y'.*

One can use the Ihara zeta function to prove the graph prime number theorem. In order to state this result, we need some definitions.

Definition 6. *The* ***prime counting function*** *is*

$$\pi(n) = \#\{\text{primes } [P]\,|n = \nu(P) = \textit{length of } P\}.$$

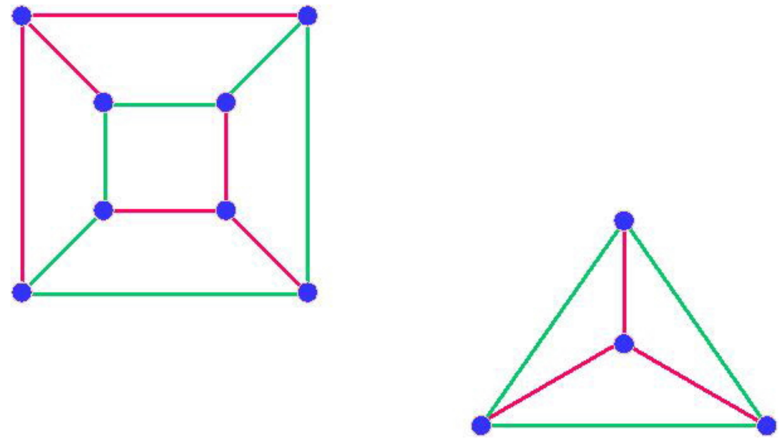


FIGURE 14. **The cube is a quadratic covering of the tetrahedron.** A spanning tree for the tetrahedron is indicated in red. Two copies of this tree are seen in the cube.

You may find it a bit surprising to note that we have replaced $\leq$ in the usual prime counting function of formula (1.2) with $=$ here.

Definition 7. *The* ***greatest common divisor of the prime path lengths*** *is*

$$\Delta_X = g.c.d.\{\nu(P) \mid [P] \text{ prime of } X\}.$$

The graph theory prime number theorem for a connected graph X satisfying the usual hypotheses says that if Δ_X divides m, then

$$\pi(m) \sim \frac{\Delta_X}{m R_X^m}, \quad \text{as } m \to \infty. \tag{2.4}$$

If Δ_X does not divide m, then $\pi(m) = 0$. Note that the theorem is clearly false for the bad graph in Figure 7.

We will see the proof in Section 10 below. It is much easier than that of the usual prime number theorem for prime integers. There are also analogs of Dirichlet's theorem on primes in progressions and the Chebotarev density theorem. We will consider this in section 22.

3. Selberg's Zeta Function

Some references for this subject are Dennis Hejhal [52], Atle Selberg [111], [112], Audrey Terras [133], and Marie-France Vignéras [139] Another reference is the collection of articles edited by Tim Bedford, Michael Keane, and Caroline Series [13].

The Selberg zeta function is a generating function for "primes" in a compact (or finite volume) Riemannian manifold M. Before we define "prime," we need to think a bit about Riemannian geometry. Assuming M has constant curvature -1 it can be realized as a quotient of the **Poincaré upper half plane**

$$H = \{x + iy \mid x, y \in \mathbb{R},\ y > 0\},$$

with **Poincaré arc length** element

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

which can be shown invariant under **fractional linear transformation**

$$z \longrightarrow \frac{az+b}{cz+d}, \quad \text{where } a, b, c, d \in \mathbb{R}, \quad ad - bc > 0.$$

The **Laplace operator** corresponding to the Poincaré arc length is

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

It too is invariant under fractional linear transformations.

It is not hard to see that **geodesics;** i.e., curves minimizing the Poincaré arc length are half lines and semicircles orthogonal to the real axis. Calling these geodesics "straight lines" creates a model for non-Euclidean geometry since Euclid's 5th postulate fails. There are infinitely many geodesics through a fixed point not meeting a given geodesic. See Figure 15.

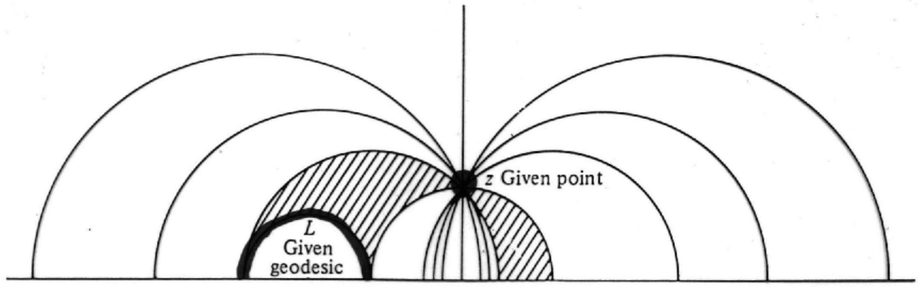


FIGURE 15. **The failure of Euclid's 5th postulate is illustrated.** All geodesics through z outside the shaded angle fail to meet L (from [133], Vol. I, p. 123).

The fundamental group Γ of M acts as a discrete group of distance-preserving transformations.

The favorite group of number theorists is the **modular group** $\Gamma = SL(2,\mathbb{Z})$ of 2×2 matrices of determinant one and integer entries or the quotient $\overline{\Gamma} = \Gamma/\{\pm I\}$.

We can identify the quotient $SL(2,\mathbb{Z})\backslash H$ with the **fundamental domain** D pictured below in Figure 16, where the sides are identified via $z \longrightarrow z+1$ and $z \longrightarrow -1/z$. These transformations generate $\overline{\Gamma}$.

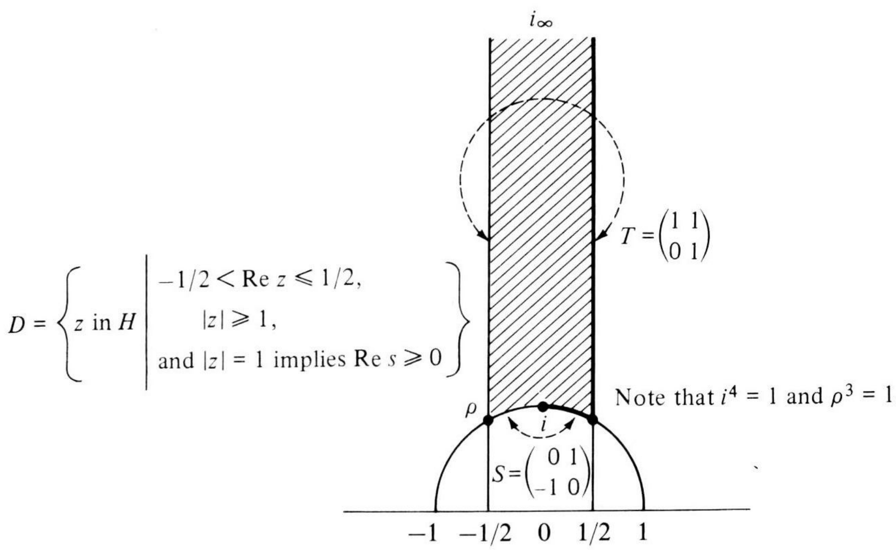


FIGURE 16. **A noneuclidean triangle** D through the points $\rho, \rho+1, i\infty$, which is a fundamental domain for H mod $SL(2,\mathbb{Z})$. The domain D is shaded. Arrows show boundary identifications by the fractional linear transformation from S and T which generate $SL(2,\mathbb{Z})/\{\pm I\}$, (from [133], p. 164).

The images of D under elements of Γ provide a **tessellation** of the upper half plane. See Figure 17.

There are 4 types of elements of $\overline{\Gamma}$. They are determined by the Jordan form of the 2x2 matrix. The **corresponding fractional linear map will be one of 4 types:**

identity	$z \longrightarrow z$
elliptic	$z \longrightarrow cz$, where $\lvert c\rvert = 1, c \neq 1$
hyperbolic	$z \longrightarrow cz$, where $c > 0, c \neq 1$
parabolic	$z \longrightarrow z + a$.

The **Riemann surface** $M = \Gamma\backslash H$ is compact and without **branch** (**ramification**) points if Γ has only the identity and hyperbolic elements. The modular group unfortunately has both elliptic and parabolic elements. It is easiest to deal with Selberg zeta functions when Γ has only the identity and hyperbolic elements. Examples of such groups are discussed in Svetlana Katok's book [67].

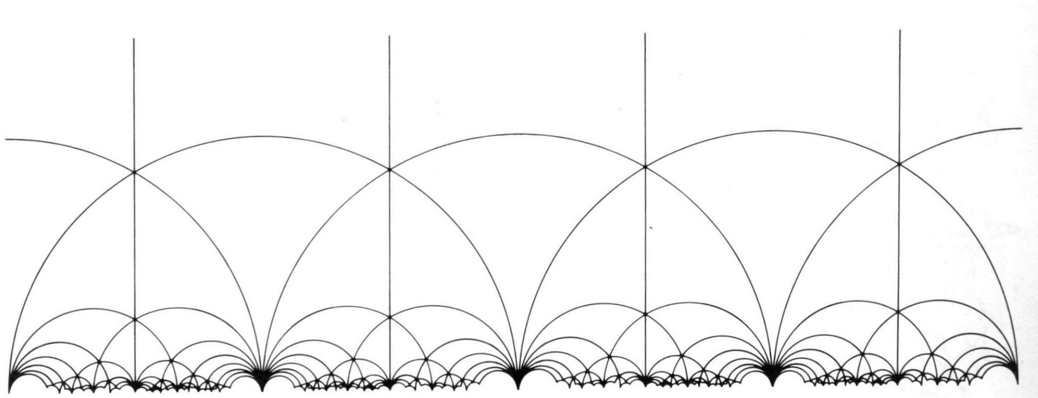


FIGURE 17. The **tessellation** of the upper half plane arising from applying the elements of the modular group to the fundamental domain D in the preceding figure. The figure is taken from [133], p. 166.

A hyperbolic element $\gamma \in \Gamma$ will have 2 fixed points on $\mathbb{R} \cup \{\infty\}$. Call these points z and w. Let $C(z,w)$ be a geodesic line or circle in H connecting points z and w in $\mathbb{R} \cup \{\infty\}$. Consider the image $\overline{C(z,w)}$ in the fundamental domain for $\Gamma\backslash H$. We say that $\overline{C(z,w)}$ is a **closed geodesic** if it is a closed curve in the fundamental domain (i.e., the beginning of the curve is the same as the end). A **primitive** closed geodesic is traversed only once. One can show that $\overline{C(z,w)}$ is a closed geodesic in $\Gamma\backslash H$ iff there is an element $\gamma \in \Gamma$ such that $\gamma C(z,w) \subset C(z,w)$. This means that z and w are the fixed points of a hyperbolic element of Γ. One can show that if a point q lies on $C(z,w)$ then so does γq and the Poincaré distance between q and γq is $\log N\gamma$, where $N\gamma = a^2$, if γ has Jordan form $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$, with a real, $a \neq \pm 1$. E. Artin was one of the first people to consider the question of whether these geodesics tend to fill up the fundamental domain as the length approaches infinity. This is related to ergodic theory and dynamical systems. It is also related to continued fractions. See Bedford et al [13].

Exercise 9. *Using a computer, graph $\overline{C(z,w)}$ for various choices of z, w. We did this in Figure 18 below and then mapped everything into the unit disc using the* ***Cayley transform***

$$z \longrightarrow \frac{i(z-i)}{z+i}.$$

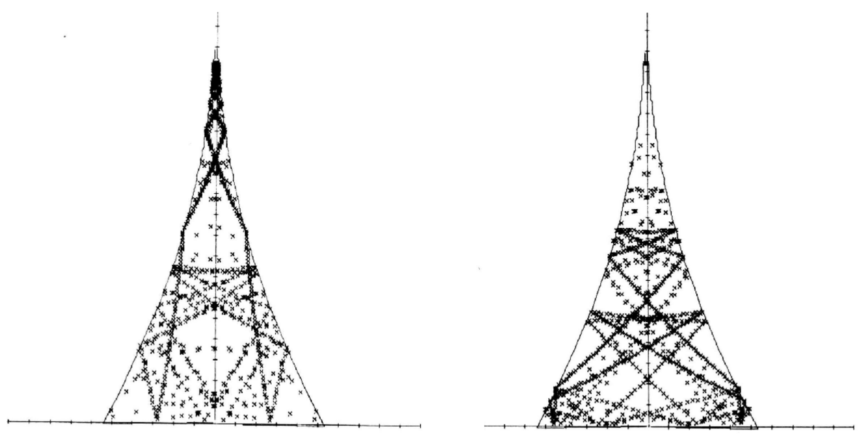


FIGURE 18. **Images of points on two geodesic circles after mapping them into the fundamental domain** of $SL(2,\mathbb{Z})$ and then by Cayley transform into the unit disc. Here the geodesic circles have center 0 and radii $\sqrt{163}$ on the left and e on the right (from [133], Vol.I, pp. 280-281).

Primes in M.

The lengths of these primitive closed geodesics coming from hyperbolic elements of Γ form the **length spectrum** of $M = \Gamma\backslash H$. These are our "primes" in M. Now we are ready to define the **Selberg zeta function** as

$$Z(s) = \prod_{[C]} \prod_{j\geq 1} \left(1 - e^{-(s+j)\nu(C)}\right). \tag{3.1}$$

The product is over all primitive closed geodesics C in $M = \Gamma\backslash H$ of length $\nu(C)$. Just as with the Ihara zeta function, this product can also be viewed as a product over conjugacy classes of primitive hyperbolic elements $\{\gamma\} \subset \Gamma$. Here "primitive hyperbolic element" means it generates its centralizer in Γ. The Selberg trace formula gives an explicit formula relating the spectrum of the Laplacian on M and the length spectrum of M.

The Selberg zeta function has many properties similar to the Riemann zeta function. Via the Selberg trace formula one shows that the logarithmic derivative of the Selberg zeta function has an analytic continuation as a meromorphic function. See Elstrodt [38], Patterson [98], and Bunke and Olbrich [22]. The non-trivial zeros of $Z(s)$ correspond to the discrete spectrum of the Poincaré Laplacian on $L^2(\Gamma\backslash H)$. This means that the Selberg zeta function satisfies the Riemann hypothesis (assuming that $\Gamma\backslash H$ is compact). Vignéras [139] proves many properties of the Selberg zeta function for the non-compact quotient $SL(2,\mathbb{Z})\backslash H$.

Sarnak has said that for nonarithmetic Γ with noncompact fundamental domain he doubts one should think of $Z(s)$ as a zeta function as he conjectures that the discrete spectrum of Δ is finite. Of course, this will be the case for the Ihara zeta function of a finite graph so perhaps some might not think it is a zeta at all.

To define "arithmetic" we must first define **commensurable subgroups** A, B of a group C. This means that $A \cap B$ has finite index both in A and B. Then suppose that Γ is an algebraic group over $\mathbb{Q}$ as in Borel's article in [20] p. 4. One says that Γ is **arithmetic** if there is a faithful rational representation ρ into the general linear group of $n \times n$ non-singular matrices such that ρ is defined over the rationals and $\rho(\Gamma)$ is commensurable with $\rho(\Gamma) \cap GL(n,\mathbb{Z})$. Roughly we are saying that the integers are hiding somewhere in the definition of Γ. See Borel's article in [20] for more information. Arithmetic and non-arithmetic subgroups of $SL(2,\mathbb{C})$ are discussed by Elstrodt, Grunewald, and Mennicke [39]. Roughly "arithmetic" Γ means that $\mathbb{Z}$ is lurking somewhere in the definition of Γ.

Exercise 10. *Show that $Z(s+1)/Z(s)$ has a product formula which looks more like that for the Ihara zeta function.*

The theory of Ihara zeta functions for $(p^e+1)-$regular qraphs, p=prime, can be understood via trace formulas for groups of 2x2 matrices over function fields. See Nagoshi [92]. One can also work out the theory of Ihara zetas on d-regular graphs based on trace formulas for groups acting on the d-regular tree. See Horton, Newland, and Terras [59], Terras [132], and Terras and Wallace [136].

4. Ruelle's Zeta Function

Some references for this subject are Ruelle [108] as well as Bedford, Keane and Series [13]. Ruelle's motivation for his definition came partially from a paper by M. Artin and B. Mazur [4]. They were in turn inspired by the definition of the **zeta function of a projective nonsingular algebraic variety** of dimension n defined over a finite field k with q elements. If N_m is the number of points of V with coordinates in the degree m extension field of k, the zeta function of V is

$$Z(z,V)=exp\left(\sum_{m=1}^{\infty}N_m\frac{z^m}{m}\right). \tag{4.1}$$

This zeta can be identified with that of a function field over k.

Example of varieties are given by taking solutions of polynomial equations over finite fields; e.g., $x^2+y^2=1$ and $y^2=x^3+ax+b$. You actually have to look at the homogeneous version of the equations in projective space. For more information on these zeta functions, see Lorenzini [76] p. 280 or Rosen [104].

Note that N_m is the number of fixed points of F^m where F denotes the **Frobenius morphism** which takes a point with coordinates (x_i) to the point (x_i^q). The **Weil conjectures**, ultimately proved in the general case by Deligne, say

$$Z(z,V)=\prod_{j=0}^{2n}P_j(z)^{(-1)^{j+1}},$$

where the P_j are polynomials whose zeros have absolute value $q^{-j/2}$. Moreover the P_j have a cohomological meaning (roughly $P_j(z)=\det\left(1-zF^*|H^j(V)\right)$. Here the Frobenius has induced an action on the ℓ-adic étale cohomology. The case that $n=1$ is very similar to that of the Ihara zeta function for a $(q+1)$-regular graph.

M. Artin and B. Mazur [4] replace the Frobenius of the algebraic variety with a diffeomorphism f of a smooth compact manifold M. They set

$$Fix(f^m)=\left\{x\in M \;\middle|\; f^m(x)=x\right\}$$

and look at the zeta function

$$\zeta(z)=\exp\left(\sum_{m=1}^{\infty}\frac{z^m}{m}\left|Fix\left(f^m\right)\right|\right). \tag{4.2}$$

The Ruelle zeta function involves a function $f:M\to M$ on a compact manifold M. Assume the set $Fix(f^m)$ is finite for all $m\geq 1$. Suppose $\varphi: M\to\mathbb{C}^{d\times d}$ is a matrix valued function. The first type of **Ruelle zeta function** is defined by

$$\zeta(z)=\exp\left\{\sum_{m\geq 1}\frac{z^m}{m}\sum_{x\in Fix(f^m)}Tr\left(\prod_{k=0}^{m-1}\varphi\left(f^k(x)\right)\right)\right\}. \tag{4.3}$$

Here we consider only the special case that $d=1$ and φ is identically 1, when formula (4.3) looks exactly like formula (4.2). Ruelle also defines a 2nd type of zeta function associated to a 1-parameter semigroup of maps $f^t: M\to M$. See the reference above for the details.

Now we consider a special case to see that the Ihara zeta function of a graph is a Ruelle zeta function. For this we consider subshifts of finite type. Let I be a finite non-empty set (our alphabet). For a graph X, let I be the set of directed edges of X. Define the **transition matrix** $t=(t_{ij})_{ij\in I}$ to be a matrix of 0's and 1's.

For the case of a graph let t denote the **edge adjacency matrix** W_1 defined below.

Definition 8. *For a graph X, define the* ***edge adjacency matrix*** W_1 *by orienting the m edges of X and labeling them as in formula (2.1). Then W_1 is the $2m\times 2m$ matrix with ij entry 1 if edge e_i feeds into e_j provided that $e_j\neq e_i^{-1}$, and ij entry 0 otherwise. By "a* ***feeds into*** *b," we mean that the terminal vertex of edge a is the same as the initial vertex of edge b.*

We will show the **two-term determinant formula**

$$\zeta_X(u)^{-1}=\det\left(I-W_1\,u\right). \tag{4.4}$$

From this we shall later derive the three-term determinant formula in Theorem 1.

Note that the product $I^{\mathbb{Z}}$ is compact and thus so is the closed subset Λ defined by

$$\Lambda=\left\{(\xi_k)_{k\in\mathbb{Z}}\middle|\; t_{\xi_k\xi_{k+1}}=1,\ \text{for all }k\right\}.$$

In the graph case $\xi\in\Lambda$ corresponds to a path without backtracking.

A continuous function $\tau:\Lambda\to\Lambda$ such that $\tau(\xi)_k=\xi_{k+1}$ is called a **subshift of finite type**. In the graph case, this shifts the path left, assuming the paths go from left to right.

Then we can find a new formula for the Ihara zeta function which shows that it is a Ruelle zeta. To understand this formula, we need a definition.

Definition 9. $N_m = N_m(X)$ *is the* ***number of closed paths of length*** m ***without backtracking and tails in the graph*** X.

From Definition 2 of the Ihara zeta, we prove in the next paragraph that

$$\log \zeta_X(u) = \sum_{m \geq 1} \frac{N_m}{m} u^m. \tag{4.5}$$

Compare this formula with formula (4.1) defining the zeta function of a projective variety over a finite field.

To prove formula (4.5), take the logarithm of Definition 2 where the product is over primes $[P]$ in the graph X to obtain

$$\begin{aligned}
\log \zeta_X(u) &= \log \left(\prod_{\substack{[P] \\ \text{prime}}} \left(1 - u^{\nu(P)}\right)^{-1} \right) = -\sum_{[P]} \log \left(1 - u^{\nu(P)}\right) \\
&= \sum_{[P]} \sum_{j \geq 1} \frac{1}{j} u^{j\nu(P)} = \sum_{P} \sum_{j \geq 1} \frac{1}{j\nu(P)} u^{j\nu(P)} = \sum_{P} \sum_{j \geq 1} \frac{1}{\nu(P^j)} u^{\nu(P^j)} \\
&= \sum_{\substack{C \text{ closed} \\ \text{backtrackless} \\ \text{tailless path}}} \frac{1}{\nu(C)} u^{\nu(C)} = \sum_{m \geq 1} \frac{N_m}{m} u^m.
\end{aligned}$$

Here we have used the power series for $\log(1-x)$ to see the third equality. Then the fourth equality comes from the fact that there are $\nu(P)$ elements in the equivalence class $[P]$, for any prime $[P]$. The fifth equality comes from $\nu(P^j) = j\nu(P)$. The sixth equality is proved using the fact that any closed backtrackless tailless path C in the graph is a power of some prime path P. The last equality comes from Definition 9 of N_m.

If the subshift of finite type τ is as defined above for the graph X, we have

$$\left|Fix(\tau^m)\right| = N_m. \tag{4.6}$$

It follows from this result and formula (4.5) that the Ihara zeta is a special case of the Ruelle zeta.

Next we claim that

$$N_m = Tr(W_1^m). \tag{4.7}$$

To see this, set $t = B = W_1$, with entries b_{ef}, for oriented edges e, f. Then

$$Tr(W_1^m) = Tr(B^m) = \sum_{e_1, \ldots, e_m} b_{e_1 e_2} b_{e_2 e_3} \cdots b_{e_m e_1},$$

where the sum is over all oriented edges of the graph. The b_{ef} are 0 unless edge e feeds into edge f without backtracking; i.e., the terminal vertex of e is the initial vertex of f and $f \neq e^{-1}$. Thus $b_{e_1 e_2} b_{e_2 e_3} \cdots b_{e_m e_1} = 1$ means that the path $C = e_1 e_2 \cdots e_m$ is closed, backtrackless, tailless of length m. Formula (4.7) follows.

Then we use formulas (4.5) and (4.7) to see that:

$$\begin{aligned}
\log \zeta_X(u) &= \sum_{m \geq 1} \frac{u^m}{m} Tr(W_1^m) = Tr \left(\sum_{m \geq 1} \frac{u^m}{m} W_1^m \right) \\
&= Tr \left(\log (I - uW_1)^{-1} \right) = \log \det (I - uW_1)^{-1}.
\end{aligned}$$

Here we have used the continuous linear property of trace. Finally we need the power series for the matrix logarithm and the following exercise.

Exercise 11. *Show that* $\exp Tr(A) = \det(\exp A)$, *for any matrix* A. *To prove this, you need to know that there is a non-singular matrix* B *such that* $BAB^{-1} = T$ *is upper triangular. See your favorite linear algebra book.*

This proves formula (4.4) for the Ihara zeta function which says $\zeta_X(u) = \det (I - uW_1)^{-1}$. More generally, this is known as the Bowen-Lanford theorem for subshifts of finite type in the context of Ruelle zeta functions. The general result is as follows. The proof is similar.

Proposition 1. ***(Bowen and Lanford)***
One has the following formula for the Ruelle zeta function of a subshift of finite type τ with transition matrix t:

$$\zeta(z) = \exp\left(\sum_{m\geq 1} \frac{z^m}{m} |Fix(\tau^m)|\right) = (\det(1 - zt))^{-1}.$$

Later, we will consider an edge zeta function attached to X which involves more than one complex variable and which has a similar determinant formula.

5. Chaos

References for this subject include Cipra [30], Haake [49], Miller and Takloo-Bighash [86] (see in particular the downloadable papers from the book's website at Princeton University Press), Rudnick [106], [107], Sarnak [110], Terras [134] and [135].

Quantum chaos is in part the study of the statistics of energy levels of quantum mechanical systems; i.e. the eigenvalues of the Schrödinger operator $\mathcal{L}\phi_n = \lambda_n \phi_n$. A good website for quantum chaos is that of Matthew W. Watkins:

www.maths.ex.ac.uk/~mwatkins.

We quote Oriol Bohigas and Marie-Joya Gionnoni [16], p. 14: "The question now is to discover the stochastic laws governing sequences having very different origins, as illustrated in ... [Figure 19]. There are displayed six spectra, each containing 50 levels ...." Note that the spectra have been rescaled to the same vertical axis from 0 to 49 and we have added 2 more columns to the original figure.

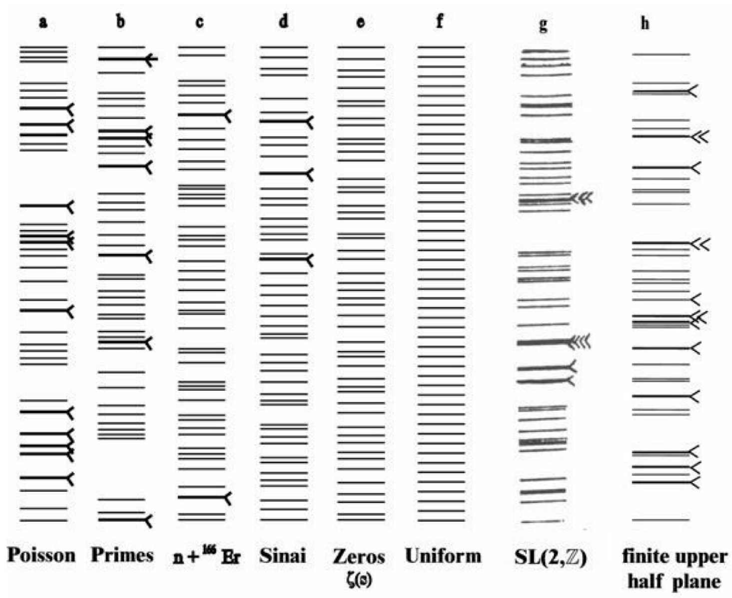


FIGURE 19. **Level Spacings in Columns a-f are from Bohigas and Giannoni** [16] **and column g is from Sarnak** [110]**.** Segments of "spectra," each containing 50 levels. The "arrowheads" mark the occurrence of pairs of levels with spacings smaller than 1/4. The labels are explained in the text. **Column h contains finite upper half plane graph eigenvalues** (without multiplicity) for the prime 53, with $\delta = a = 2$.

In Figure 19, column a represents a Poisson spectrum, meaning that of a random variable with spacings of probability density e^{-x}. Column b represents primes between 7791097 and 7791877. Column c represents the resonance energies of the compound nucleus observed in the reaction $n +^{166} Er$. Column d comes from eigenvalues corresponding to transverse vibrations of a membrane whose boundary is the Sinai billiard which is a square with a circular hole cut out centered at the center of the square. Then column e is from the positive imaginary part of zeros of the Riemann zeta function from the 1551th to the 1600th zero. Column f is equally spaced - the picket fence or uniform distribution. Column g comes from Sarnak [110] and corresponds to eigenvalues of the Poincaré Laplacian on the fundamental domain of the modular group $SL(2, \mathbb{Z})$ consisting of 2×2 integer matrices of determinant 1. From the point of view of randomness, columns g and h should be moved to lie next to column b. Column h is the spectrum of a finite upper half plane graph for p=53 ($a = \delta = 2$), without multiplicity. See Terras [132] for the definition of finite upper half plane graphs.

Exercise 12. *Produce your own versions of as many columns of Figure 19 as possible for poles/zeros of various zeta functions or eigenvalues of various matrices or operators.*

Quantum mechanics says the energy levels E of a physical system are the eigenvalues of a Schrödinger equation $\mathcal{H}\phi = E\phi$, where $\mathcal{H}$ is the Hamiltonian (a differential operator), ϕ is the state function (eigenfunction of $\mathcal{H}$), and E is the energy level (eigenvalue of $\mathcal{H}$). For complicated systems, physicists decided that it would usually be impossible to know all the energy levels. So they investigate the statistical theory of these energy levels. This sort of thing happens in ordinary statistical mechanics as well. Of course symmetry groups (i.e., groups of motions commuting with $\mathcal{H}$) have a big effect on the energy levels.

In the 1950's Wigner (see [143]) considered modelling $\mathcal{H}$ with a large real symmetric $n \times n$ matrices whose entries are independent Gaussian random variables. He found that the histogram of the eigenvalues looks like a semi-circle (or, more precisely, a semi-ellipse). This has been named the **Wigner semi-circle distribution**. For example, he considered the eigenvalues of 197 "random" real symmetric 20x20 matrices. The top graph in Figure 20 below shows the results of an analog of Wigner's experiment using Matlab. We take 200 random (normally distributed) real symmetric 50x50 matrices with entries that are chosen according to the normal distribution. Wigner notes on p. 5: "What is distressing about this distribution is that it shows no similarity to the observed distribution in spectra." This may be the case in physics, but the semi-circle distribution is well known to number theorists as the Sato-Tate distribution.

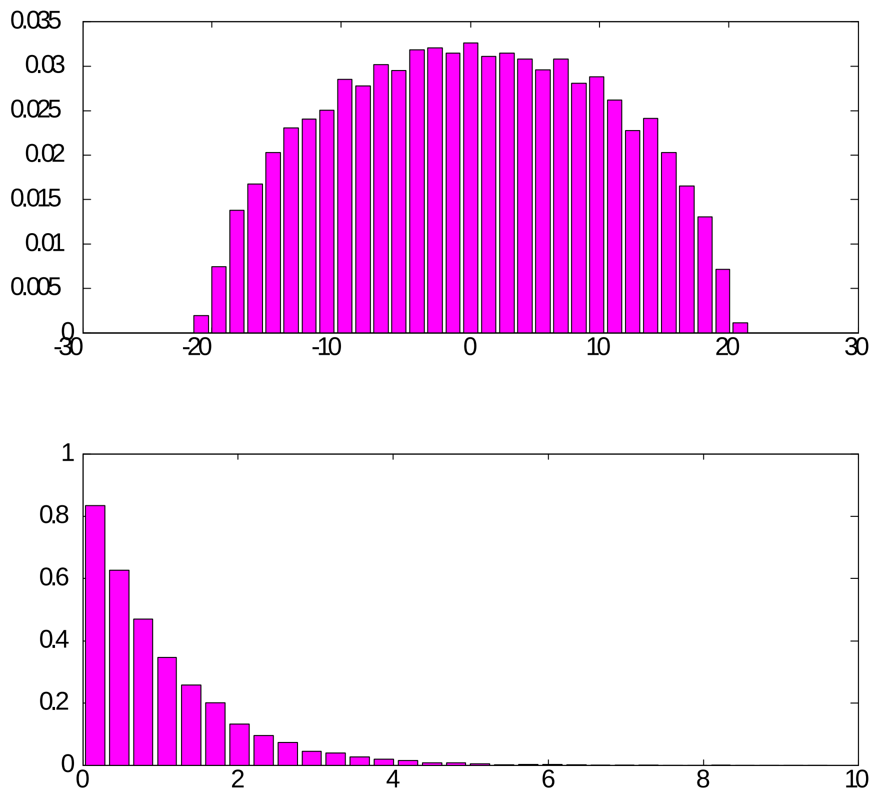


FIGURE 20. **The top histogram represents the spectra of 200 random real 50x50 symmetric matrices** created using Matlab. **The bottom histogram represents the spacings of the spectra** of the matrices from the top histogram. It appears to be e^{-x}, the spacings of Poisson random variables.

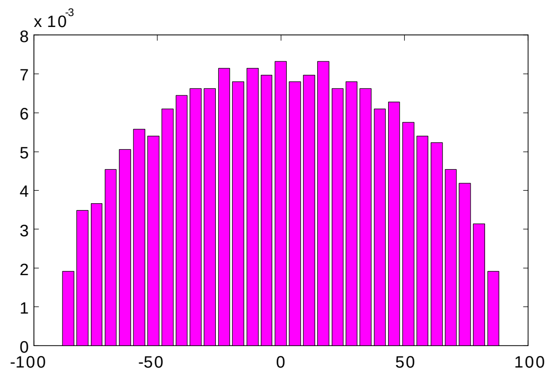
Exercise 13. *Repeat the experiment that produced Figure 20 using uniformly distributed matrices rather than normally distributed ones. This is a problem best done with Matlab which has commands rand(50), giving a random* 50×50 *matrix with uniformly distributed entries, and randn(50) giving a normal random* 50×50 *matrix. You must normalize the eigenvalues to have mean spacing one by considering them in batches.*

So physicists have devoted more attention to histograms of level spacings rather than levels. This means that you arrange the energy levels (eigenvalues) E_i in decreasing order:

$$E_1 \geq E_2 \geq \cdots \geq E_n.$$

Assume that the eigenvalues are normalized so that the mean of the level spacings $|E_i - E_{i+1}|$ is 1. Then one can ask for the shape of the histogram of the normalized level spacings. There are (see Sarnak [110]) two main sorts of answers to this question: **Poisson level spacings**, meaning e^{-x}, and **GOE spacings** (see Mehta [85]) which is more complicated to describe exactly but looks like $\frac{\pi}{2}xe^{-\frac{\pi x^2}{4}}$ (the **Wigner surmise**). In 1957 Wigner (see [143]) gave an argument for the surmise that the level spacing histogram for levels having the same values of all quantum numbers is given by $\frac{\pi}{2}xe^{-\frac{\pi x^2}{4}}$ if the mean spacing is 1. In 1960 Gaudin and Mehta found the correct distribution function which is surprisingly close to Wigner's conjecture but different. The correct graph is labeled GOE in Figure 22. Note the level repulsion indicated by the vanishing of the function at the origin. Also in Figure 22, we see the Poisson spacing density which is e^{-x}. See Bulmer [21], p. 102.

The spacing histogram in Figure 20 is the lower half of the Figure and it looks Poisson. It should be compared with that in Figure 21 below the top of which represents the spectrum histogram for one 1001×1001 matrix while the bottom is the level spacings histogram for the same matrix. This illustrates an important aspect of the dichotomy between GOE and Poisson behavior. If you throw lots of random symmetric matrices together you get Poisson spacing but it you take just one large symmetric matrix, you see GOE spacing. Later we will say more about this dichotomy.



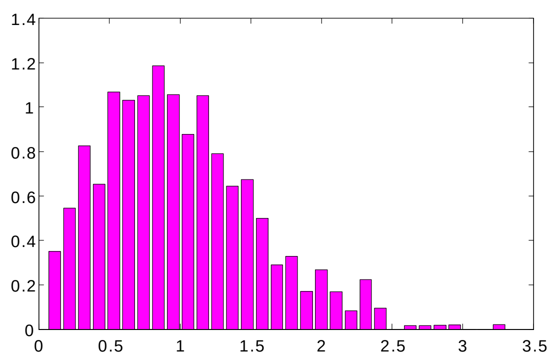


FIGURE 21. **The spectrum of a random normal symmetri**c real 1001×1001 **matrix** is **on top and the normalized level spacings on the bottom.**

You can find a Mathematica program to compute the GUE function at the website in [41]. Sarnak [110], p. 160 says: "It is now believed that for integrable systems the eigenvalues follow the Poisson behavior while for chaotic systems they follow the GOE distribution." Here GOE stands for **Gaussian Orthogonal Ensemble** - the eigenvalues of a random $n \times n$ symmetric real matrix as n goes to infinity. And GUE stands for the **Gaussian Unitary Ensemble** (the eigenvalues of a random $n \times n$ Hermitian matrix).

There are many experimental studies comparing GOE prediction and nuclear data. Work on atomic spectra and spectra of molecules also exists. In Figure 22, we reprint a figure of Bohigas, Haq, and Pandey [17] giving a comparison of histograms of level spacings for (a) ^{166}Er and (b) a nuclear data ensemble (or NDE) consisting of about 1700 energy levels corresponding to 36 sequences of 32 different nuclei. Bohigas et al say: "The criterion for inclusion in the NDE is that the individual sequences be in general agreement with GOE."

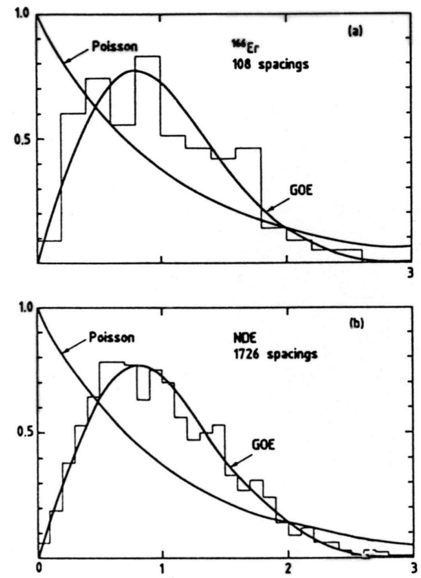


FIGURE 22. (f**rom Bohigas, Haq, and Pandey** [17]) Level spacing histogram for (a) ^{166}Er and (b) a nuclear data ensemble.

Andrew Odlyzko (see www.dtc.umn.edu/~odlyzko/doc/zeta.html) has investigated the level spacing distribution for the non-trivial zeros of the Riemann zeta function. He considers only zeros which are high up on the Re $s = \frac{1}{2}$ line**.** Assume the Riemann hypothesis and look at the zeros ordered by imaginary part

$$\left\{ \gamma_n \middle| \zeta(\frac{1}{2} + i\gamma_n) = 0,\ \gamma_n > 0 \right\}.$$

For the normalized level spacings, replace γ_n by $\widetilde{\gamma_n} = \frac{1}{2\pi}\gamma_n log\, \gamma_n$, since we want the mean spacing to be one. Here one needs to know that the number of γ_n such that $\gamma_n \le T$ is asymptotic to $\frac{1}{2\pi} T \log T$ as $T \longrightarrow \infty$.

Historically the connections between the statistics of the Riemann zeta zeros γ_n and the statistics of the energy levels of quantum systems were made in a dialogue of Freeman Dyson and Hugh Montgomery over tea at the Institute for Advanced Study, Princeton. Odlyzko's experimental results show that the level spacings $|\gamma_n - \gamma_{n+1}|$, for large n, look like that of the Gaussian unitary ensemble (GUE); i.e., the eigenvalue distribution of a random complex Hermitian matrix. See Figure 23.

The level spacing distribution for the eigenvalues of Gaussian unitary matrices is not a standard function in Matlab, Maple or Mathematica. Sarnak [110] and Katz and Sarnak [69] proceed as follows. Let $K_s : L^2[0,1] \longrightarrow L^2[0,1]$ be the integral operator with kernel defined by

$$h_s(x,y) = \frac{sin\left(\frac{\pi s(x-y)}{2}\right)}{\frac{\pi(x-y)}{2}}, \text{ for } \ s \ge 0.$$

Approximations to this kernel have been investigated in connection with the uncertainty principle (see Terras [133], Vol. I, p. 51). The eigenfunctions are spheroidal wave functions. Let $E(s)$ be the Fredholm determinant $det(I - K_s)$ and let $p(s) = E''(s)$. Then $p(s) \ge 0$ and $\int_0^\infty p(s)ds = 1$. The Gaudin-Mehta distribution ν is defined in the GUE case by $\nu(I) = \int_I p(x)ds$. For the GOE case the kernel h_s is replaced by $\{h_s(x,y) + h_s(-x,y)\}/2$. See also Mehta [85].

Katz and Sarnak and others (see [69], p. 23) have investigated many zeta and L-functions of number theory and have found that "the distribution of the high zeroes of any L-function follow the universal GUE Laws, while the distribution of the low-lying zeroes of certain families follow the laws dictated by symmetries associated with the family. The function field analogs of these phenomena can be established...." More precisely (see [69], p. 11) they show that "the zeta functions of almost all curves C [over a finite field $\mathbb{F}_q$] satisfy the Montgomery-Odlyzko law [GUE] as q and g [the genus] go to infinity."

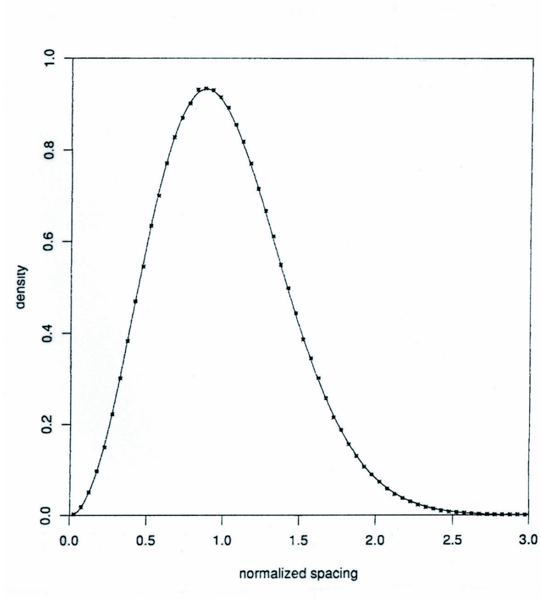


FIGURE 23. **From Cipra** [30]**. Odlyzko's comparison of level spacing of zeros of the Riemann zeta function and that for GUE (Gaussian unitary ensemble).** See Odlyzko and Forrester [41]. The fit is good for the 1,041,000 zeros near the 2×10^{20} zero.

These statistical phenomena are as yet unproved for most of the zeta functions of number theory; e.g., Riemann's. But the experimental evidence of Rubinstein [105] and Strömbergsson [126] and others is strong. Figures 3 and 4 of Katz and Sarnak [69] show the level spacings for the zeros of the L-function corresponding to the modular form Δ and the L-function corresponding to a certain elliptic curve and compare them with GUE. Strömbergsson's web site has similar pictures for L-functions corresponding to Maass wave forms (http://www.math.uu.se/~andreas/zeros.html). All these pictures look GUE. See also the L-function wiki.

The conjectured dichotomy of quantum chaos is illustrated in the following table. The first row gives the Bohigas-Giannoni-Schmit conjecture from 1984 on the left and the Berry-Tabor-Gutzwiller Conjecture from 1977 on the right. Sarnak invented the term "arithmetical quantum chaos" to describe the 2nd row of the table. Recall the definition of arithmetic group from Section 3. Using terminology from André Weil's Columbia U. lectures in 1971, the Laplacians on Riemann surfaces can smell number theory, if it is present.

RMT Spacings (GOE etc.)	**Poisson Spacings**
quantum spectra of a system with chaos in classical counterpart	energy levels of quantum system with integrable system for clasical counterpart
eigenvalues of Laplacian for non-arithmetic manifold	eigenvalues of Laplacian for arithmetic manifold
zeros Riemann zeta	
poles Ihara zeta random regular graph	poles Ihara zeta Cayley graph of an abelian group

Table 1. The Conjectural Dichotomy of Quantum Chaos

Figure 20 shows the Wigner semicircle distribution of the spectrum of a large random symmetric real matrix. When the matrix is the adjacency matrix of a large regular graph under suitable hypotheses, surprisingly, one also sees an approximate semicircle. This was proved by McKay [84]. See Theorem 2 below. In [59] we give a proof due to Nagoshi [93], [94] which uses the Selberg trace formula on the $(q+1)$-regular tree and the Weyl equidistribution theorem. See also Chung et al [28], Mehta [85], and Sunada [130].

Before stating McKay's result, we need to recall Weyl's equidistribution criterion (see Weyl [142] or Iwaniec and Kowalski [64], Chapter 21).

Definition 10. *A sequence $\{x_n\}$ in an interval I on the real line is said to be* ***equidistributed*** *with respect to a measure $d\mu$ iff for every open set B in interval I*

$$\lim_{N\to\infty}\frac{1}{N}\left|\{n\leq N \;\mid\; x_n\in B\}\right|=\mu(B).$$

This equidistribution property is equivalent to the statement that for any continuous function with compact support f on I:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leq N}f(x_n)=\int_I f(x)d\mu(x). \tag{5.1}$$

Definition 11. *The* ***Plancherel measure*** *$d\mu_q$ on $\left[-2\sqrt{q},2\sqrt{q}\right]$ is given by*

$$d\mu_q=\frac{q+1}{2\pi}\frac{\sqrt{4q-\lambda^2}}{(q+1)^2-\lambda^2}d\lambda.$$

The measure $d\mu_q$ is that of the Plancherel theorem, as in [59] where a proof of the following theorem is to be found.

Theorem 2. *(**McKay** [84]) Let $\{X_m\}_{m\geq 1}$ be a sequence of $(q+1)$-regular graphs such that for each $r>0$ we have $\lim\limits_{n\to\infty}\frac{N_r(X_n)}{|V(X_n)|}=0$, where $N_r(X_n)$ is from Definition 9. Then, if $A_{X_m}=$ adjacency matrix of X_m, the spectrum of A_{X_m} becomes equidistributed with respect to the measure $d\mu_q$ on $\left[-2\sqrt{q},2\sqrt{q}\right]$, from Definition 11, as n $\to\infty$. More explicitly, this means that if $[\alpha,\beta]\subset\left[-2\sqrt{q},2\sqrt{q}\right]$,*

$$\lim_{m\to\infty}\frac{\#\left\{\lambda\in Spec\left(A_{X_m}\right)\;\mid\alpha\leq\lambda\leq\beta\right\}}{|X_m|}=\frac{q+1}{2\pi}\int_\alpha^\beta\frac{\sqrt{4q-\lambda^2}}{(q+1)^2-\lambda^2}d\lambda.$$

Derek Newland [95] has investigated the spacings of the poles of the Ihara zeta function for various kinds of graphs. He finds that the pole spacings for large random k-regular graphs appear to be derived from the adjacency matrix eigenvalue spacings being GOE. On the other hand, pole spacings for the Cayley graphs of abelian groups (in particular, the Euclidean graphs in Example below and Terras [132]) appear to be spacings of Poisson random variables. Newland's experiments give the last line for Table 1, the conjectural dichotomy table, namely spacings of poles of zeta of a random graph on the left (essentially GOE) versus spacings of poles of zeta of a Cayley graph of an abelian group (Poisson). Figure 24 shows the result of one such experiment for a random regular graph as given by Mathematica. See Skiena [114] for more information on the way Mathematica deals with graphs. See also the article of Jakobson, Miller, Rivin, and Rudnick in [53], pp. 317-327.

Note: For the level spacing to look GOE or GUE one expects the zeta function not to be a product of other zetas or L-functions. The zeta function of a Cayley graph such as the Euclidean graph in Figure 24 will be a product over the representations of the group. Thus it behaves more like a lot of different zeta functions rather than one zeta. This is the sort of behavior we saw when comparing Figures 20 and 21. See Farmer [40] for some information about random matrix theory and families of L-functions.

Exercise 14. *Compute the change of variables between an element of the spectrum of the adjacency matrix and the imaginary part of s when q^{-s} is a pole of ζ_X for a $(q+1)$-regular graph X.*

Exercise 15. *Do a similar experiment to that of Figure 24 for a Cayley graph of your choice or for an n-cover of $X=K_4-edge$.*

As a final project for Part I, you might try to list all the zeta functions that you can find and figure out what they are good for. There are lots of them. We have left out zeta functions of codes for example.

Later we will consider the spacings of poles of irregular graphs and covering graphs.

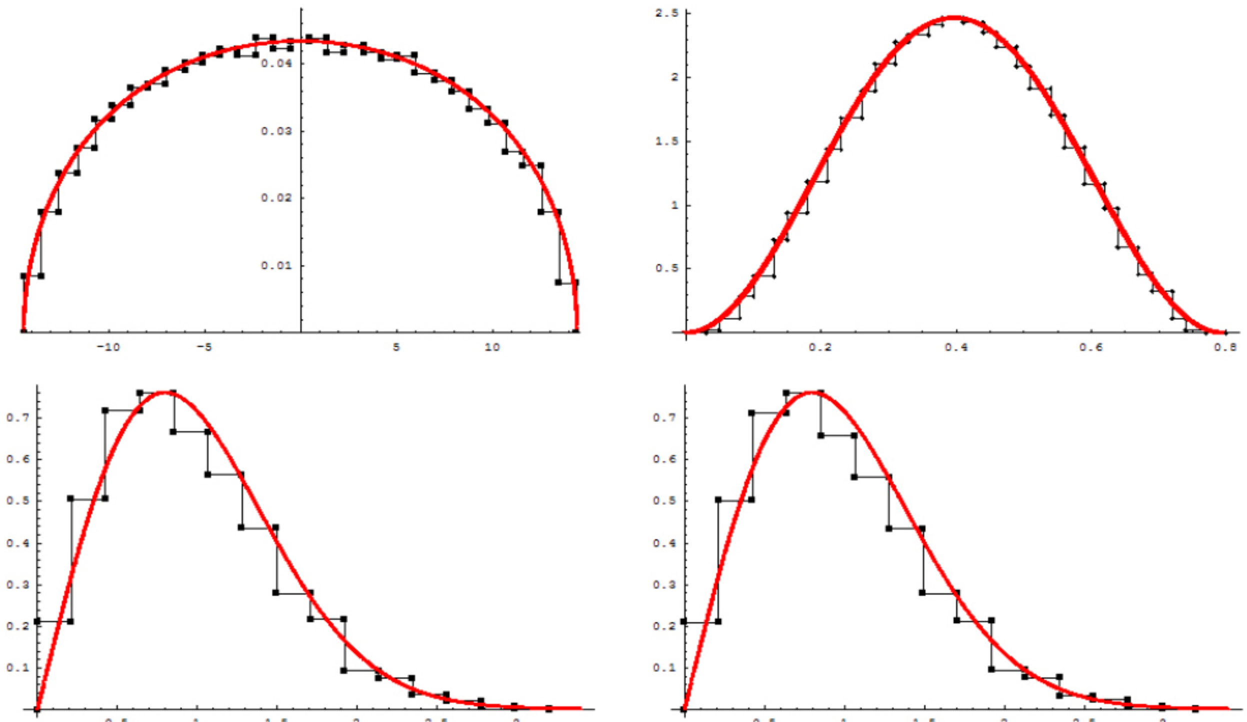


FIGURE 24. Taken from Newland [95]. For a pseudo-random regular graph with degree 53 and 2000 vertices, generated by Mathematica, the **top row shows the distributions for eigenvalues of adjacency matrix on left and imaginary parts of the Ihara zeta poles on right. The bottom row contains their respective level spacings.** The red line on bottom left is the Wigner surmise for the GOE $y = (\frac{\pi x}{2})e^{-\frac{\pi x^2}{4}}$.

Part 2. The Ihara Zeta Function and the Graph Theory Prime Number Theorem

The graph theory zetas first appeared in work of Ihara on p-adic groups in the 1960s (see [62]). Serre (see [113]) made the connection with graph theory. The main authors on the subject in the 1980s and 90s were Sunada [128], [129], [130], Hashimoto [50], [51], and Bass [12]. Other references are Venkov and Nikitin [138] and Northshield's paper in the volume of Hejhal et al [53]. The main properties of the Riemann zeta function have graph theory analogs, at least for regular graphs. For irregular graphs there is no known functional equation and it is difficult to formulate the Riemann hypothesis, but we will try. In later sections, we will consider the multivariable zeta functions known as edge and path zeta functions of graphs. We will show how to specialize the path zeta to the edge zeta and the edge zeta to the original one variable Ihara (vertex) zeta.

Much of our discussion can be found in the papers of the author and Harold Stark [119], [120], [121]. Topics for this section will include the graph theory prime number theorem and our version of Bass's proof of the determinant formula (which was Theorem 1 in the previous section). We will see how to modify the definitions to obtain zeta functions of weighted or metric graphs. We will find out what happens to the edge zeta if you delete an edge (fission) and what happens to the path zeta if you collapse an edge to a point (fusion or contraction). We will cover Artin L-functions of Galois graph coverings from [120] in later sections.

We do not consider zeta functions of infinite graphs here. Such zeta functions are discussed, for example, by Bryan Clair and Shahriar Mokhtari-Sharghi [31], Rostislav Grigorchuk and Andrzej Zuk [46], and Daniele Guido, Tommaso Isola, and Michel Lapidus [48]. Nor do we consider directed graphs. Zeta functions for such graphs are discussed, for example, by Matthew Horton [57], [58]. There are also extensions to hypergraphs (see Cristopher Storm [124]) and buildings (see Ming-Hsuan Kang, Wen-Ching Winnie Li, and Chian-Jen Wang [66]).

Throughout this section we will assume Theorem 1 of Ihara. It will be proved in the next part.

6. The Ihara Zeta Function of a Weighted Graph

Many applications involve weighted or metric graphs; that is, graphs with positive real numbers attached to the edges to represent lengths or resistance or some other physical attribute. In particular, quantum graphs are weighted (see [14]). Other references for weighted graphs are Fan Chung and S.T. Yau [29] or Osborne and Severini [96]. For the most part we will not consider weighted graphs here but let us at least give a natural extension of the definition of the Ihara zeta function to weighted graphs.

Definition 12. *For a graph X with oriented edge set $\overrightarrow{E}$, consisting of $2|E|$ oriented edges, suppose we have a weighting function $L : \overrightarrow{E} \to \mathbb{R}^+$. Then define the **weighted length** of a closed path $C = a_1 a_2 \cdots a_s$, where $a_j \in \overrightarrow{E}$, by*

$$\nu(C, L) = \nu_X(C, L) = \sum_{i=1}^{s} L(a_i).$$

Definition 13. *The **Ihara zeta function of a weighted (undirected) graph** for $|u|$ small and $u \notin (-\infty, 0)$ is*

$$\zeta_X(u, L) = \prod_{[P]} \left(1 - u^{\nu(P,L)}\right)^{-1}.$$

Clearly when $L = 1$, meaning the function such that $L(e) = 1$ for all edges e in X, we have $\zeta_X(u, 1) = \zeta_X(u)$, our original Ihara zeta function.

Definition 14. *Given a graph X with positive integer-valued weight function L, define the **inflated graph** X_L in which each edge e is replaced by an edge with $L(e) - 1$ new degree 2 vertices.*

Then clearly $\nu_X(C, L) = \nu_{X_L}(C, 1)$,where the 1 means again that $1(e) = 1$, for all edges e. It follows that **for positive integer-valued weights L, we have the identity relating the weighted zeta and the ordinary Ihara zeta**:

$$\zeta_X(u, L) = \zeta_{X_L}(u).$$

It follows that $\zeta_X(u, L)^{-1}$ is a polynomial for integer valued weights L.

For non-integer weights, it is possible to obtain a determinant formula using the edge zeta functions in section 11. See also Mizuno and Sato [89].

Example 5. ***Inflation of K_5.*** *Suppose $Y = K_5$, the complete graph on 5 vertices. Let $L(e) = 5$ for each of the 10 edges of X. Then $X = Y_L$ is the graph on the left in Figure 26. The new graph X has 45 vertices (4 new vertices on the 10 edges of K_5). One sees easily that*

$$\zeta_X(u)^{-1} = \zeta_{K_5}(u^5)^{-1} = (1 - u^{10})^5 (1 - 3u^5)(1 - u^5)(1 + u^5 + 3u^{10}).$$

Exercise 16. *What happens to $\zeta_Y(u)$ if $Y = X_{L_n}$, for $L_n = n$, as $n \to \infty$?*

For the most part, we shall restrict our discussion to non-weighted graphs from now on.

7. Regular Graphs, Location of Poles of Zeta, Functional Equations

Next we want to consider the Ihara zeta function for regular graphs (which are unweighted and satisfying our usual hypotheses for the most part). We need some facts from graph theory first. References for the subject include Biggs [15], Bollobas [18], [19], Cvetković, Doob and Sachs [32].

Definition 15. *A graph is a **bipartite graph** iff the set of vertices can be partitioned into 2 disjoint sets S, T such that no vertex in S is adjacent to any other vertex in S and no vertex in T is adjacent to any other vertex in T.*

Exercise 17. *Show that an example of a bipartite graph is the cube of Figure 14.*

Proposition 2. ***Facts about Spectrum(A), when A is the adjacency operator of a connected $(q+1)$-regular graph X.*** *Assume that X is a connected $(q+1)$-regular graph and that A is its adjacency matrix.*
1) $\lambda \in \mathrm{Spectrum}(A)$ implies $|\lambda| \leq q+1$.
2) $q+1 \in \mathrm{Spectrum}(A)$ and it has multiplicity 1.
3) $-(q+1) \in \mathrm{Spectrum}(A)$ iff the graph X is bipartite.

To prove fact 1), note that $(q+1)$ is clearly an eigenvalue of A corresponding to the constant vector. Suppose $Av = \lambda v$, for some vector $v = {}^t(v_1, ..., v_n) \in \mathbb{R}^n$. And suppose that the maximum of the $|v_i|$ occurs at $i = a$. Then, using the notation $b \sim a$, to mean the bth vertex is adjacent to the ath, we have

$$|\lambda|\,|v_a| = \left|(Av)_a\right| = \left|\sum_{b\sim a} v_b\right| \leq (q+1)\,|v_a|\,.$$

Fact 1) follows.

To prove fact 2), suppose $Av = (q+1)v$, for some non-0 vector $v = {}^t(v_1, ..., v_n) \in \mathbb{R}^n$. Again suppose that the maximum of the $|v_i|$ occurs at $i = a$. We can assume $v_a > 0$, by multiplication of the vector v by -1. As in the proof of fact 1,

$$(q+1)v_a = (Av)_a = \sum_{b\sim a} v_b \leq (q+1)v_a.$$

To have equality, there can be no cancellation in this sum and $v_b = v_a$, for each b adjacent to a. Since we assume that X is connected, we can iterate this argument and conclude that v must be the constant vector.

Exercise 18. *a) Prove fact 3) above.*
b) Show that, if $(q+1)$ has multiplicity 1 as an eigenvalue of the adjacency matrix of a $(q+1)$-regular graph, then this graph must be connected.

Definition 16. *Suppose that X is a connected $(q+1)$-regular graph (without degree 1 vertices). We say that the Ihara zeta function $\zeta_X(q^{-s})$ satisfies the **Riemann hypothesis** iff when $0 < \mathrm{Re}\, s < 1$,*

$$\zeta_X(q^{-s})^{-1} = 0 \implies \mathrm{Re}\, s = \frac{1}{2}.$$

Note that if $u = q^{-s}$, $\mathrm{Re}\, s = \frac{1}{2}$ corresponds to $|u| = \frac{1}{\sqrt{q}}$.

Theorem 3. *For a connected $(q+1)$-regular graph X, $\zeta_X(u)$ satisfies the Riemann hypothesis iff the graph X is Ramanujan in the sense of Definition 4.*

Proof. Use theorem 1 to see that

$$\zeta_X(q^{-s})^{-1} = (1-u^2)^{r-1} \prod_{\lambda \in spectrum(A)} \left(1 - \lambda u + qu^2\right).$$

Write $1 - \lambda u + qu^2 = (1-\alpha u)(1-\beta u)$, where $\alpha\beta = q$ and $\alpha + \beta = \lambda$. Note that α, β are the reciprocals of poles of $\zeta_X(u)$. Using the facts in Proposition 2 above, we have 3 cases.
Case 1. $\lambda = \pm(q+1)$ implies $\alpha = \pm q$ and $\beta = \pm 1$.
Case 2. $|\lambda| \leq 2\sqrt{q}$ implies $|\alpha| = |\beta| = \sqrt{q}$.
Case 3. $2\sqrt{q} < |\lambda| < q+1$ implies $\alpha, \beta \in \mathbb{R}$ and $1 < |\alpha| = |\beta| < q$, $\quad |\alpha| = |\beta| \neq \sqrt{q}$.

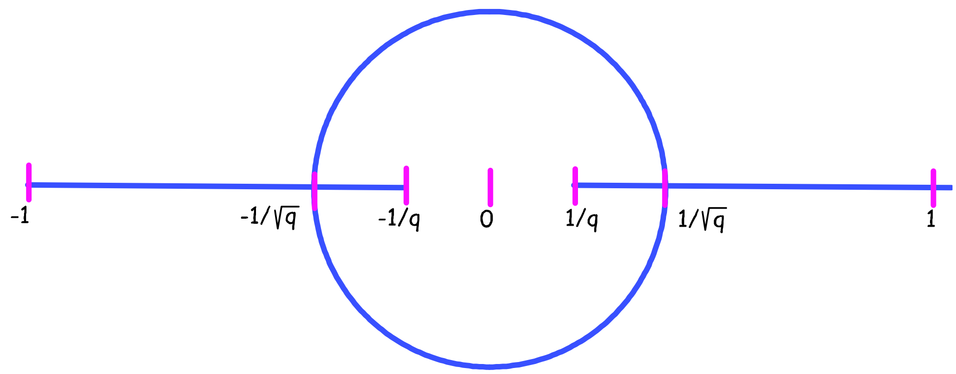


FIGURE 25. **Possible locations for poles of $\zeta_X(u)$ for a regular graph** are marked in blue. The circle corresponds to the part of the spectrum of the adjacency matrix satisfying the Ramanujan inequality. The real poles correspond to the non-Ramanujan eigenvalues of A, except for the two poles on the circle itself and the endpoints of the intervals.

To see these things, let u be either α^{-1} or β^{-1}. Then by the quadratic formula, we have α or $\beta = u^{-1}$ where

$$u = \frac{\lambda \pm \sqrt{\lambda^2 - 4q}}{2q}.$$

Cases 1 and 2 are easily seen. We leave them as **Exercises**.

To understand case 3, first assume $\lambda > 0$ and note that $u = \frac{\lambda + \sqrt{\lambda^2 - 4q}}{2q}$ is a monotone increasing function of λ. This implies that the larger root u is in the interval $\left(\frac{1}{\sqrt{q}}, 1\right)$. Where is the smaller root $u' = \frac{\lambda - \sqrt{\lambda^2 - 4q}}{2q}$? Answer: $|u'| \in \left(\frac{1}{q}, \frac{1}{\sqrt{q}}\right)$. Here we use the fact that $uu' = \frac{1}{q}$. A similar argument works for negative λ (**Exercise**).

The proof of the theorem is finished by noting that when $u = q^{-s}$, case 2 is $\operatorname{Re} s = \frac{1}{2}$. $\square$

Figure 25 shows the possible locations of poles of the Ihara zeta function of a $(q+1)$-regular graph. The poles satisfying the Riemann hypothesis are those on the circle. The circle basically corresponds to case 2 in the preceding proof. The real axis corresponds to Cases 1 and 3.

Exercise 19. *Fill in the details in the proof of the preceding theorem. Then show that Figure 25 shows the possible locations of poles of the Ihara zeta function of a $(q+1)$-regular graph. Label the places on the figure corresponding to the three cases in the proof.*

Exercise 20. *Show that the radius of convergence of the product defining the Ihara zeta function of a $(q+1)$-regular graph is $R_X = \frac{1}{q}$.*

The following proposition gives some functional equations of the Ihara zeta function for a regular graph. If we set $u = q^{-s}$, the functional equations relate the value at s with that at $1-s$, just as is the case for the Riemann zeta function.

Proposition 3. *Suppose that X is a $(q+1)$-regular connected graph without degree 1 vertices with $n = |V|$. Then we have the following **functional equations** among others.*

1) $\Lambda_X(u) = (1-u^2)^{r-1+\frac{n}{2}} \left(1 - q^2u^2\right)^{\frac{n}{2}} \zeta_X(u) = (-1)^n \Lambda_X\left(\frac{1}{qu}\right).$

2) $\xi_X(u) = (1+u)^{r-1} (1-u)^{r-1+n} (1-qu)^n \zeta_X(u) = \xi_X\left(\frac{1}{qu}\right).$

3) $\Xi_X(u) = (1-u^2)^{r-1} (1+qu)^n \zeta_X(u) = \Xi_X\left(\frac{1}{qu}\right).$

Proof. We will prove part 1) and leave the rest as an **Exercise**. To see part 1), write

$$\begin{aligned}\Lambda_X(u) &= (1-u^2)^{\frac{n}{2}}\left(1-q^2u^2\right)^{\frac{n}{2}}\det\left(I-Au+qu^2I\right)^{-1}\\ &= \left(\frac{q^2}{q^2u^2}-1\right)^{\frac{n}{2}}\left(\frac{1}{q^2u^2}-1\right)^{\frac{n}{2}}\det\left(I-A\frac{1}{qu}+\frac{q}{(qu)^2}I\right)^{-1}\\ &= (-1)^n\Lambda_X(\frac{1}{qu}).\end{aligned}$$

□

Exercise 21. *Prove parts 2 and 3 of Proposition 3.*

Look at figure 25. What sort of symmetry is indicated by the functional equations which imply that if $\zeta_X(u)$ has a pole at u, then it must also have a pole at $\frac{1}{qu}$? If u is on the circle then $\frac{1}{qu}$ is the complex conjugate of u. If u is in the interval $\left(\frac{1}{\sqrt{q}},1\right)$, then $\frac{1}{qu}$ is in the interval $\left(\frac{1}{q},\frac{1}{\sqrt{q}}\right)$.

To produce examples of regular graphs, the easiest method is to start with a generating set S of your favorite finite group G. Assume that S is symmetric, meaning that $s\in S$ implies $s^{-1}\in S$. Create a graph called a Cayley graph. It allows you to visualize the group. You can add colors and directions on the edges to get an even better picture but we won't do that. The name comes from the mathematician Arthur Cayley.

Definition 17. *The* ***Cayley graph,*** *denoted $X(G,S)$, has as its vertices the elements of G and has edges between vertex g and gs for all $g\in G$ and $s\in S$. Here S is a symmetric generating set for G.*

Cayley graphs are always regular with degree $|S|$. We want S to be a generating set because we want a connected graph. We want S to be symmetric for the graph to be undirected.

The cube is $X(\mathbb{F}_2^3,S)$, where $\mathbb{F}_2$ denotes the field with 2 elements, $\mathbb{F}_2^3$ is the additive group of 3-vectors with entries in this field, and $S=\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$. Another example is the Paley graph (named for another mathematician whose name rhymes with Cayley) considered in subsection 9.2 below.

We considered a large number of Cayley graphs in Terras [132]. One example is $X(\mathbb{F}_q^n,S)$, where S consists of solutions $x\in\mathbb{F}_q^n$ of the equation $x_1^2+\cdots+x_n^2=a$ for some $a\in\mathbb{F}_q$. We called such graphs "Euclidean." There are also "non-Euclidean" graphs associated to finite fields where the distance is replaced by a finite analog of the Poincaré distance in the upper half plane. The question of whether these Euclidean and non-Euclidean graphs are Ramanujan can be translated into a question about bounds on exponential sums. See Subsection 9.2 below.

More examples of regular graphs come from Lubotzky, Phillips and Sarnak [79]. See Subsection 9.2 below and [35]. Mathematica will create "random" regular graphs with the command X=RegularGraph[d,n]; where d=degree and n=number of vertices.

Exercise 22. *Consider examples of regular graphs such as those mentioned above and find out whether they are Ramanujan graphs. Then plot the poles of the Ihara zeta function. You might also look at the level spacings of the poles as in Figure 24.*

Exercise 23. *Show that, if X is a non-bipartite k-regular graph with $k\geq 3$, then $1=\Delta_X=g.c.d.$ of the prime lengths, as in Definition 7.*

Exercise 24. *Consider the zeta function of the graph X obtained by removing one edge from the tetrahedron graph. Does the Ihara zeta function of X satisfy a functional equation of the form in Proposition 3 with $u\to u/q$ replaced by $u\to Ru$, where the radius of convergence of the Ihara zeta $R=R_X$ is from Definition 3.*

8. Irregular Graphs: What is the RH?

Next let us speak about irregular graphs (again unweighted and satisfying our usual hypotheses). Kotani and Sunada [72] prove the following theorem.

Theorem 4. *(**Kotani and Sunada**). Suppose the graph X satisfies our usual hypotheses (see the paragraphs before Definition 1) and has vertices with maximum degree $q+1$ and minimum degree $p+1$.*

1) Every pole u of $\zeta_X(u)$ satisfies $R_X \leq |u| \leq 1$, with R_X from Definition 3, and

$$q^{-1} \leq R_X \leq p^{-1}. \tag{8.1}$$

2) Every non-real pole u of $\zeta_X(u)$ satisfies the inequality

$$q^{-1/2} \leq |u| \leq p^{-1/2}. \tag{8.2}$$

3) The poles of ζ_X on the circle $|u| = R_X$ have the form $R_X e^{2\pi i a/\Delta_X}$, where $a = 1, ..., \Delta_X$. Here Δ_X is from Definition 7.

Proof. We postpone this proof until the section on the edge zeta function. □

Horton [57] gives examples of graphs such that R_X is as close as you want to a given positive real number such as π or e.

Now let us define two constants associated to the graph X.

Definition 18.

$$\rho_X = \max\{|\lambda| \mid \lambda \in \text{spectrum}(A_X)\},$$
$$\rho'_X = \max\left\{|\lambda| \;\middle|\; \lambda \in \text{spectrum}(A_X),\ |\lambda| \neq \rho_X\right\}.$$

We will say that the **naive Ramanujan inequality** is

$$\rho'_X \leq 2\sqrt{\rho_X - 1}. \tag{8.3}$$

Lubotzky [78] has defined X to be **Ramanujan** if

$$\rho'_X \leq \sigma_X. \tag{8.4}$$

where σ_X is the **spectral radius of the adjacency operator on the universal covering tree** of X. Recall that the spectral radius of the operator A is the supremum of $|\lambda|$ such that $A - \lambda I$ has no inverse (as a bounded linear operator on the tree). See Terras [133] for more information on spectral theory and Terras [132]. Both inequalities (8.3) and (8.4) reduce to the usual definition of Ramanujan for connected regular graphs.

Definition 19. $\overline{d_X}$ *denotes the* ***average degree of the vertices*** *of X.*

Hoory [54] has proved the following theorem.

Theorem 5.

$$2\sqrt{\overline{d_X} - 1} \leq \sigma_X.$$

Proof. For the special case that the graph is regular, the proof will essentially be given below when we prove the result of Alon and Boppana which is Theorem 8. For the irregular case, the reader is referred to Hoory [54]. □

From Theorem 5 one has a criterion for a graph X to be Ramanujan in Lubotzky's sense. It need only satisfy the **Hoory inequality**

$$\rho'_X \leq 2\sqrt{\overline{d_X} - 1}. \tag{8.5}$$

To develop the RH for irregular graphs, the natural change of variable is $u = R_X^s$ with R_X from Definition 3. All poles of $\zeta_X(u)$ are then located in the "critical strip", $0 \leq \text{Re}(s) \leq 1$ with poles at $s = 0$ ($u = 1$) and $s = 1$ ($u = R_X$). The examples below show that, for irregular graphs, one cannot expect a functional equation relating $f(s) = \zeta(R_X^s, X)$ and $f(1-s)$. Therefore it is natural to say that the Riemann hypothesis for X should require that $\zeta_X(u)$ has no poles in the open strip $1/2 < \text{Re}(s) < 1$. This is the graph theory RH below. After looking at examples, it seems that one rarely sees an Ihara zeta satisfying this RH (although random graphs do seem to approximately satisfy the RH). Thus we also consider the weak graph theory RH below.

Graph theory RH $\zeta_X(u)$ is pole free for

$$R_X < |u| < \sqrt{R_X}. \tag{8.6}$$

Weak graph theory RH $\zeta_X(u)$ is pole free for

$$R_X < |u| < 1/\sqrt{q}. \tag{8.7}$$

Note that (8.6) and (8.7) are the same if the graph is regular. We have examples (such as Example 6 below) for which $R_X > q^{-\frac{1}{2}}$ and in such cases the weak graph theory RH is true but vacuous. In [121] we give a longer discussion of the preceding 2 versions of the RH for graphs showing the connections with the versions for the Dedekind zeta function and the existence of Siegel zeros. We will define Siegel zeros for the Ihara zeta function later.

Sometimes number theorists state a modified GRH = Generalized Riemann Hypothesis for the Dedekind zeta function and this just ignores all possible real zeros while only requiring the non-real zeros to be on the line $\mathrm{Re}(s) = \frac{1}{2}$. The graph theory analog of the modified weak GRH would just ignore the real poles and require that there are no non-real poles of $\zeta_X(u)$ in $R_X < |u| < q^{-1/2}$. But this is true for all graphs by Theorem 4: if μ is a pole of $\zeta_X(u)$ and $|\mu| < q^{-1/2}$ then μ is real!

One may ask about the relations between the constants $\rho_X, \overline{d_X}, R_X$. One can show (**Exercise**) that

$$\rho_X \geq \overline{d_X}. \tag{8.8}$$

This is easily seen using the fact that ρ_X is the maximum value of the Rayleigh quotient $\langle Af, f\rangle / \langle f, f\rangle$, for any non-0 vector f in $\mathbb{R}^n$, while $\overline{d_X}$ is the value when f is the vector all of whose entries are 1. In all the examples to date we see that

$$\rho_X \geq 1 + \frac{1}{R_X} \geq \overline{d_X} \tag{8.9}$$

but can only show (see Proposition 5 in Subsection 11.2 below) that

$$\rho_X \geq \frac{p}{q} + \frac{1}{R_X}.$$

As a **research problem**, the reader might want to investigate the possible improvement to formula (8.9).

Next we give some examples including answers to the questions: Do the spectra of the adjacency matrices satisfy the naive Ramanujan inequality (8.3) or the Hoory inequality (8.5)? Do the Ihara zeta functions for the graphs have the pole-free region (8.7) of the weak graph theory RH or the pole-free region (8.6) of the full graph theory RH?

Example 6. *Let X be **the graph obtained from the complete graph on 5 vertices by adding 4 vertices to each edge** as shown on the left in Figure 26.*

For the graph X, we find that $\rho' \approx 2.32771$ and

$$\{\rho, 1 + 1/R, \overline{d_X}\} \approx \{2.39138, 2.24573, 2.22222\}.$$

This graph satisfies the naive Ramanujan inequality (8.3) but not the Hoory inequality (8.5). The magenta points in the picture on the right in Figure 26 are the poles not equal to -1 of $\zeta_X(u)$. Here

$$\zeta_X(u)^{-1} = \zeta_{K_5}(u^5)^{-1} = \left(1 - u^{10}\right)^5 \left(1 - 3u^5\right) \left(1 - u^5\right) \left(1 + u^5 + 3u^{10}\right).$$

The circles in the picture on the right in Figure 26 are centered at the origin with radii

$$\left\{q^{-\frac{1}{2}}, R, R^{\frac{1}{2}}, p^{-\frac{1}{2}}\right\} \approx \{0.57735, 0.802742, 0.895958, 1\}.$$

The zeta function satisfies the RH and thus the weak RH. However the weak RH is vacuous.

Example 7. ***Random graph with probability 1/2 of an edge.***

The magenta points in figure 27 are the poles not equal to ± 1 of the Ihara zeta function of a random graph produced by Mathematica with the command RandomGraph[100,1/2]. This means there are 100 vertices and the probability of an edge between any 2 vertices is 1/2. The graph satisfies the Hoory inequality (8.5) and it is thus Ramanujan in Lubotzky's sense. It also satisfies the naive Ramanujan inequality (8.3). We find that $\rho' \approx 10.0106$ and $\{\rho, 1 + 1/R, \overline{d_X}\} \approx \{50.054, 50.0435, 49.52\}$. The circles in Figure 27 are centered at the origin and have radii given by $\{q^{-\frac{1}{2}}, R^{\frac{1}{2}}, p^{-\frac{1}{2}}\} \approx \{0.130189, 0.142794, 0.166667\}$. The poles of the zeta function satisfy the weak RH but not the RH. However, the RH seems to be approximately true. See Skiena [114] *for more information on the model that Mathematica uses to produce random graphs.*

Example 8. ***Torus minus some edges.***

From the torus graph T which is the product of a 10-cycle and a 20-cycle, we delete 6 edges to obtain a graph we will call N which is on the left in Figure 28. The spectrum of the adjacency matrix of N satisfies neither the Hoory inequality (8.5) nor the naive Ramanujan inequality (8.3). We find that $\{\rho, 1 + 1/R, \overline{d}\} \approx \{3.98749, 3.98568, 3.98\}$, and $\rho' \approx 3.90275$. The right hand side of Figure 28 shows the poles of the Ihara zeta for N as magenta points. The circles are centered at the origin and have radii $\{q^{-\frac{1}{2}}, R^{\frac{1}{2}}, p^{-\frac{1}{2}}\} \approx \{0.57735, 0.57873, 0.70711\}$. The zeta poles satisfy neither the graph theory weak RH nor the RH.

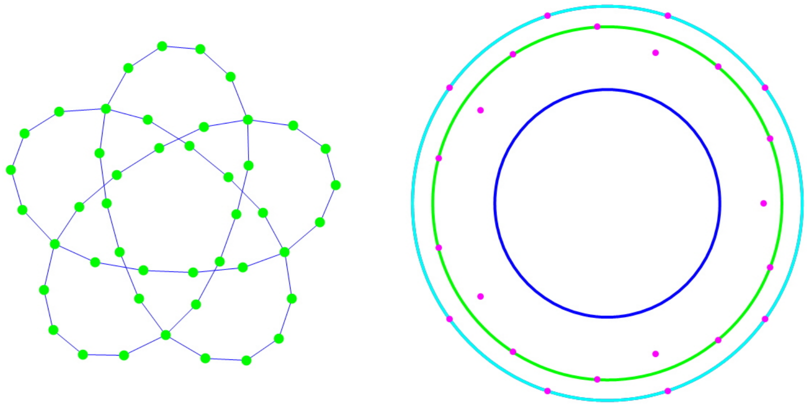


FIGURE 26. On the left is the graph $X = Y_5$ obtained by adding 4 vertices to each edge of $Y = K_5$, the complete graph on 5 vertices. On the right the **poles** ($\neq -1$) **of the Ihara zeta function** of X are the magenta points. The circles have centers at the origin and radii $\left\{q^{-\frac{1}{2}}, R^{\frac{1}{2}}, p^{-\frac{1}{2}}\right\}$. Note the 5-fold rotational symmetry of the poles.

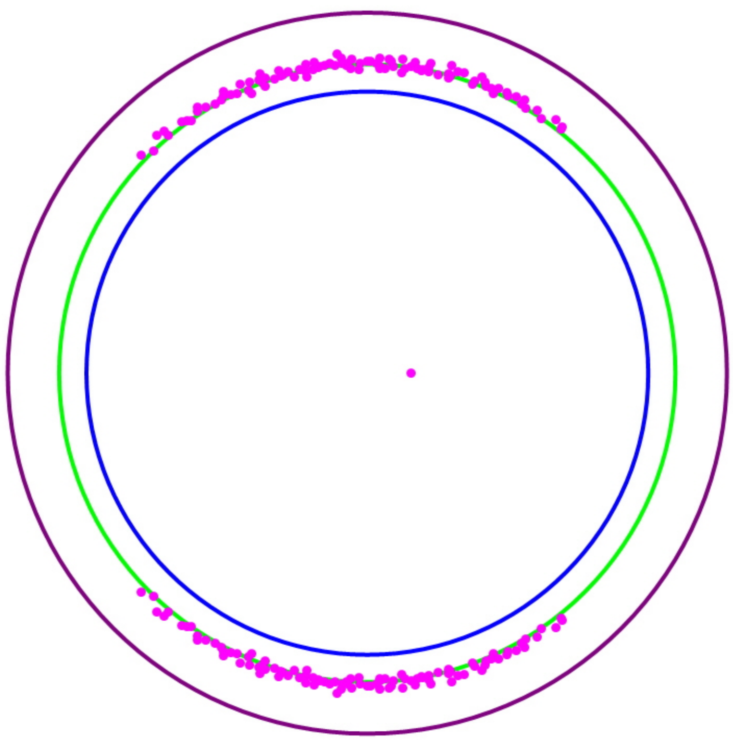


FIGURE 27. The magenta points are **poles** ($\neq \pm 1$) **of the Ihara zeta function for a random graph produced by Mathematica** with the command RandomGraph[100,$\frac{1}{2}$]. The circles have centers at the origin and radii $\left\{q^{-\frac{1}{2}}, R^{\frac{1}{2}}, p^{-\frac{1}{2}}\right\}$. The RH looks approximately true but is not exactly true. The weak RH is true.

Exercise 25. *Consider the graph $X = K_n - edge$.*

a) Does the zeta function for X satisfy the RH?

b) How about the weak RH?

c) Does X satisfy the naive Ramanujan inequality, formula (8.3)?

d) Does X satisfy the Hoory inequality, formula (8.5)?

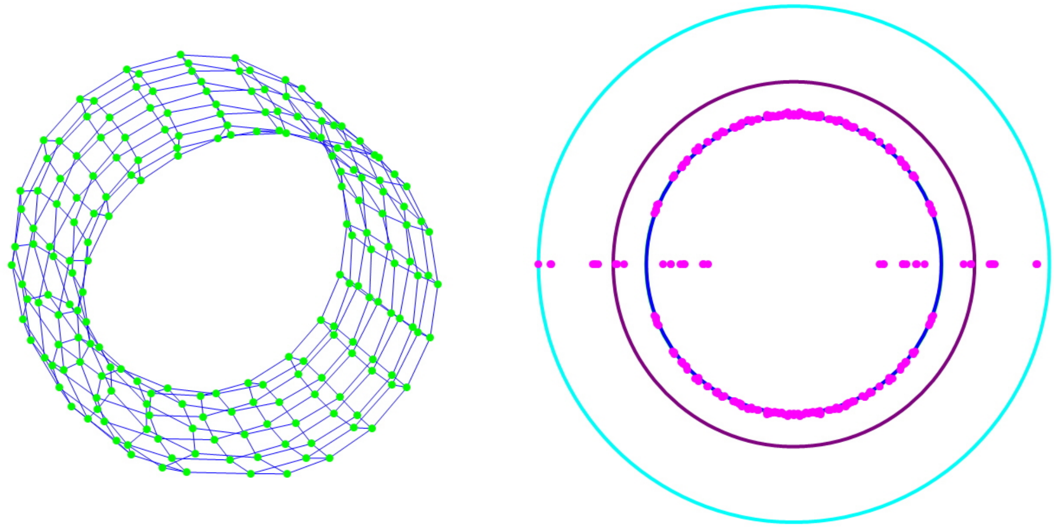


FIGURE 28. The graph N on the left results from deleting 6 edges from the product of a 10-cycle and a 20-cycle. In the picture on the right, magenta points indicate the **poles ($\neq \pm 1$) of the Ihara zeta function** of N. The circles are centered at the origina with radii $\{1/\sqrt{q}, \sqrt{R}, 1/\sqrt{p}\}$. The Ihara zeta function satisfies neither the RH nor the weak RH.

e) Is there a functional equation relating $\zeta_X(u)$ *and* $\zeta_X(R_X/u)$?

Hint. See [121] where most of these questions are answered explicitly.

Example 9. *Figures 29 and 30 show the results of some Mathematica experiments on the distribution of the poles of zeta for 2 graphs. The graphs were constructed using the RealizeDegreeSequence command as well as the commands GraphUnion and Contract. The top row show the graphs. The second row shows the histogram of degrees. The magenta points in the last row are poles of the Ihara zetas corresponding to the graphs on the top row. Many poles violate the RH by being inside the green circle rather than outside. Those poles violating the weak RH are inside the inner circle. No such poles occur for the graph in Figure 30 but there are such poles for the graph in Figure 29.*

Exercise 26. *Do more examples in the spirit of the preceding figures. The pole locations in the Figures 27 and 28 do not appear too different from those for a regular graph in Figure 25. The easiest way to see a more 2-dimensonal pole picture, would be to use the command RandomGraph[100,.1]. There are many other ways to construct new graphs in Mathematica, for example, those used to obtain Figures 29 and 30, where we used the RealizeDegreeSequence command as well as the commands GraphUnion and Contract. There are also many examples of covering graphs in later sections of this book. Or you could construct some zig-zag products (see Hoory et al* [55] *).*

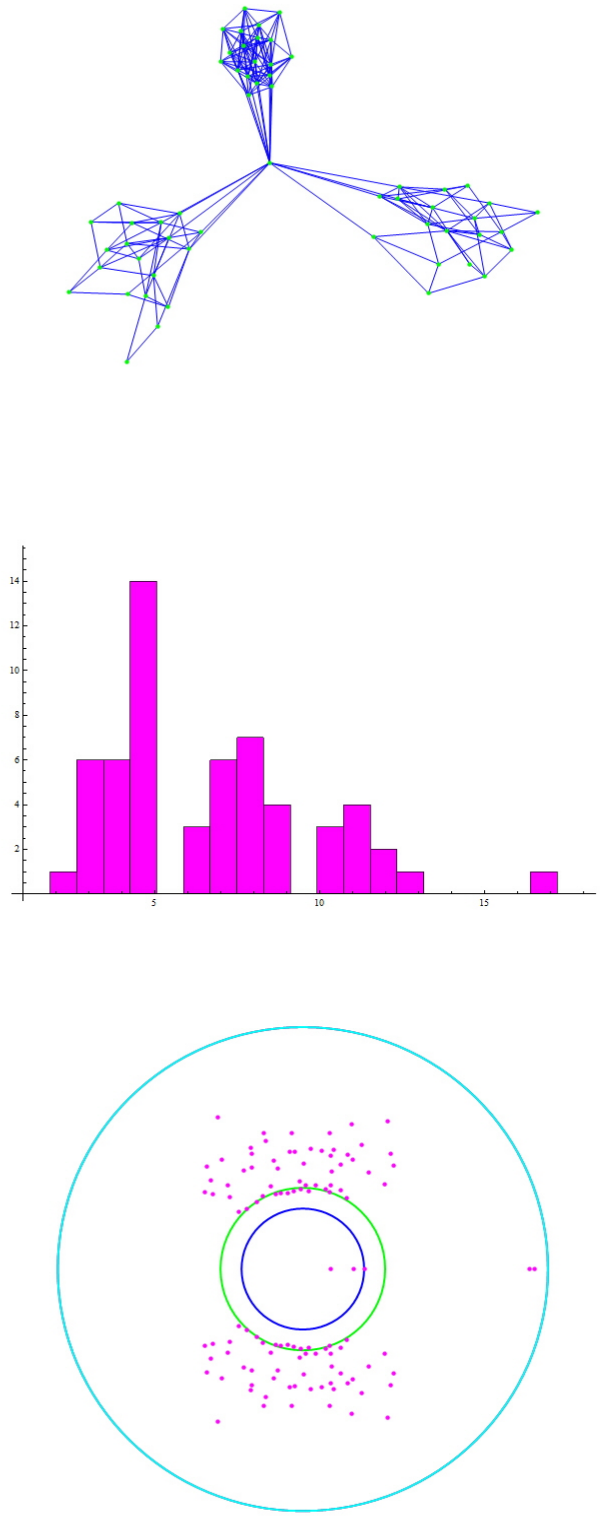


FIGURE 29. **A Mathematica Experiment. The top row shows the graph. The middle row shows the histogram of degrees. In the bottom row, the magenta points are poles of the Ihara zeta function of the graph.** The middle green circle is the Riemann hypothesis circle with radius $\sqrt{R}$, where R is the closest pole to 0. The inner circle has radius $\frac{1}{\sqrt{q}}$, where $q+1$ is the maximum degree of the graph. The outer circle has radius 1. For this graph $p = 1$ and thus the circle of radius $\frac{1}{\sqrt{p}}$ coincides with the circle of radius 1. Many poles are inside the green middle circle and thus violate the Riemann hypothesis. For this graph, the Riemann hypothesis and the weak Riemann hypothesis are false as is the naive Ramanujan inequality. The probability of an edge is 0.119177.

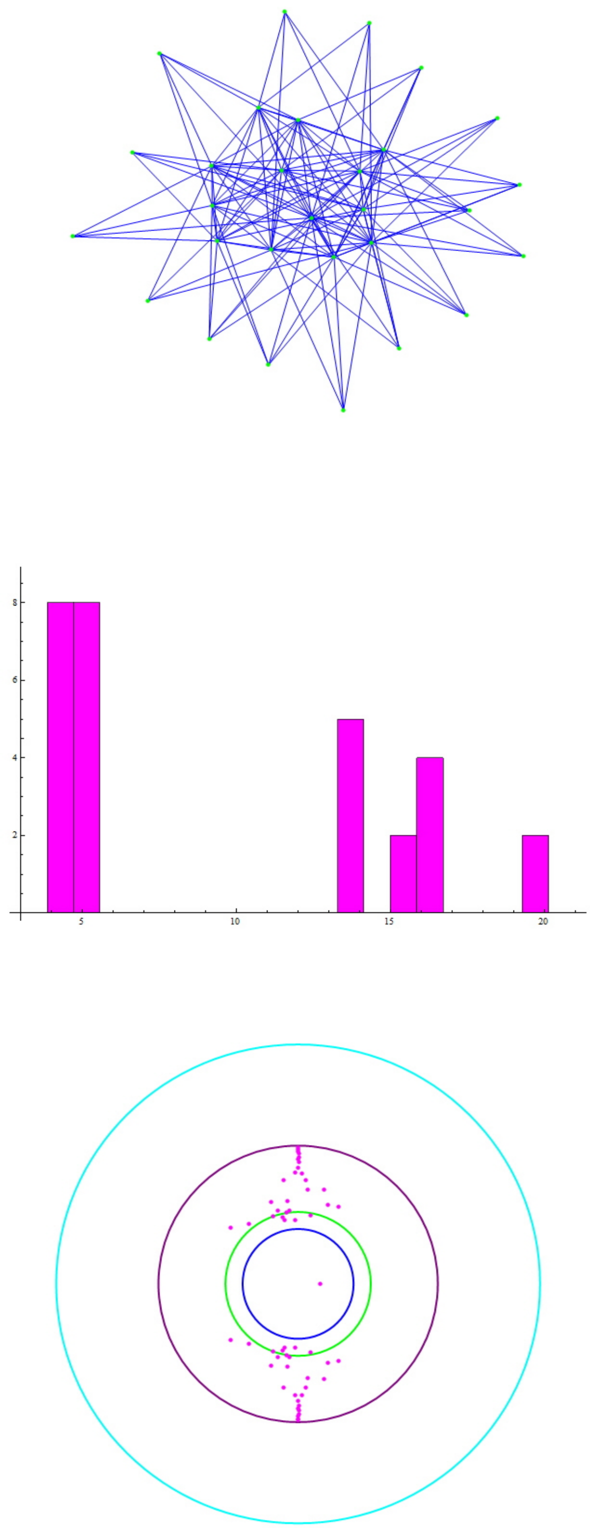


FIGURE 30. **A Mathematica Experiment. The top row shows the graph. The middle row shows the histogram of degrees. In the bottom row, the magenta points are poles of the Ihara zeta function of the corresponding graph.** The inner circle has radius $1/\sqrt{q}$, where $q+1$=maximum degree of the graph. The next circle out (the green circle) is the Riemann hypothesis circle with radius $\sqrt{R}$, where R is the closest pole to 0. The outer circle has radius 1. The circle just inside this one has radius $\frac{1}{\sqrt{p}}$, where $p+1$ is the minimum degree of the graph. For this graph, the Riemann hypothesis is false, but the weak Riemann hypothesis is true as well as the naive Ramanujan inequality. The probability of an edge is 0.339901 for this graph.

9. Discussion of Regular Ramanujan Graphs

In this section we restrict ourselves to regular graphs. Our goals are

(1) to explain why a random walker gets lost fast on a Ramanujan graph;
(2) give examples of regular Ramanujan graphs;
(3) show why the Ramanujan bound is best possible;
(4) explain why Ramanujan graphs are good expanders;
(5) give a diameter bound for a Ramanujan graph.

9.1. **Random Walks on Regular Graphs.** Suppose that A is the adjacency matrix of a k-regular graph X with n vertices. We get a **Markov chain** from A as follows. The states are the vertices of X. At time t, the process (walker) goes from the ith state to the jth state with probability p_{ij} given by $\frac{1}{k}$ if vertex i is adjacent to vertex j and with 0 probability otherwise. A **probability vector** $p \in \mathbb{R}^n$ has non-negative entries p_i such that $\sum_{i=1}^{n} p_i = 1$. Here p_i represents the probability that the random walker is at vertex i of the graph.

Notation 1. *All our vectors in $\mathbb{R}^n$ are column vectors and we write ${}^t p$ to denote the* ***transpose*** *of a column vector p in $\mathbb{R}^n$. The same notation will also be used for matrices.*

The **Markov transition matrix** is $T = \left(p_{ij}\right)_{1\leq i.i \leq n} = \frac{1}{k}A$. Let $p_i^{(m)}$ denote the probability that the walker is a vertex i at time m. The **probability vector** is $p^{(m)} = {}^t\left(p_1^{(m)}, ..., p_n^{(m)}\right)$. Then

$$p^{(m+1)} = Tp^{(m)} \text{ and } p^{(m)} = T^m p^{(0)}.$$

Theorem 6. ***(A Random Walker Gets Lost).*** *Suppose that X is a connected non-bipartite k-regular graph with n vertices and adjacency matrix A. If $T = \frac{1}{k}A$, for every initial probability vector $p^{(0)}$, we have*

$$\lim_{m\to\infty} p^{(m)} = \lim_{m\to\infty} T^m p^{(0)} = u = {}^t\left(\frac{1}{n}, ..., \frac{1}{n}\right);$$

i.e., the limit is the uniform probability vector.

Proof. Since T is a real symmetric matrix, the spectral theorem from linear algebra says that there is a real orthogonal matrix U; i.e., ${}^tUU = I$, the identity matrix, such that ${}^tUTU = D$, where D is a diagonal matrix with the eigenvalues λ_i of T down the diagonal. Let $U = (u_1, ..., u_n)$, with column vectors u_i. Then these columns are orthonormal, meaning the inner products of the columns satisfy

$$< u_i, u_j >= {}^tu_i u_j = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

Now any vector v can be written as a linear combination of the column vectors of U

$$v = \sum_{i=1}^{n} < v, u_i > u_i, \text{ and } Tv = \sum_{i=1}^{n} < v, u_i > \lambda_i u_i.$$

Then

$$T^m v = \sum_{i=1}^{n} < v, u_i > \lambda_i^m u_i.$$

Assume that u_1 is a constant vector of norm 1 (with eigenvalue $\lambda_1 = 1$). Let the entries of u_1 be $\frac{1}{\sqrt{n}}$. By the hypothesis on X, we know (via Proposition 2) that $|\lambda_i| < 1$, for $i > 1$. It follows that

$$\lim_{m\to\infty} \lambda_i^m = \begin{cases} 1, & i = 1; \\ 0, & i \neq 1. \end{cases}$$

Thus

$$\lim_{m\to\infty} T^m v =< v, u_1 > u_1 = \frac{1}{n}.$$

This proves the theorem. $\square$

But we want to know how long it takes the random walker to get lost. This depends on the second largest eigenvalue in absolute value of the adjacency matrix, assuming the graph is non-bipartite. The next theorem answers the question. If the graph is bipartite, one can modify the random walk to make the walker get lost, by allowing the walker to stay in place with equal probability. We will use the $1-$norm $\|\|_1$ to measure distances between vectors in $\mathbb{R}^n$. Statisticians seem to prefer this to the 2-norm. See Diaconis [36]. Define

$$\|v\|_1 = \sum_{i=1}^{n} |v_i| . \tag{9.1}$$

Theorem 7. ***(How long to get lost?)*** *Suppose that X is a connected non-bipartite k-regular graph with n vertices and adjacency matrix A. If $T = \frac{1}{k}A$, for every initial probability vector $p^{(0)}$, we have*

$$\left\| T^m p^{(0)} - u \right\|_1 \leq \sqrt{n} \left(\frac{\mu}{k}\right)^n,$$

where $u = {}^t\left(\frac{1}{n}, ..., \frac{1}{n}\right)$ and

$$\mu = \max\left\{|\lambda| \,|\lambda \in Spectrum(A), |\lambda| \neq q+1\right\}.$$

1

Proof. See my book [132], pp. 104-106. The proof is in the same spirit as that of the preceding theorem. □

Corollary 1. *If the graph in Theorem 7 is Ramanujan as in Definition 4, then $\mu \leq 2\sqrt{k-1}$, and*

$$\left\| T^m p^{(0)} - u \right\|_1 \leq \sqrt{n} \left(\frac{2\sqrt{k-1}}{k}\right)^n.$$

The moral of this story is that for large values of the degree k, it does not take a very long time before the walker is lost.

Exercise 27. *Redo the preceding results for irregular graphs.*

9.2. **Examples, The Paley Graph, 2D Euclidean Graphs, and Graphs of Lubotzky, Phillips and Sarnak.** We consider various examples, all of which are Cayley graphs $X(G,S)$. When G is the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, its **characters** are of the form $\chi_a(y) = e^{\frac{2\pi i a y}{n}}$, where $a, y \in G$. These form a basis for the eigenfunctions of the adjacency matrix A viewed an an operator on functions: $f : G \to \mathbb{C}$ via the formula

$$Af(y) = \sum_{x \in S} f(x+y),$$

for $x \in G$. That is,

$$A\chi_a = \lambda_a \chi_a, \quad \text{where} \quad \lambda_a = \sum_{x \in S} \chi_a(x). \tag{9.2}$$

See Terras [132] for more information on this subject.

Exercise 28. *Prove formula (9.2).*

Example 10. ***The Paley Graph.*** *$P = X(\mathbb{Z}/p\mathbb{Z}, \square)$ is a Cayley graph for the group $\mathbb{Z}/p\mathbb{Z}$, where p is an odd prime of the form $p = 1+4n$, $n \in \mathbb{Z}$.*[2] *The vertices of the graph are elements of $\mathbb{Z}/p\mathbb{Z}$ and two vertices a, b are connected iff $a-b$ is a non-zero square in $\mathbb{Z}/p\mathbb{Z}$. If $p \equiv 1 (\mathrm{mod}\, 4)$, then -1 is a square and conversely (**Exercise**). It follows that when $p \equiv 1 (\mathrm{mod}\, 4)$ the Paley graph is undirected.*

*The **characters** of the group $\mathbb{Z}/p\mathbb{Z}$ are of the form $\chi_a(y) = e^{\frac{2\pi i a y}{p}}$, for $a, y \in \mathbb{Z}/p\mathbb{Z}$. They form a complete orthogonal set of eigenfunctions of the adjacency operator of the Paley graph.*

$$A\chi_a(y) = \sum_{x \sim y} e^{\frac{2\pi i a x}{p}} = \frac{1}{2} \sum_{\substack{x = y+u^2 \\ 0 \neq u \in \mathbb{Z}/p\mathbb{Z}}} e^{\frac{2\pi i a x}{p}} = \lambda_a \chi_a(y).$$

The eigenvalues λ_a have the form

$$\lambda_a = \frac{1}{2} \sum_{u=1}^{p-1} e^{\frac{2\pi i a u^2}{p}}.$$

[1] The constant $\mu = \rho'$ from Definition 18.

[2] Paley and Cayley are 2 different mathematicians. The Paley graph is a special case of a Cayley graph.

*Recall that the **Gauss sum** is*

$$G_a = \sum_{u=0}^{p-1} e^{\frac{2\pi i a u^2}{p}}. \tag{9.3}$$

Thus $\lambda_a = \frac{1}{2}(G_a - 1)$. *Use the Exercise below then to see that if* a *is not congruent to* $0 (\mathrm{mod}\ p)$*,*

$$|\lambda_a| \leq \frac{1+\sqrt{p}}{2}.$$

Thus the graph is Ramanujan if $p \geq 5$*, since the degree is* $\frac{p-1}{2}$*.*

Exercise 29. *Show that when* a *is not congruent to* $0 (\mathrm{mod}\ p)$*, the Gauss sum defined in the preceding example satisfies* $|G_a| = \sqrt{p}$*.*

Hint. This can be found in most elementary number theory books and in my book [132].

Exercise 30. *Fill in the details showing that the Paley graph* $P = X(\mathbb{Z}/p\mathbb{Z}, \square)$ *above is Ramanujan when* $p \geq 5$*. Then find out how large* p *must be in order that* $\|T^m v - u\|_1 \leq \frac{1}{100}$*, where* $v = {}^t(1,0,0,...,0)$ *and* $u = {}^t\left(\frac{1}{p},...,\frac{1}{p}\right)$.

Example 11. ***2 Dimensional Euclidean Graphs.***

Suppose that p *is an odd prime. Define the Cayley graph* $X(G,S)$ *for the group* $G = \mathbb{F}_p^2$ *consisting of 2-vectors with entries in* $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$*, with the operation of vector addition. The generating set* S *is the set of vectors* $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{F}_p^2$ *satisfying* $x^2 + y^2 = 1$*. This is a special case of the Euclidean graphs considered in Terras* [132] *where they are connected with finite analogs of symmetric spaces.*

The characters of $G = \mathbb{F}_p^2$ *are* $\psi_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \exp\left(\frac{2\pi i(ax+by)}{p}\right)$*, for* (a,b) *and* $(x,y) \in G$*. They form a complete orthogonal set of eigenfunctions of the adjacency operator of the 2D Euclidean graph:*

$$A\psi_{a,b}\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{\begin{pmatrix} x \\ y \end{pmatrix} \sim \begin{pmatrix} u \\ v \end{pmatrix}} e^{\frac{2\pi i ax+by}{p}} = \sum_{r^2+s^2=1} e^{\frac{2\pi i(a(r+u)+b(s+v))}{p}} = \lambda_{a,b}\psi_{a,b}\begin{pmatrix} u \\ v \end{pmatrix}.$$

The corresponding eigenvalues $\lambda_{a,b}$ *are*

$$\lambda_{a,b} = \sum_{r^2+s^2=1} e^{\frac{2\pi i(ar+bv)}{p}}.$$

These numbers can be identified with a sum which is a favorite of number theorists called a Kloosterman sum.

If κ is a character of the multiplicative group $\mathbb{F}_p^* = \mathbb{F}_p - 0$ and $a, b \in \mathbb{F}_p^*$, define the **generalized Kloosterman sum** as

$$K(\kappa|a,b) = \sum_{t \in \mathbb{F}_p^*} \kappa(t) e^{\frac{-2\pi i(at+b/t)}{p}}.$$

It turns out that the non-trivial eigenvalues of the adjacency matrix for the 2D Euclidean Cayley graph are

$$\lambda_{a,b} = \frac{1}{p} G_1^2 K\left(\varepsilon^2 \,\middle|\, 1, a^2+b^2\right),$$

where G_1 is the Gauss sum in formula (9.3) and the quadratic character is

$$\varepsilon(t) = \begin{cases} 1, & t \equiv u^2 (\mathrm{mod}\ p), \text{ for some } u \in \mathbb{F}_p^* \\ 0, & t \equiv 0 (\mathrm{mod}\ p) \\ -1, & \text{otherwise.} \end{cases} \tag{9.4}$$

As a consequence of the Riemann hypothesis for zeta functions of curves over finite fields one has a bound on the Kloosterman sums. This was proved by A. Weil. See Rosen [104] for more information. The bound implies that for $(a,b) \neq (0,0)$, we have

$$\left|\lambda_{a,b}\right| \leq 2\sqrt{q}.$$

The degrees of the 2D Euclidean Cayley graph may be computed exactly to be $p - \varepsilon(-1)$, where ε is defined by formula (9.4). See Rosen [104].

It follows that if $p \equiv 3 (\mathrm{mod}\, 4)$, the graphs are Ramanujan. But when $p = 17, 53$, for example, the graphs are not Ramanujan. Katz has proved that these Kloosterman sums do not vanish. He also proved that the distribution of the Kloosterman sums approaches the semicircle distribution as $p \to \infty$. See Terras [132] for the references. Derek Newland [95] has found that for these Euclidean graphs, the level spacings of $\mathrm{Im}\,(s)$ corresponding to poles of $\zeta(q^{-s}, X)$ look Poisson for large p. The contour maps of the eigenfunctions are beautiful pictures of the finite circles $x^2 + y^2 \equiv a (\mathrm{mod}\, p)$. There are movies on my website of these pictures as p runs through an increasing sequence of primes.

In Terras [132] we also define non-Euclidean finite upper half plane graphs where the Euclidean distance is replaced by a finite analog of the Poincaré distance. These also give Ramanujan graphs which provide interesting spectra of their adjacency matrices. It takes quite a bit of knowledge of group representations plus Weil's result proving the Riemann hypothesis for curves over finite fields in order to prove that the finite upper half plane graphs are Ramanujan.

Exercise 31. *Consider some examples of the finite upper half plane graphs in* [132]. *Experiment with the spectra of the adjacency matrices to see whether the graphs are Ramanujan. Look at the level spacings of the poles of Ihara zeta.*

Such examples are not really the expander graphs sought after by computer scientists since the degree blows up with the number of vertices. It is more difficult to find families of Ramanujan graphs of fixed degree with number of vertices approaching infinity. The first examples were due to Margulis [82] and independently Lubotzky, Phillips and Sarnak [79]. J. Friedman [42] proves that for fixed degree k and and $\epsilon > 0$, the probability that $\lambda_1(X_{m,k}) \leq 2\sqrt{k-1} + \epsilon$ approaches 1 as $n \to \infty$. J. Miller et al [86] gives evidence for the conjecture that the probability that a regular graph is exactly Ramanujan is approximately 27%.

Let us finish by presenting the example of Lubotzky, Phillips and Sarnak [79].

Example 12. ***The Lubotzky, Phillips and Sarnak Graphs $\mathbf{X}_{p,q}$.***

Let p and q be distinct primes congruent to 1 modulo 4. The graphs $\mathbf{X}_{p,q}$ are Cayley graphs for the group $G = PGL(2, \mathbb{F}_q) = GL(2, \mathbb{F}_q)/Center$. Here $GL(2, \mathbb{F}_q)$ is the group of non-singular 2×2 matrices with elements in the field with q elements. The center consists of matrices which are non-0 scalar multiples of the identity. Fix some integer i so that $i^2 \equiv -1 (\mathrm{mod}\, q)$. Define S to be

$$S = \left\{ \left(\begin{array}{cc} a_o + i a_1 & a_2 + i a_3 \\ -a_2 + i a_3 & a_0 - i a_1 \end{array} \right) \middle| \; a_0^2 + a_1^2 + a_2^2 + a_3^2 = p, \;\; \textit{for odd} \;\; a_0 > 0 \;\; \textit{and even} \;\; a_1, a_2, a_3 \right\}.$$

A theorem of Jacobi says there are exactly $p + 1$ integer solutions to $a_0^2 + a_1^2 + a_2^2 + a_3^2 = p$ so that $|S| = p + 1$. One can show that S is closed under matrix inverse. The graph $X_{p,q}$ is then the connected component of the identity in the Cayley graph $X(G, S)$. It can be proved that either $X(G, S)$ is connected or it has 2 connected components of equal size. Using Weil's proof of the Riemann hypothesis for zeta functions of curves over finite fields, Lubotzky, Phillips and Sarnak show that these graphs are Ramanujan. For fixed p we then have a family of Ramanujan graphs of degree $p + 1$ having $O(q^3)$ vertices as $q \to \infty$.

Exercise 32. *Compute the Ihara zeta functions for some of the graphs in this section.*

9.3. **Why the Ramanujan Bound is Best Possible (Alon and Boppana).** We want to prove the following theorem.

Theorem 8. ***(Alon and Boppana)*** *Suppose that X_n is a sequence of k-regular connected graphs with the number of vertices of X_n approaching infinity with n. Let $\lambda_1(X_n)$ denote the second largest eigenvalue of the adjacency matrix of X_n. Then*

$$\lim_{n\to\infty} \left(\inf_{m \geq n} \lambda_1(X_m) \right) \geq 2\sqrt{k-1}.$$

Proof. **(Lubotzky, Phillips, and Sarnak).** Let the set of eigenvalues of the adjacency matrix A_n of X_n be

$$Spec(A_n) = \left\{ \lambda_0 = k > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|V(X_n)|-1} \right\}.$$

Let $N_v(m, X_n)$ be the number of paths of length m going from vertex v to v in graph X_n. Note that these paths can have backtracking and tails. Then

$$\sum_{j=0}^{|V(X_n)|-1} \lambda_j^m = Tr(A_n^m) = \sum_{v \in X} N_v(m, X_n).$$

The universal covering space of X_n is the **k-regular tree T_k** (meaning it is an infinite graph which is k-regular, connected and having no cycles). Part of T_4 is pictured in Figure 10. The lower bound we seek is actually the spectral radius of the adjacency operator on T_k.

Let τ_m be the number of paths of length m on T_k going from any vertex $\widetilde{v}$ back to $\widetilde{v}$. Since T_k is the k-regular tree, τ_m is 0 unless m is even and τ_m is independent of $\widetilde{v}$. Then $N_v(m, X_n) \geq \tau_m$, since any path on T_k projects down 1-1 to a path on X_n. Therefore

$$\sum_{j=0}^{|V(X_n)|-1} \lambda_j^m = \sum_{v \in X_n} N_v(m, X_n) \geq |V(X_n)|\, \tau_m.$$

It follows that

$$k^m + (|V(X_n)| - 1)\, \lambda_1^m \geq |V(X_n)|\, \tau_m.$$

We will be done if we can show that

$$\tau_{2m}^{1/2m} \to 2\sqrt{k-1}, \text{ as } m \to \infty. \tag{9.5}$$

For then we would have

$$\lambda_1 \geq \left(\frac{|V(X_n)|\,\tau_{2m} - k^{2m}}{|V(X_n)| - 1}\right)^{1/2m} = \left(\frac{|V(X_n)|}{|V(X_n)| - 1}\right)^{1/2m} \tau_{2m}^{1/2m} \left(1 - \frac{k^{2m}}{\tau_{2m}\,|V(X_n)|}\right)^{1/2m}.$$

The first factor approaches 1 as $n \to \infty$. The second factor approaches $2\sqrt{k-1}$. The third factor approaches 1.

Now we must prove formula (9.5). For this part of the proof we follow the reasoning of H. Stark. Let x and y be any two points of T_k such that the distance between them $d(x, y) = j$; i.e., the number of edges in the unique path in T_k joining x and y is j.

We define $\tau(n, j)$ to be the number of ways starting at x to get to a point y at distance j from x by a path of length n in T_k. It is $\tau(m, 0) = \tau_m$ that we want to study. It is an **Exercise** to see that $\tau(n, j) \neq 0$ implies that $j \equiv n (\mathrm{mod}\, 2)$ and $n \geq j$. For $j > 0$ and $n > 1$, we have the recursion

$$\tau(n, j) = (k-1)\tau(n-1, j+1) + \tau(n-1, j-1). \tag{9.6}$$

For one must be at one of the k neighbors of y at the $(n-1)$st step, and $(k-1)$ of these neighbors are a distance $(j+1)$ from x, while the last neighbor is at a distance $(j-1)$ from x.

The recursion (9.6) is reminiscent of Pascal's triangle. It is an **Exercise** to show that

$$\tau(2m, 0) \geq a(2m, 0)(k-1)^m,$$

where $a(n, j)$ is defined by the following recursive definition

$$\begin{aligned} &a(n, j) = a(n-1, j-1) + a(n-1, j+1);\\ &a(0, 0) = 1;\\ &a(n, 0) = 0, \quad \text{unless } \ 0 \leq j \leq n, \quad a(0, 0) = 1. \end{aligned}$$

Set $a_{2m} = a(2m, 0)$. Note that a_{2m} satisfies the recursion

$$a_{2m} = \sum_{k=1}^{m} a_{2k-2}a_{2m-2k}.$$

This recursion arises in many ways in combinatorics. See Vilenkin [140]. For example a_{2m} is the number of permutations of $2m$ letters, m of which are b's and m of which are f's, such that for every r with $1 \leq r \leq 2m$, the number of b's in the first r terms of the permutations is $\geq$ the number of f's. The solution is the **Catalan number**

$$a_{2m} = \frac{1}{m+1}\binom{2m}{m}.$$

Stirling's formula implies that

$$\binom{2m}{m}^{1/2m} \sim 2, \text{ as } m \to \infty.$$

Formula (9.5) follows and thus the theorem. □

Exercise 33. *Fill in the details in the preceding proof. It may help to look at Figure 31 showing some of the values of* $a(n, j)$.

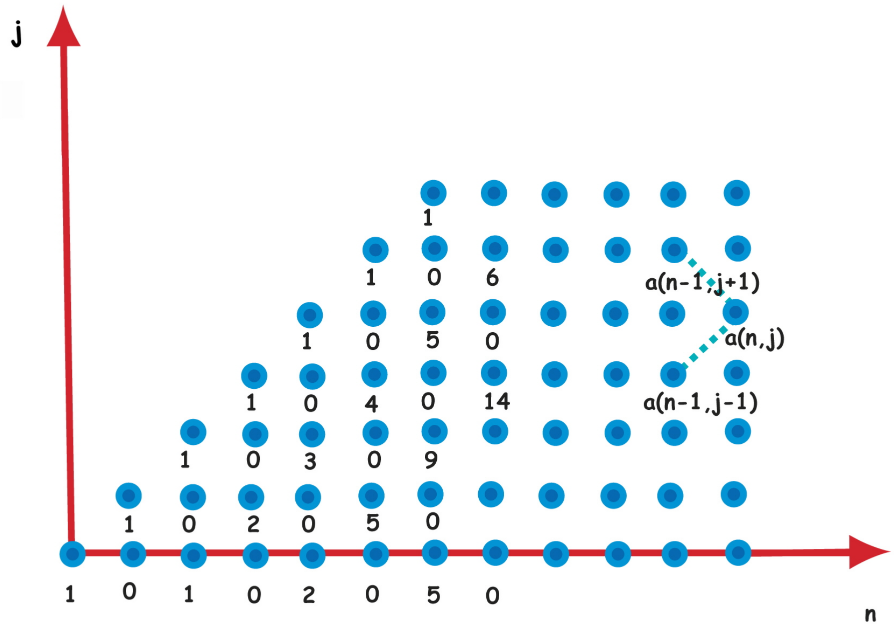


FIGURE 31. **Part of the Proof of Theorem 8.** Values of $a(n,j)$ defined by the recursion $a(n,j) = a(n-1,j-1)+a(n-1,j+1)$, with $a(0,0)=1$, $a(n,j)=0$ unless $0 \leq j \leq n$.

9.4. **Why are Ramanujan Graphs Good Expanders?** First, what is an expander graph? Roughly it means that the graph is highly connected but sparse (meaning that there are relatively few edges). Such graphs are quite useful in computer science - for building efficient communication networks, for creating error-correcting codes with efficient encoding and decoding. See Davidoff et al [35], Hoory et al [55], Lubotzky [77], Sarnak [109] for more information. Fan Chung [27] provides a discussion of expansion in irregular graphs.

Suppose that X is an undirected k-regular graph satisfying our usual assumptions.

Definition 20. *For sets of vertices S, T of X, define*

$$E(S,T) = \{e \,|\, e \text{ is an edge of } X \text{ with one vertex in } S \text{ and the other vertex in } T\}.$$

Definition 21. *If S is a set of vertices of X, we say the* ***boundary*** *is $\partial S = E(S, X - S)$.*

Definition 22. *A graph X with vertex set V and $n = |V|$ has* ***expansion ratio***

$$h(X) = \min_{\{S \subset V \,|\, |S| \le \frac{n}{2}\}} \frac{|\partial S|}{|S|}.$$

Note that there are many variations on this definition. We follow Sarnak [109] and Hoory et al [55] here. The expansion constant is the discrete analog of the Cheeger constant in differential geometry. See Lubotzky [77].

Definition 23. *A sequence of $(q+1)$-regular graphs $\{X_j\}$ such that $|V(X_j)| \to \infty$, as $j \to \infty$, is called an* ***expander family*** *if there is an $\varepsilon > 0$ such that the expansion ratio $h(X_j) \ge \varepsilon$, for all j.*

For connected $k-$regular graphs X whose adjacency matrix has spectrum $k = \lambda_1 > \lambda_2 \ge \cdots \ge \lambda_n$, one can prove that

$$\frac{k - \lambda_2}{2k} \le h(X). \tag{9.7}$$

See my book [132], p. 80. There is also a discussion in Hoory, Lineal and [55], pages 474-476, who prove an upper bound as well. Such results were originally proved by Dodziuk, Alon, and Milman. It follows that for large expansion constant, one needs small λ_2.

Next we prove the expander-mixing lemma (from Alon and Fan Chung [1]) which implies that $E(S,T)$ will be closer to the expected number of edges between S and T in a random $k-$regular graph X of edge density $\frac{k}{n}$, (where $n = |V|$) provided that μ, the 2nd largest eigenvalue (in absolute value) of the adjacency matrix of X is small as possible.

Lemma 1. ***The Expander Mixing Lemma.***

Suppose X is a connected k-regular non bipartite graph with n vertices and

$$\mu = \max\{|\lambda| \,|\lambda \in Spectrum(A), |\lambda| \ne k\}.$$

Then for all sets S, T of vertices of X, we have

$$\left| E(S,T) - \frac{k\,|S|\,|T|}{n} \right| \le \mu\sqrt{|S|\,|T|}.$$

Proof. By our hypotheses $\mu < k$.

Let δ_S denote the vector whose entries are 1 for vertices of S and 0 otherwise. Recall the spectral theorem for the symmetric matrix A =the adjacency matrix of X. This says there is a complete orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors ϕ_j of A, with $A\phi_j = \lambda_j \phi_j$ and giving

$$(A)_{a,b} = \sum_{j=1}^{n} \lambda_j \phi_j(a)\phi_j(b).$$

Here we write $\phi_j(a)$ to denote the entry of ϕ_j corresponding to vertex a of X. We may assume that we have numbered things so that $\phi_1(a) = \frac{1}{\sqrt{n}}$ and $\lambda_1 = k$. Then, pulling out the 1st term of the sum gives

$$|E(S,T)| = {}^{T}\delta_S A \delta_T = \sum_{j=1}^{n} \lambda_j \sum_{\substack{a\in S\\ b\in T}} \phi_j(a)\phi_j(b) = \frac{k}{n}\,|S|\,|T| + \sum_{j=2}^{n} \lambda_j \sum_{\substack{a\in S\\ b\in T}} \phi_j(a)\phi_j(b).$$

Now, by the definition of μ, since our graph is not bipartite, there is only one eigenvalue with absolute value equal to k, and

$$\left| \sum_{j=2}^{n} \lambda_j \sum_{\substack{a\in S\\ b\in T}} \phi_j(a)\phi_j(b) \right| \le \mu \sum_{j=2}^{n} \sum_{\substack{a\in S\\ b\in T}} \left|\phi_j(a)\phi_j(b)\right|.$$

To finish the proof, use the Cauchy-Schwarz inequality. Note that the Fourier coefficients of δ_S with respect to the basis ϕ_j are

$$<\delta_S, \phi_j> = \sum_{a \in S} \phi_j(a).$$

This implies by Bessel's equality that

$$|S| = \|\delta_S\|_2^2 = \sum_{j=1}^{n} <\delta_S, \phi_j>^2 .$$

So Cauchy-Schwarz says

$$\left| \sum_{j=2}^{n} \sum_{\substack{a \in S \\ b \in T}} \phi_j(a)\phi_j(b) \right| \leq \left| \sum_{j=1}^{n} \sum_{\substack{a \in S \\ b \in T}} \phi_j(a)\phi_j(b) \right| \leq \sqrt{|S|\,|T|}.$$

This completes the proof of the lemma. □

Exercise 34. *Suppose the graph X represents a gossip network. Explain how you can use Lemma 1 to estimate how many people you need to tell to make sure that over one-half of the people hear a given rumor after 1 iteration. What about 2 iterations?*

9.5. **Why do Ramanujan graphs have small diameters?** In this section, we present a theorem of Fan Chung [26] which bounds the diameter of a connected k-regular graph in terms of the second largest eigenvalue in absolute value. We assume the graph is not bipartite to avoid the problem that $-k$ could also be an eigenvalue. From the theorem, we see that Ramanujan graphs will have as small diameter as possible for sequences of k-regular graphs with number of vertices approaching infinity. Thus, the Ramanujan graphs found by Lubotzky, Phillips and Sarnak [79] were shown to have small diameters.

Definition 24. *Define the distance $d(x, y)$ between 2 vertices x, y of a graph X to the the length of a shortest path connecting the vertices. Then the* ***diameter*** *of X is*

$$\max_{x,y \in V(X)} d(x, y).$$

Theorem 9. *(**Fan Chung** [26]). Suppose that X is a connected, non bipartite $k-$regular graph with n vertices and*

$$\mu = \max\left\{|\lambda| \mid \lambda \in Spectrum(A),\ |\lambda| \neq k\right\}.$$

Then

$$\text{diameter}(X) \leq 1 + \frac{\log(n-1)}{\log \frac{k}{\mu}}.$$

Proof. As in the proof in the last subsection, we will use the spectral theorem for the adjacency matrix A of X. This says there is a complete orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors ϕ_j of A, with $A\phi_j = \lambda_j \phi_j$ and giving

$$(A)_{a,b} = \sum_{j=1}^{n} \lambda_j \phi_j(a)\phi_j(b).$$

Write $\phi_j(a)$ to denote the entry of ϕ_j corresponding to vertex a of X. Assume that we have numbered things so that $\phi_1(a) = \frac{1}{\sqrt{n}}$ and $\lambda_1 = k$.

Note that for vertices a, b of X, we have $\left(A^t\right)_{a,b} = \#\{\text{paths of length } t \text{ connecting } a \text{ to } b\}$. If d is the diameter of X, then $\left(A^d\right)_{a,b} \neq 0$, for some a, b with $d(a, b) = d$. Then $\left(A^{d-1}\right)_{a,b} = 0$, as there is no shorter path connecting a and b. Therefore, if $t = d - 1$, and

$$0 = \left(A^t\right)_{a,b} = \sum_{j=1}^{t} \lambda_j^t \phi_j(a)\phi_j(b).$$

Use the Cauchy-Schwarz inequality to see that

$$0 \geq \frac{k^t}{n} - \mu^t \sum_{j=2}^{n} \left|\phi_j(a)\right| \left|\phi_j(b)\right| \geq \frac{k^t}{n} - \mu^t \left(\sum_{j=2}^{n} \left|\phi_j(a)\right|^2 \right)^{1/2} \left(\sum_{j=2}^{n} \left|\phi_j(b)\right|^2 \right)^{1/2}$$

$$= \frac{k^t}{n} - \mu^t \sqrt{1 - \phi_1(a)^2}\sqrt{1 - \phi_1(b)^2} = \frac{k^t}{n} - \mu^t \left(1 - \frac{1}{n}\right).$$

This implies that

$$\frac{k^t}{n} \le \mu^t \left(1 - \frac{1}{n}\right).$$

Thus

$$\left(\frac{k}{\mu}\right)^t \le n - 1.$$

Taking logs,

$$t \log \frac{k}{\mu} \le \log(n-1).$$

So recalling that $t = d - 1$, we have

$$d - 1 \le \frac{\log(n-1)}{\log \frac{k}{\mu}}.$$

The theorem follows. □

Exercise 35. *Compute the diameters of your favorite graphs such as K_4, $K_4 - e$, the Paley graphs, the 2 dimensional Euclidean graphs, the icosahedron, ...*

10. The Graph Theory Prime Number Theorem

The main application of the Ihara zeta function is to give an asymptotic estimate for $\pi(m)$ the number of primes of length m in our graph. This is the content of the next the theorem. We will use results proved in previous section on the Ruelle zeta. Before we do this, we need to consider the generating function obtained from the logarithmic derivative of the Ihara zeta function. First recall Definition 9 of the numbers N_m and the generating function

$$u \frac{d}{du} \log \zeta_X(u) = \sum_{m \ge 1} N_m u^m. \tag{10.1}$$

This follows from formula (4.5) from our earlier section on the Ruelle zeta function.

Theorem 10. ***Graph Prime Number Theorem.*** *We assume that the graph X satisfies our usual hypotheses (stated before Definition 1). Suppose that R_X is as in Definition 3. If $\pi(m)$ and Δ_X are as in Definitions 6 and 7, then $\pi(m) = 0$ unless Δ_X divides m. If Δ_X divides m, we have*

$$\pi(m) \sim \Delta_X \frac{R_X^{-m}}{m}, \quad \text{as} \quad m \to \infty.$$

Proof. We imitate the proof of the analogous result for zeta functions of function fields in Rosen [104]. Observe that the defining formula for the Ihara zeta function can be written as

$$\zeta_X(u) = \prod_{n \ge 1} (1 - u^n)^{-\pi(n)}.$$

Then

$$u \frac{d}{du} \log \zeta_X(u) = \sum_{n \ge 1} \frac{n\pi(n) u^n}{1 - u^n} = \sum_{m \ge 1} \sum_{d|m} d\pi(d) u^m.$$

Here the inner sum is over all positive divisors of m. Thus from formula (10.1) we obtain the **relation between N_m and $\pi(n)$**.

$$N_m = \sum_{d|m} d\pi(d). \tag{10.2}$$

This sort of relation occurs frequently in number theory and combinatorics. It is inverted using the **Möbius function** $\mu(n)$ defined by

$$\mu(n) = \begin{cases} 1, & n = 1 \\ (-1)^r, \quad n = p_1 \cdots p_r, & \text{for distinct primes } p_i \\ 0, & \text{otherwise.} \end{cases}$$

Then by the **Möbius inversion formula**

$$\pi(m) = \frac{1}{m} \sum_{d|m} \mu\left(\frac{m}{d}\right) N_d. \tag{10.3}$$

Next we look at the 2-term determinant formula (4.4) where W_1 is from Definition 8 in the section on the Ruelle zeta function. This gives

$$u\frac{d}{du}\log\zeta_X(u) = u\frac{d}{du}\sum_{\lambda\in Spec(W_1)}\log(1-\lambda u) = \sum_{\lambda\in Spec(W_1)}\sum_{n\geq 1}(\lambda u)^n.$$

It follows from formula (10.1) that we have the **formula relating N_m and the spectrum of the edge adjacency matrix W_1 :**

$$N_m = \sum_{\lambda\in Spec(W_1)}\lambda^m. \tag{10.4}$$

The dominant terms in this last sum are those coming from $\lambda \in Spec(W_1)$ such that $|\lambda| = R^{-1}$, with $R = R_X$ from Definition 3.

By Theorem 4 of Kotani and Sunada, the largest absolute value of an eigenvalue λ occurs Δ_X times with these eigenvalues having the form $e^{2\pi ia/\Delta_X}R^{-1}$, where $a = 1, ..., \Delta_X$. Using the orthogonality relations for exponential sums (see [132]) which are basic to the theory of the finite Fourier transform, we see that

$$\pi(n) \sim \frac{1}{n}\sum_{|\lambda|\ \text{maximal}}\lambda^n = \frac{R^{-n}}{n}\sum_{a=1}^{\Delta_X}e^{\frac{2\pi ian}{\Delta_X}} = \frac{R^{-n}}{n}\begin{cases}0, & \Delta_X \text{ does not divide } n\\ \Delta_X, & \Delta_X \text{ divides } n.\end{cases} \tag{10.5}$$

The graph prime number theorem follows from formulas (10.3), (10.4), and (10.5). □

Exercise 36. *Prove* $\sum_{a=1}^{\Delta_X}e^{\frac{2\pi ian}{\Delta_X}} = \begin{cases}0, & \Delta_X \text{ does not divide } n\\ \Delta_X, & \Delta_X \text{ divides } n.\end{cases}$

Example 13. ***Primes in $K_4 - e$, the graph obtained from K_4 by deleting an edge e.*** *See Figure 11. We have seen that*

$$\zeta_X(u)^{-1} = \left(1-u^2\right)(1-u)\left(1+u^2\right)\left(1+u+2u^2\right)\left(1-u^2-2u^3\right).$$

From this, we have

$$u\frac{d}{du}\log\zeta_X(u) = 12u^3+8u^4+24u^6+28u^7+8u^8+48u^9+120u^{10}+44u^{11}+104u^{12}+416u^{13}+280u^{14}+O\left(u^{15}\right).$$

Now we want to use $N_m = \sum_{d|m} d\pi(d)$ *to compute the small values of* $\pi(m)$. *First* $3\pi(3) = 12$ *implies that* $\pi(3) = 4$. *Similarly we find* $\pi(4)$ *and* $\pi(5)$ *in the list.*

$$\pi(3) = 4,\ \pi(4) = 2,\ \pi(5) = 0.$$

Then $6\pi(6)+3\pi(3) = 24$ *implies that* $\pi(6) = 2$. *Next we find that* $\pi(7) = 4$, *while* $\pi(8) = 0$. *Then* $9\pi(9)+3\pi(3) = 48$ *implies* $\pi(9) = 36/9 = 4$. *Finally* $10\pi(10)+5\pi(5) = 120$ *says that* $\pi(10) = 12$.

So the rest of our list says

$$\pi(6) = 2, \pi(7) = 4, \pi(8) = 0, \pi(9) = 4, \pi(10) = 12.$$

The reader should look at the graph for examples of the primes of lengths 3, 4, 6, 7, 9, 10.

Exercise 37. *If the graph* $X = K_4$, *the tetrahedron, find* $\pi(m)$ *for* $m = 3, 4, 5, , , , 11$.

If the Riemann hypothesis (either version for irregular graphs) holds for $\zeta_X(u)$, then one has a good bound on the error term in the prime number theorem by formula (4.4). By Theorem 8 of Alon and Boppana, the bound on the error term will be best possible for a family of connected $(q+1)$-regular graphs with number of vertices approaching infinity.

Question. What properties of the graph are determined by the Ihara zeta?

There are many papers on this question. See Yaim Cooper [24], Debra Czarneski [33], Matthew Horton [57], [58], Christopher Storm [125], for example. Some things are obvious. The degree of the reciprocal of the Ihara zeta is the number of directed edges. The number of vertices is found by noting that the rank r of the fundamental group is determined in the next paragraph and $r - 1 = |E| - |V|$. We have also seen that the zeta function tells us the numbers N_m and thus the **girth** (=length of shortest cycle) of the graph, which is the first m with nonzero N_m.

Horton [57], [58] finds a simple formula for the **girth** (=length of shortest cycle) of the graph from the reciprocal of zeta. He also shows that the chromatic polynomial of the graph cannot be determined from zeta alone. Storm [125] finds that zeta determines the clique number, the number of Hamiltonian cycles. A **clique** is a complete graph which is an induced subgraph. A **Hamiltonian cycle** is a cycle which visits every vertex exactly once. Later (see Section 21), we will find that it is possible for $\zeta_X(u) = \zeta_Y(u)$ without graph X being isomorphic to graph Y. Thus zeta does **not** determine the graph up to graph isomorphism.

The Ihara zeta function determines the rank of the fundamental group, for it is the order of the pole of the Ihara zeta function at $u = 1$. The **complexity** κ_X of a graph is defined to be the number of spanning trees in X. One can use the matrix-tree theorem (see Biggs [15]) to prove that

$$\left[\frac{d^r}{du^r}\zeta_X^{-1}(u)\right]\Big|_{u=1} = r!(-1)^{r+1}2^r(r-1)\kappa_X. \tag{10.6}$$

This result is an exercise on the last page of Terras [132], where some hints are given. It is an analog of the formula for the Dedekind zeta function of a number field at 0 (a formula involving the class number and the regulator of the number field). See Figure 4 and Lang [73].

Exercise 38. *Prove formula (10.6).*

Exercise 39. *Prove the prime number theorem for a (q+1)-regular graph using Theorem 1 of Ihara with the 3-term determinant rather than the* $\det(I - W_1)^{-1}$ *formula.*

We have found that the Ihara zeta function possesses many analogous properties to the Dedekind zeta function of an algebraic number field. There are other analogs as well. For example there is an analog of the ideal class group called the Jacobian of a graph. It has order equal to κ_X, the complexity. It has been considered by Bacher, de la Harpe and Tatiana Nagnibeda [5] as well as Baker and Norine [7].

Part 3. Edge and Path Zeta Functions

In this part, we consider 2 multivariable zeta functions associated to a finite graph, the edge zeta and the path zeta. We will give a matrix analysis version of the Bass proof of Ihara's determinant formula. This implies that there is a determinant formula for the vertex zeta function of weighted graphs even if the weights are not integers. We will discuss what deleting an edge of a graph (**fission**) does to the edge zeta function. We will also discuss what happens if a graph edge is **fused** ; i.e,. shrunk to a point. There is an application of the edge zeta to error correcting codes which will be discussed in the last part of this book. See also Koetter et al [70] and [71].

11. The Edge Zeta Function

11.1. Definitions and Bass Proof of Ihara 3-Term Determinant Formula.

Notation 2. *From now on, we change our notation for the Ihara zeta function of the last section, replacing $\zeta_X(u)$ by $\zeta(u,X)$ (or even $\zeta_V(u,X)$, where the "V" is for* ***vertex)****. We may call the Ihara zeta a* ***"vertex zeta"*** *although we will try to avoid this unless it might lead to confusion.*

Definition 25. *The* ***edge matrix*** *W for graph X is a $2m \times 2m$ matrix with a,b entry corresponding to the oriented edges a and b. This a,b entry is the complex variable w_{ab}, if edge a feeds into edge b and $b \neq a^{-1}$, while the a,b entry is 0, otherwise.*

Note that W_1 from Definition 8 is obtained from the edge matrix W by setting all non-zero entries of W equal to 1.

Definition 26. *Given a closed path C in X, which is written as a product of oriented edges $C = a_1 a_2 \cdots a_s$, the* ***edge norm*** *of C is*

$$N_E(C) = w_{a_1a_2} w_{a_2a_3} \cdots w_{a_{s-1}a_s} w_{a_s a_1}.$$

The ***edge zeta function*** *is*

$$\zeta_E(W,X) = \prod_{[P]} (1 - N_E(P))^{-1},$$

where the product is over primes in X. Here assume that all $|w_{ab}|$ are sufficiently small for convergence.

Specializing Variables to Obtain other Zetas

1) Clearly if you set all non-zero variables in W equal to $u \in \mathbb{C}$, the edge norm $N_E(C)$ specializes to $u^{\nu(C)}$. Therefore (by the definitions of the Ihara zeta function and the edge zeta function) the **edge zeta function specializes to the Ihara (vertex) zeta function**; i.e,

$$\zeta_E(W,X)\big|_{0 \neq w_{ab} = u} = \zeta(u,X). \tag{11.1}$$

2) If X is a **weighted graph** with weight function L, and you specialize the non-zero variables

$$w_{ab} = u^{(L(a)+L(b))/2}, \tag{11.2}$$

you get the weighted Ihara zeta function of Definition 13. Or you could specialize

$$w_{ab} = u^{L(a)}. \tag{11.3}$$

As in Mizuno and Sato [89], one can also associated non-negative values w_e to each directed edge e and then write $w_{ab} = \sqrt{w_a}\sqrt{w_b}$, if directed edge a leads into directed edge b without backtracking, and 0 otherwise. This leads to a nice version of the zeta for weighted graphs.

3) To obtain the Hashimoto edge zeta function discussed in Stark and Terras [119], specialize $w_{ab} = u_a$. This is the zeta in the application to error-correcting codes in Part 5.

4) If you cut or delete an edge of a graph (something we think of as "**fission**"), you can compute the edge zeta for the new graph with one less edge by setting any variables equal to 0 if the cut or deleted edge or its inverse appear in one of its subscripts. Note that graph theorists usually call an edge a "cut edge" only if its removal disconnects the graph.

5) You can also use the variables w_{ab} in the edge matrix W corresponding to $b = a^{-1}$ to produce a zeta function that keeps track of paths with backtracking or tails. See Bartholdi [11].

6) Finally one can consider edge zetas of directed graphs. See Horton [57], [58].

The edge zeta again has a determinant formula and is the reciprocal of a polynomial in the w_{ab} variables. This is the following theorem whose proof should be compared with that of Proposition 1.

Theorem 11. ***(Determinant Formula for the Edge Zeta).***

$$\zeta_E(W,X) = \det(I - W)^{-1}.$$

Proof. First note that, from the Euler product for the edge zeta function, we have

$$-\log \zeta_E(W,X) = \sum_{[P]} \sum_{j\geq 1} \frac{1}{j} N_E(P)^j.$$

Since there are $\nu(P)$ elements in the prime $[P]$, we have

$$-\log \zeta_E(W,X) = \sum_{\substack{m\geq 1\\ j\geq 1}} \frac{1}{jm} \sum_{\substack{P\\ \nu(P)=m}} N_E(P)^j.$$

Here the sum is over primitive cycles P.

It follows that

$$-\log \zeta_E(W,X) = \sum_{C} \frac{1}{\nu(C)} N_E(C).$$

Here we sum over paths C which need not be prime paths, but are still closed without backtracking or tails. This comes from the fact that such a path C has the form P^j, for some prime path P and $j = 1, 2, 3, ...$ Then, by the Exercise below, we see that

$$-\log \zeta_E(W,X) = \sum_{m\geq 1} \frac{1}{m} Tr(W^m).$$

Finally, again using the Exercise below, we see that the right hand side of the preceding formula is $\log\det(I-W)^{-1}$ This proves log (determinant formula) and taking exp of both sides gives the theorem. □

Exercise 40. *Prove that*

$$\sum_{C} \frac{1}{\nu(C)} N_E(C) = \sum_{m\geq 1} \frac{1}{m} Tr(W^m) = -Tr\log(I-W) = \log\det(I-W)^{-1}.$$

Hints. 1) For the first equality, you need to think about $Tr(W^m)$ as an $(m+1)$-fold sum of products of w_{ij} in terms of closed paths C of length m.

2) For the second equality, use the power series for $\log(I-W)$.

3) Recall Exercise 11.

By formula (11.1), we have the following Corollary, since specializing all the non-zero variables in W to be u, yields the matrix uW_1, where W_1 is from Definition 8. We also proved this Corollary in the Section on Ruelle zeta functions.

Corollary 2. $\zeta_X(u) = \zeta_V(u,X) = \det(I-uW_1)^{-1}$.

Moral: The poles of $\zeta_X(u)$ are the reciprocals of the eigenvalues of W_1.

Exercise 41. *Write W_1 in block form with $|E| \times |E|$−blocks:*

$$W_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

1) Show that $D = {}^tA,\ B = {}^tB,\ C = {}^tC$. The diagonal entries of B and C are zero.

2) Show that the sum of the entries of the ith row of W_1 is the degree of the vertex which is the starting vertex of edge i.

Exercise 42. *Consider some weighted graphs and their zeta functions. Can you expect them to have all the properties of the ordinary Ihara zetas, no matter what sort of weights are involved?*

Example 14. ***Dumbbell Graph.***

Figure 32 shows the labeled picture of the dumbbell graph X. For this graph we find that $\zeta_E(W,X)^{-1} =$

$$\det\begin{pmatrix} w_{11}-1 & w_{12} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{23} & 0 & 0 & w_{26} \\ 0 & 0 & w_{33}-1 & 0 & w_{35} & 0 \\ 0 & w_{42} & 0 & w_{44}-1 & 0 & 0 \\ w_{51} & 0 & 0 & w_{54} & -1 & 0 \\ 0 & 0 & 0 & 0 & w_{65} & w_{66}-1 \end{pmatrix}.$$

Note that if we cut or delete the vertical edges which are edges e_2 and e_5, we should specialize all the variables with 2 or 5 in them to be 0. This yields the edge zeta function of the subgraph with the vertical edge removed, and incidentally diagonalizes the matrix W. We call this **"fission"**. The edge zeta is particularly suited to keeping track of such fission.

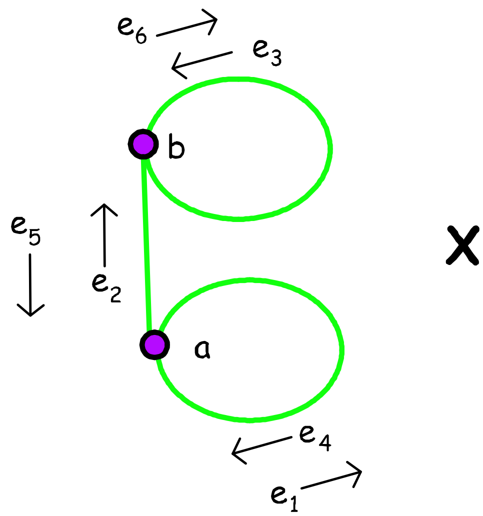


FIGURE 32. **The dumbell graph**

Exercise 43. *Do another example computing the edge zeta function of your favorite graph. Then see what happens if you delete an edge.*

Next we give a version of **Bass's proof of the Ihara determinant formula** (Theorem 1) using the preceding theorem. In what follows, n is the number of vertices of X and m is the number of unoriented edges of X.

First define some matrices. Set $J = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$. Then define the $n \times 2m$ **start matrix** S and the $n \times 2m$ **terminal matrix** T by setting

$$s_{ve} = \begin{cases} 1, & \text{if } v \text{ is the starting vertex of edge } e, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$t_{ve} = \begin{cases} 1, & \text{if } v \text{ is the terminal vertex of edge } e, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $S = (MN)$, $T = (NM)$, where M and N are $|V| \times |E|$ matrices, thanks to our numbering system for the directed edges, where $e_{j+|E|} = e_j^{-1}$. Here $j = 1, 2, ..., |E|$.

Exercise 44. *Write $S = (MN)$, $T = (NM)$, where M and N are $|V| \times |E|$ matrices of 0's and 1's. Use the following proposition to create random graphs and plot the poles of their zeta functions.*

Hint: *Make use of the Matlab commands creating random permutation matrices P_i to build up $M = (P_1 \cdots P_k)$. Similarly build up N. Obtain W_1 from Proposition 4 below. We created Figure 33 below this way.*

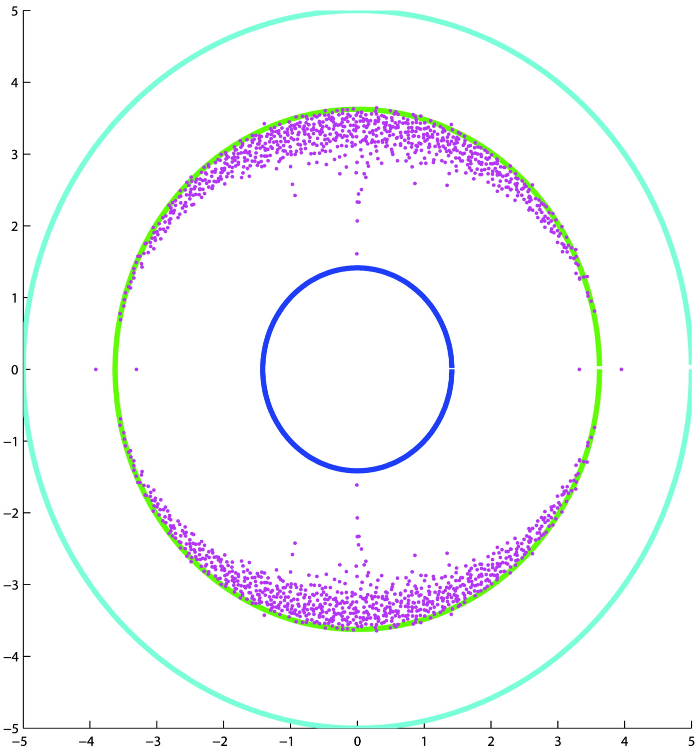


FIGURE 33. **A Matlab Experiment. The eigenvalues ($\neq \pm 1$ or $1/R$) of the edge adjacency matrix W_1 for a random graph** are the purple points. The inner circle has radius $\sqrt{p}$. the middle green circle has radius $1/\sqrt{R}$. The outer circle has radius $\sqrt{q}$. The green (middle) circle is the Riemann hypothesis circle. Because the eigenvalues of W_1 are reciprocals of the poles of zeta, now the RH says the spectrum should be inside the green circle. The Riemann hypothesis looks approximately true. The graph has 800 vertices, mean degree $\cong 13.125$, edge probability $\cong .0164$.

Proposition 4. ***Some Matrix Identities*** *Using the preceding definitions, the following formulas hold. We write* ${}^tM$ *for the transpose of the matrix* M.

1) $SJ = T$, $TJ = S$.

2) If A *is the adjacency matrix of* X *and* $Q + I_n$ *is the diagonal matrix whose jth diagonal entry is the degree of the jth vertex of* X, *then* $A = S\,{}^tT$, *and* $Q + I_n = S\,{}^tS = T\,{}^tT$.

3) The edge adjacency matrix W_1 *from Definition 8 satisfies* $W_1 + J = {}^tTS$.

Proof. 1) This comes from the fact that the starting (terminal) vertex of edge e_j is the terminal (starting) vertex of edge $e_{j+|E|}$, according to our edge numbering system from formula (2.1).

2) Consider

$$(S\,{}^tT)_{a,b} = \sum_e s_{ae} t_{be}.$$

The right hand side is the number of oriented edges e such that a is the initial vertex and b is the terminal vertex of e, which is the a, b entry of A. Note that $A_{a,a}$ =2*number of loops at vertex a. Similar arguments prove the second formula.

3) We have

$$({}^tTS)_{ef} = \sum_v t_{ve} s_{vf}.$$

The sum is 1 iff edge e feeds into edge f, even if $f = e^{-1}$. So, recalling our directed edge labeling convention, if $f \neq e^{-1}$, we get $(W_1)_{e,f} = (W_1 + J)_{ef}$, but when $f = e^{-1}$ we get $(J)_{e,f} = (W_1 + J)_{ef}$. □

Exercise 45. *a) Prove part 1) of Proposition 4.*

b) Prove that $Q + I_n = S\,{}^tS$.

Finally we come to the proof we have advertised for so long.

Bass's Proof of the Ihara Determinant Formula Theorem 1.

Proof. We seek to derive the Ihara determinant formula $\zeta_X(u)^{-1} = (1-u^2)^{r-1}\det\left(I - Au + Qu^2\right)$ from the identity saying $\zeta_X(u)^{-1} = \det(I - W_1 u)$, which was Corollary 2 above. This will be done using some simple block matrix identities.

In the following identity all matrices are $(n+2m)\times(n+2m)$, where the 1st block is $n \times n$, if n is the number of vertices of X and m is the number of unoriented edges of X. Use the preceding proposition to see that

$$\begin{pmatrix} I_n & 0 \\ {}^tT & I_{2m} \end{pmatrix}\begin{pmatrix} I_n(1-u^2) & Su \\ 0 & I_{2m} - W_1 u \end{pmatrix}$$
$$= \begin{pmatrix} I_n - Au + Qu^2 & Su \\ 0 & I_{2m} + Ju \end{pmatrix}\begin{pmatrix} I_n & 0 \\ {}^tT - {}^tSu & I_{2m} \end{pmatrix}.$$

Exercise 46. *Check this equality. Relate it to the Schur complement of a block in a matrix.*

Take determinants to obtain

$$(1-u^2)^n \det(I - W_1 u) = \det\left(I_n - Au + Qu^2\right)\det(I_{2m} + Ju).$$

To finish the proof of Theorem 1, observe that

$$I + Ju = \begin{pmatrix} I & Iu \\ Iu & I \end{pmatrix}$$

implies

$$\begin{pmatrix} I & 0 \\ -Iu & I \end{pmatrix}(I + Ju) = \begin{pmatrix} I & Iu \\ 0 & I\left(1-u^2\right) \end{pmatrix}.$$

Thus $\det(I + Ju) = \left(1-u^2\right)^m$. Since $r - 1 = m - n$, for a connected graph, Theorem 1 follows. □

11.2. **Properties of W_1 and a Proof of the Theorem of Kotani and Sunada.** Next we want to prove Theorem 4 of Kotani and Sunada. First we will need some facts from linear algebra as well as some facts about the W_1 matrix. In the next definition, a **permutation matrix** is a square matrix such that exactly one entry in each row and column is 1 and the rest of the entries are 0.

Definition 27. *An $s\times s$ matrix A, with $s>1$, whose entries are nonnegative is* ***irreducible*** *iff there does* **not** *exist a permutation matrix P such that $A={}^tP\begin{pmatrix} B & C \\ 0 & D\end{pmatrix}P$, where B is a $t\times t$ matrix with $1\leq t<s$.*

The following Theorem is proved in Horn and Johnson [56], p. 361. The Perron-Frobenius theorem concerns such matrices.

Theorem 12. *An $n\times n$ matrix with all nonnegative entries is irreducible iff $(I+A)^{n-1}$ has all positive entries.*

Theorem 13. ***Facts About W_1***

Assume X satisfies the usual hypotheses stated before Definition 1. Let $n=|V|$ the number of vertices of graph X and $m=|E|$ the number of undirected edges of X.

1) The jth row sum of the entries of W_1 is $q_j=-1+$degree (vertex which is the starting vertex of the jth edge).

2) (Horton) The singular values of W_1 (i.e., the square roots of the eigenvalues of $W_1\,{}^tW_1$) are $\left\{q_1,...,q_n,\underbrace{1,...,1}_{2m-n}\right\}$.

3) The matrix $(I+W_1)^{2m-1}$ has all positive entries. This says that the matrix W_1 is irreducible.

Proof. 1) We leave this as an **Exercise**.

2) (Horton [57], [58]). Modify W_1 to list all edges ending at the same vertex together. Note that then

$$W_1\,{}^tW_1=\begin{pmatrix} A_1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & A_n\end{pmatrix}, \text{ and } A_j=(q_j-1)J+I,$$

where J is a $(q_j+1)\times(q_j+1)$ matrix of ones. Since the spectrum of J is $\{q_j+1,0,...,0\}$, the spectrum of A_j is $\{q_j^2,1,...,1\}$. The result follows.

3) This follows from Lemma 2 below. □

Exercise 47. *Consider the graph which consists of one vertex with 2 loops and another vertex on one of the loops. Modify W_1 to list all edges ending at the same vertex together and compute $W_1\,{}^tW_1$.*

Lemma 2. *Suppose X satisfies our usual hypotheses stated before Definition 1. Given a directed edge e_1 starting at a vertex v_1 and a directed edge e_2 ending at a vertex v_2 in X ($v_1=v_2$, $e_1=e_2$, $e_1=e_2^{-1}$ are allowed), there exists a backtrackless path $P=P(e_1,e_2)$ from v_1 to v_2 with initial edge e_1, terminal edge e_2, and length $\leq 2|E|$.*

Proof. See Figure 34 which shows our construction of $P(e_1,e_2)$ in two cases. First we construct a path P without worrying about its length. This construction is not minimal, but it has relatively few cases to consider.

Choose a spanning tree T of X. Define "**cut**" edge of X to mean an edge left out of T. Begin by creating 2 backtrackless paths P_1f_1 and P_2f_2 with initial edges e_1 and e_2^{-1} and terminal edges f_1 and f_2 such that f_1 and f_2 are cut edges (i.e., non-tree edges of X). If e_1 is a cut edge, we let P_1 have length 0 and $f_1=e_1$ (i.e., $P_1f_1=e_1$). If e_1 is not a cut edge, take P_1 to be a backtrackless path in T with initial edge e_1 which proceeds along T until it is impossible to go any further along the tree. Symbolically we write $P_1=e_1T_1$, where T_1 is a path along the tree, possibly of length zero. Let v_1' be the terminal vertex of P_1. With respect to the tree T, v_1' is a dangler or leaf (vertex of degree 1), but X has no danglers. Thus there must be a directed cut edge in X, which we take to be f_1, with initial vertex v_1'. By construction, P_1f_1 is backtrackless also since P_1 is in the tree and f_1 isn't.

Similarly, if e_2 is a cut edge, let P_2 have length 0 and $f_2=e_2^{-1}$ (i.e., $P_2f_2=e_2^{-1}$). If e_2 is not a cut edge, then as above form a backtrackless path $P_2f_2=e_2^{-1}T_2f_2$ where T_2 is in the tree, possibly of length 0 and f_2 is a cut edge. In all cases, we let v_1' and v_2' be the initial vertices of f_1 and f_2.

Now, if we can find a path P_3 beginning at the terminal vertex of f_1 and ending at the terminal vertex of f_2 such that the path $f_1P_3f_2^{-1}$ has no backtracking, then $P=P_1f_1P_3f_2^{-1}P_2^{-1}$ will have no backtracking, with e_1 and e_2 as its initial and terminal edges, respectively. Of course, creating the path $f_1P_3f_2^{-1}$ was the original problem in proving Lemma. However, we now have the additional information that f_1 and f_2 are cut edges of the graph X.

We now have two cases. Case 1 is the case that $f_1\neq f_2$, which is pictured at the top of Figure 34. In this case we can take $P_3=T_3=$ the path within the tree T running from the terminal vertex of f_1 to the terminal vertex of f_2. Then, even if the length of T_3 is 0, the path $f_1T_3f_2^{-1}$ has no backtracks and we have created P.

Case 2 is $f_1 = f_2$. Thus f_1 and f_2 are the same cut edge of X. In the worst case scenario, we would have $e_2 = e_1^{-1}$, $T_2 = T_1$, $f_2 = f_1$. See the lower part of Figure 34. Since the usual hypotheses say X has rank at least 2, there is another cut edge f_3 of X with $f_3 \neq f_1$ or f_1^{-1}. Let T_3 be the path along the tree T from the terminal vertex of f_1 to the initial vertex of f_3 and let T_4 be the path along the tree T from the terminal vertex of $f_2 = f_1$ to the terminal vertex of f_3. Then $P_3 = T_3 f_3 T_4^{-1}$ has the desired property that $f_1 P_3 f_1^{-1}$ has no backtracking, even if T_3 and/or T_4 have length 0. Thus we have created in all cases a backtrackless path P with initial edge e_1 and terminal edge e_2.

One can create a path P of length $\leq 2|E|$ as follows. If an edge is repeated, it is possible to delete all the edges in between the 1st and 2nd versions of that edge as well as the 2nd version of the edge without harming the properties of P. □

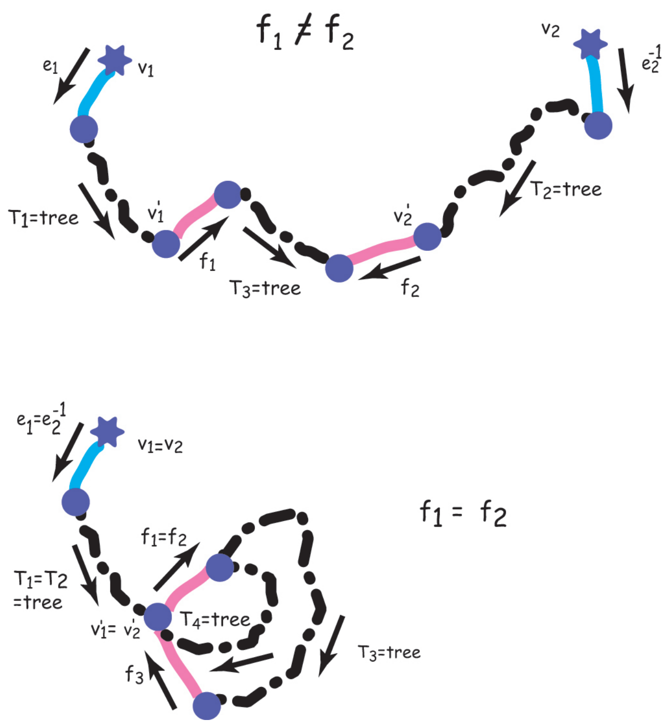


FIGURE 34. **The paths in Lemma 2**. Here dashed black paths are along the spanning tree of X. The edges e_1 and e_2 may not be edges of X which are cut to get the spanning tree T. But f_1, f_2 and (in the second case) f_3 are cut or non-tree edges. The lower figure does not show the most general situation as f_3 need not touch $f_1 = f_2$.

Exercise 48. *Prove Lemma 2 if the graph X is a bouquet of n loops with $n \geq 2$.*

Corollary 3. *Suppose X satisfies our usual hypotheses. In particular, X is finite, connected, with no degree 1 vertices, and its fundamental group has rank at least 2. Then the edge adjacency matrix W_1 is irreducible.*

Proof. It follows from the preceding Lemma that all entries of $(I + W_1)^{2|E|-1}$ are positive. To see this, we look at the e, f entry of $W_1^{\nu-1}$. Take a backtrackless path P starting at e and ending at f. The Lemma says that we can assume the length of P is $\nu = \nu(P) \leq 2|E|$. The e, f entry of the matrix $W_1^{\nu-1}$ is a sum of terms of the form $w_{e_1 e_2} \cdots w_{e_{\nu-1} e_\nu}$, where each e_{i_j} denotes an oriented edge and $e_1 = e$, $e_\nu = f$. The term corresponding to the path P will be positive and the rest of the terms are non-negative. Then use Theorem 12. □

Example 15. ***Example of a Shortest Path from Lemma 2 in Dumbbell.***

Consider the dumbbell graph in Figure 32. The shortest possible path $P(e_1, e_1^{-1})$, using the terminology of Lemma 2, is $e_1 e_2 e_3 e_2^{-1} e_1^{-1}$ with length $5 = 2|E| - 1$.

Exercise 49. *Show that the hypotheses on X in Lemma 2 are necessary. In particular, what happens when the rank of the fundamental group is 1? And what happens if some vertices have degree 1?*

We have the following Corollary to the preceding Theorem.

Corollary 4. *The poles of the Ihara zeta function of X are contained in the region* $\frac{1}{q} \leq R_X \leq |u| \leq 1$, *where* $q+1$ *is the maximum degree of a vertex of* X.

Proof. The poles are reciprocals of the eigenvalues of the W_1 matrix. The singular values of W_1 are $\left\{ q_1, ..., q_n, \underbrace{1, ..., 1}_{2m-n} \right\}$. Assume that $q_j \geq q_{j+1}$. This means that q_j^2 and 1 are successive maxima of the Rayleigh quotient $\frac{{}^t\overline{(W_1 v)} W_1 v}{{}^t\overline{v} v}$ over $v \in \mathbb{C}^n$ orthogonal to the vectors at which the preceding maxima are taken. If $W_1 v = \lambda v$, for non-0 v, then the Rayleigh quotient is $|\lambda|^2$. □

Next recall the Perron-Frobenius Theorem. A proof can be found in Horn and Johnson [56]. We first recall a definition.

Definition 28. *The **spectral radius** $\rho = \rho(A)$ of a matrix A is defined to be the maximum of all $|\lambda|$, for λ an eigenvalue of A.*

Theorem 14. ***(Perron and Frobenius)*** *Let A be an $s \times s$ matrix all of whose entries are nonnegative. Assume that A is irreducible. Then we have the following facts.*

1) The spectral radius $\rho(A)$ is positive and is an eigenvalue of A which is simple (both in the algebraic and geometric senses). There is a corresponding eigenvector of A all of whose entries are positive.

2) Let $S = \{\lambda_1, ..., \lambda_k\}$ be the eigenvalues of A having maximum modulus. Then

$$S = \left\{ \rho(A) e^{2\pi i a/k} \middle| a = 1, ..., k \right\}.$$

3) The spectrum of A is invariant under rotation by $\frac{2\pi}{k}$.

If $A = W_1$, the spectral radius $\rho(W_1) = R^{-1}$ and the number $k = \Delta = g.c.d.$ of the lengths of the primes of X.

Exercise 50. *Prove the last statement.*

Hint. Part 3) of the Perron-Frobenius theorem implies that $\zeta_X(u)^{-1} = \det(I - W_1 u) = f(u^k)$. *By the definition of* Δ, *we have* $\log \zeta_X(u)^{-1} = F(u^\Delta)$. *Thus* $\pi(m) = 0$ *unless* Δ *divides* m. *Recall that, if* $\zeta = e^{2\pi i/k}$,

$$\prod_{j=1}^{k} \left(1 - \zeta^j u\right) = 1 - u^k.$$

Now we proceed to prove the Kotani and Sunada Theorem.

Proof of Theorem 4 of Kotani and Sunada.

Let us restate what we are proving. Suppose $q+1$ **is the maximum degree of** X **and** $p+1$ **is the minimum degree of a graph** X satisfying our usual hypotheses stated before Definition 1.

1) Every pole u **of** $\zeta_X(u)$ **satisfies** $R_X \leq |u| \leq 1$, **with** R_X **from Definition 3, and** $q^{-1} \leq R_X \leq p^{-1}$.

2) For a graph X, **every non-real pole** u **of** $\zeta_X(u)$ **satisfies the inequality** $q^{-1/2} \leq |u| \leq p^{-1/2}$.

3) The poles of ζ_X **on the circle** $|u| = R_X$ **have the form** $R_X e^{2\pi i a/\Delta_X}$, **where** $a = 1, ..., \Delta_X$. **Here** Δ_X **is from Definition 7.**

Proof. The 2nd inequality in Part 1) comes from a result of Frobenius saying that $\rho(W_1) = R^{-1}$ is bounded above and below by the maximum and minimum row sums of W_1, respectively. See Minc [88], p. 24 or Horn and Johnson [56], p. 492.

We know that $R_X \leq |u| \leq 1$ by Corollary 4. Here we will give the Kotani and Sunada proof that $|u| \leq 1$.

If u is a pole of $\zeta_X(u)$ with $|u| \neq 1$, then there is a non-zero vector f so that $\left(I - Au + u^2 Q\right) f = 0$.

We denote the inner product $< f, g >= {}^t\overline{g} f$, for column vectors f, g in $\mathbb{C}^n$. Then

$$0 =< \left(I - uA + u^2 Q\right) f, f >= \|f\|^2 - u < Af, f > + u^2 < Qf, f > .$$

Set $\lambda = \frac{<Af, f>}{\|f\|^2}$, $\delta = \frac{<Df, f>}{\|f\|^2}$, and $D = Q + I$. So we have $1 - u\lambda + u^2(\delta - 1) = 0$. The quadratic formula gives $u = \frac{\lambda \pm \sqrt{\lambda^2 - 4(\delta-1)}}{2(\delta-1)}$.

Clearly $p \leq \delta - 1 \leq q$. We also have $|\lambda| \leq \delta$. To prove this, we can make use of the S and T matrices in Proposition 4. Note that $S = (M \ N)$ and $T = (N \ M)$ where M and N have $m = |E|$ columns. Then it is an Exercise using the matrix identities in Proposition 4 to show that

$$D - A = (M - N)\,{}^t(M - N) \text{ and } D + A = (M + N)\,{}^t(M + N).$$

So $| < Af, f > | \leq < Df, f >$. It follows that $|\lambda| \leq \delta$.

There are now two cases.

Case 1. **The pole u is real.**

Then

$$\frac{\lambda + \sqrt{\lambda^2 - 4(\delta - 1)}}{2(\delta - 1)} \leq \frac{\delta + \sqrt{\delta^2 - 4(\delta - 1)}}{2(\delta - 1)} = 1$$

and

$$\frac{\lambda - \sqrt{\lambda^2 - 4(\delta - 1)}}{2(\delta - 1)} \geq \frac{-\delta - \sqrt{\delta^2 - 4(\delta - 1)}}{2(\delta - 1)} = -1.$$

Thus $|u| \leq 1$.

Case 2. **The pole u is not real.**

Then

$$|u|^2 = \frac{\lambda^2 + \left(4(\delta - 1) - \lambda^2\right)}{4(\delta - 1)^2} = \frac{1}{\delta - 1}.$$

The second statement of Theorem 4 follows from this and the fact that $p \leq \delta - 1 \leq q$.

Finally the third part of Theorem 4 follows from the Perron - Frobenius Theorem 14. □

Proposition 5. *Make our usual hypotheses on the graph X. Recall Definitions 3, 28, Then we have the inequality*

$$\rho(A) \geq \frac{p}{q} + \frac{1}{R},$$

where $p + 1$=minimum degree of a vertex of X and $q + 1$=maximum degree of a vertex of X.

Proof. Let $u = R$, the radius of convergence of the Ihara zeta or reciprocal of the Perron-Frobenius eigenvalue of W_1. Then, as in the preceding proof,

$$0 = \| f \|^2 - u < Af, f > + u^2 < Qf, f > .$$

Set $\lambda = \frac{<Af, f>}{\| f \|^2}$, $\delta = \frac{<Df, f>}{\| f \|^2}$, and $D = Q + I$. So we have $1 - R\lambda + R^2 (\delta - 1) = 0$. Thus $\lambda = \frac{1}{R} + R(\delta - 1)$. It follows that $\rho \geq \frac{1}{R} + \frac{p}{q}$, since

$$R \geq \frac{1}{q} \text{ and } \delta - 1 \geq p.$$

□

Problem 1. *It is a research problem to see if it is possible to improve the inequality in this proposition by replacing $\frac{p}{q}$ with* 1.

12. Path Zeta Functions

Here we look at a zeta function invented by Stark. It has several advantages over the edge zeta. It can be used to compute the edge zeta with smaller determinants. It gives the edge zeta for a graph in which an edge has been fused; i.e., shrunk to one vertex.

First recall that the fundamental group of X can be identified with the group generated by the edges left out of a spanning tree T of X. Then T has $|V| - 1 = n - 1$ edges. We label the oriented versions of these **edges left out of the spanning tree** T **(**or "**cut**" or**"deleted" edges** of T**)** (and their inverses)

$$e_1, \ldots, e_r, e_1^{-1}, \ldots, e_r^{-1}.$$

Label the remaining (oriented) **edges in the spanning tree** T

$$t_1, \ldots, t_{n-1}, t_1^{-1}, \ldots, t_{n-1}^{-1}.$$

Any backtrackless, tailless cycle on X is uniquely (up to starting point on the tree between last and first e_k) determined by the ordered sequence of e_k's it passes through. In particular, if e_i and e_j are 2 consecutive e_k's in this sequence, then the part of the cycle between e_i and e_j is the unique backtrackless path on T joining the last vertex of e_i to the first vertex of e_j. For such e_i and e_j, we know that e_j is not the inverse of e_i, as the cycle is backtrackless. Nor is the last edge the inverse of the first. Conversely, if we are given any ordered sequence of edges from the e_k's with no 2 consecutive edges being inverses of each other and with the last edge not being inverse to the 1st edge, there is a unique (up to starting point on the tree between the last and first e_k) backtrackless tailless cycle on X whose sequence of e_k's is the given sequence.

The free group of rank r generated by the e_k's puts a group structure on backtrackless tailless cycles which is completely equivalent to the fundamental group of X. When dealing with the fundamental group of X, any closed path starting at a fixed vertex v_0 on X is completely determined up to homotopy by the ordered sequence of e_k's that it passes through. If we do away with backtracking, such a path will be composed of a tail on the tree and then a backtrackless, tailless cycle corresponding to the same sequence of e_k's, followed by the original tail in the reverse direction, ending at v_0 again. Thus the free group of rank r generated by the e_k's is identified with the fundamental group of X. We will therefore refer to the free group generated by the e_k's as the **fundamental group** of X.

There are 2 elementary reduction operations for paths written down in terms of directed edges just as there are elementary reduction operations for words in the fundamental group of X. This means that if $a_1, ..., a_s$ and e are taken from the e_k's and their inverses, the **2 elementary reduction operations** are:

i) $a_1 \cdots a_{i-1} e e^{-1} a_{i+2} \cdots a_s \cong a_1 \cdots a_{i-1} a_{i+2} \cdots a_s$;

ii) $a_1 \cdots a_s \cong a_2 \cdots a_s a_1$.

Using the 1st elementary reduction operation, each equivalence class of words corresponds to a group element and a word of minimum length in an equivalence class is a **reduced** word in group theory language. Since the second operation is equivalent to conjugating by a_1, an equivalence class using both elementary reductions corresponds to a conjugacy class in the fundamental group. A word of minimum length using both elementary operations corresponds to finding words of minimum length in a conjugacy class in the fundamental group. If $a_1, ..., a_s$ are taken from $e_1, ..., e_{2r}$, a word $C = a_1 \cdots a_s$ is of minimum length in its conjugacy class iff $a_{i+1} \neq a_i^{-1}$, for $1 \leq i \leq s-1$ and $a_1 \neq a_s^{-1}$. This is equivalent to saying that C corresponds to a **backtrackless, tailless** cycle under the correspondence above. Equivalent cycles correspond to conjugate elements of the fundamental group. A conjugacy class $[C]$ is **primitive** if a word of minimal length in $[C]$ is not a power of another word. We will say that a word of minimal length in its conjugacy class is **reduced in its conjugacy class**. From now on, we assume a representative element of $[C]$ is chosen which is reduced in $[C]$.

Definition 29. *The $2r \times 2r$ **path matrix** Z has ef entry given by the complex variable z_{ef} if $e \neq f^{-1}$ and by 0 if $e = f^{-1}$.*

Note that the path matrix Z has only one zero entry in each row unlike the edge matrix W from Definition 25 which is rather sparse unless the graph is a bouquet of loops. Next we imitate the definition of the edge zeta function.

Definition 30. *Define the **path norm** for a path $C = a_1 \cdots a_s$ reduced in its conjugacy class $[C]$, where $a_i \in \left\{e_1^{\pm 1}, \ldots, e_s^{\pm 1}\right\}$ as*

$$N_P(C) = z_{a_1 a_2} \cdots z_{a_{s-1} a_s} z_{a_s a_1}.$$

*Then the **path zeta** is defined for small $|z_{ij}|$ to be*

$$\zeta_P(Z, X) = \prod_{[C]} (1 - N_P(C))^{-1},$$

where the product is over primitive reduced conjugacy classes $[C]$ other than the identity class.

We have similar results to those for the edge zeta.

Theorem 15. ***Determinant Formula for Path Zeta.***

$$\zeta_P(Z, X)^{-1} = \det(I - Z).$$

Proof. Imitate the proof of Theorem 11 for the edge zeta. **Exercise**. □

Next we want to find a way to get the edge zeta out of the path zeta. To do this requires a procedure called **specializing the path matrix to the edge matrix.** Use the notation above for the edges e_i left out of the spanning tree T and the edges t_j of T. A closed backtrackless tailless path C is first written as a product of generators of the fundamental group and then as a product of actual edges e_i and t_k. Do this by inserting $t_{k_1} \cdots t_{k_s}$ which is the unique non backtracking path on T joining the terminal vertex of e_i and the starting vertex of e_j if e_i and e_j are successive deleted or cut edges in C. Now **specialize the path matrix Z to $Z(W)$ with entries**

$$z_{ij} = w_{e_i t_{k_1}} w_{t_{k_1} t_{k_2}} \cdots w_{t_{k_{s-1}} t_{k_s}} w_{t_{k_s} e_j}. \tag{12.1}$$

Then the path zeta function at $Z(W)$ specializes to the edge zeta function

Theorem 16. *Using the specialization procedure defined above, we have*

$$\zeta_P(Z(W), X) = \zeta_E(W, X).$$

Proof. The result should be clear since the two defining infinite products coincide. □

M. Horton [57] has a Mathematica program to do the specialization in formula (12.1).

Note that $\zeta_P(Z, X) = \zeta_E(Z, X^\#)$, where $X^\#$ is the graph obtained from X by fusing all the edges of the spanning tree T to a point. Thus $X^\#$ consists of a bouquet of r loops.

Example 16. ***The Dumbbell Again*** *Recall that the edge zeta of the dumbbell graph of Figure 32 was evaluated by a* 6×6 *determinant. The path zeta requires a* 4×4 *determinant. Take the spanning tree to be the vertical edge. That is really the only choice here. One finds using the determinant formula for the path zeta and the specialization of the path to edge zeta:*

$$\zeta_E(W,X)^{-1} = \det \begin{pmatrix} w_{11}-1 & w_{12}w_{23} & 0 & w_{12}w_{26} \\ w_{35}w_{51} & w_{33}-1 & w_{35}w_{54} & 0 \\ 0 & w_{42}w_{23} & w_{44}-1 & w_{42}w_{26} \\ w_{65}w_{51} & 0 & w_{65}w_{54} & w_{66}-1 \end{pmatrix}. \tag{12.2}$$

If we shrink the vertical edge to a point (which we call **"fusion"** or contraction), the edge zeta of the new graph is obtained by replacing any $w_{x2}w_{2y}$ (for $x, y = 1, 3, 4, 6$) which appear in formula (12.2) by w_{xy} and any $w_{x5}w_{5y}$ (for $x, y = 1, 3, 4, 6$) by w_{xy}. This gives the zeta function of the new graph obtained from the dumbbell, by fusing the vertical edge.

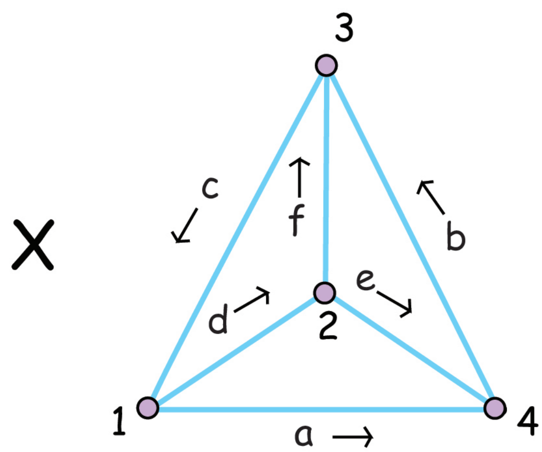


FIGURE 35. **Labeling the edges of the tetrahedron.**

Example 17. ***The Path Zeta Function of the Tetrahedron Specializes to the Edge Zeta Function of the Tetrahedron.***

Refer to Figure 35 and label the inverse edges with the corresponding capital letters. List the edges that index the entries of the matrix Z as a, b, c, A, B, C. You will then find that the matrix $Z(W)$ for the tetrahedron is

$$\begin{pmatrix} w_{aE}w_{ED}w_{Da} & w_{ab} & w_{aE}w_{Ef}w_{fc} & 0 & w_{aE}w_{Ef}w_{fB} & w_{aE}w_{ED}w_{DC} \\ w_{bF}w_{FD}w_{Fa} & w_{bF}w_{Fe}w_{eb} & w_{bc} & w_{bF}w_{Fe}w_{eA} & 0 & w_{bF}w_{FD}w_{DC} \\ w_{ca} & w_{cd}w_{de}w_{eb} & w_{cd}w_{df}w_{fc} & w_{cd}w_{de}w_{eA} & w_{cd}w_{df}w_{fB} & 0 \\ 0 & w_{Ad}w_{de}w_{eb} & w_{Ad}w_{df}w_{fc} & w_{Ad}w_{de}w_{eA} & w_{Ad}w_{df}w_{fB} & w_{AC} \\ w_{BE}w_{ED}w_{Da} & 0 & w_{BE}w_{Ef}w_{fc} & w_{BA} & w_{BE}w_{Ef}w_{fB} & w_{BE}w_{ED}w_{DC} \\ w_{CF}w_{FD}w_{Da} & w_{CF}w_{Fe}w_{eb} & 0 & w_{CF}w_{Fe}w_{eA} & w_{CB} & w_{CF}w_{FD}w_{DC} \end{pmatrix}.$$

Exercise 51. *As a check on the preceding example, specialize all the variables in the* $Z(W)$ *matrix to* $u \in \mathbb{C}$ *and call the new matrix* $Z(u)$. *Check that* $det(I - Z(u))$ *is the reciprocal of the Ihara zeta function* $\zeta_X(u)$.

Exercise 52. *Compute the path zeta function for your favorite graph.*

Part 4. **Finite Unramified Galois Coverings of Connected Graphs**

Once again,we assume the usual hypotheses for all graphs. These hypotheses were stated before Definition 1. The unweighted graph X has vertex set V and (undirected) edge set E. It is possibly irregular and possibly has loops and multiple edges. We view a graph covering as an analog of an extension of algebraic number fields or function fields. It is also an analog of a covering of Riemann surfaces. Coverings of weighted graphs have been considered by Chung and Yau [29] as well as Osborne and Severini [96]. The latter paper applies graph coverings to quantum computing.

All the coverings considered here will be unramified unless stated otherwise.

At the end of subsection 13.3 we will consider some ramified graph coverings.

13. Finite Unramified Coverings and Galois Groups

In this section we begin the study of Galois theory for finite unramified covering graphs. It leads to a generalization of Cayley and Schreier graphs and it provides factorizations of zeta functions of normal coverings into products of Artin L-functions associated to representations of the Galois group of the covering. Coverings can also be used in constructions of Ramanujan graphs and in constructions of pairs of graphs that are isospectral but not isomorphic. Most of this section is taken from Stark and Terras [120]. Other references are Sunada [128] and Hashimoto [51]. Another theory of graph covering which is essentially equivalent can be found in Gross and Tucker [47]. Our coverings differ in that we require all our graphs to be connected and our aim is to find analogs of the basic properties of finite degree extensions of algebraic number fields and their zeta functions. It is also possible to consider infinite coverings such as the universal covering tree T of a finite graph X. We will not do so here except in passing. This is mostly a book about finite graphs after all.

13.1. **Definitions.** If our graphs had no multiple edges and loops, our definition of covering would be Definition 5. First we need to think about directed coverings. If we want to prove the fundamental theorem of Galois theory for graphs with loops and multiple edges, Definition 5 will not be sufficient. We need to make more a more complicated definition of graph covering involving neighborhoods in directed graphs. This definition is necessary for the proof of the unique lifting property. See Proposition 6. See Figure 42 in the next subsection for an example illustrating the need for our definitions from the point of view of being able to develop the fundamental theorem of Galois theory. See also Massey [83], p. 201, for the same definitions. The one third in the definition of (directed) neighborhood could be replaced by any $\varepsilon > 0$.

Definition 31. *A* ***neighborhood*** *N of a vertex v in a directed graph X is obtained by taking one–third of each edge at v. The labels and directions are to be included. See Figure 36.*

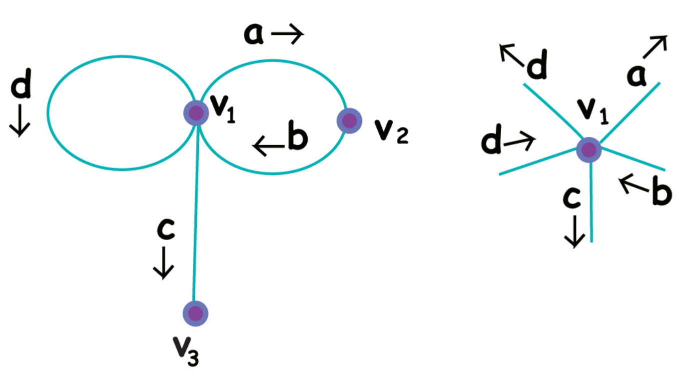


FIGURE 36. **a directed graph and a neighborhood of vertex** v_1

Definition 32. *An undirected finite graph Y is a* ***covering*** *of an undirected graph X if, after arbitrarily directing the edges of X, there is an assignment of directions to the edges of Y and an onto* ***covering map*** *$\pi : Y \longrightarrow X$ sending neighborhoods of Y 1-1, onto neighborhoods of X preserving directions.*

Note that a covering map π not only takes vertices of Y to vertices of X, but also edges of Y to edges of X. The fact that Y is a covering of X is independent of the choice of directions on X. In coverings Y over X (written Y/X) involving loops and multiedges, it is useful to **label** the edges of X and then give the edges of the cover analogous labels in order to see that we really have a covering map. We have attempted to do this for all the examples that follow. See Figure 42.

See Figure 37 for an example of an invalid assignment of directions in Y over X. Note also that if you lift a loop you may get a graph with multiple edges. Thus, once you allow loops, you cannot discuss the general covering without allowing multiple edges. The example in Figure 37 is an illegal covering map since a neighborhood of vertex v of X has one edge going in and one going out (once you take $1/3$ of each edge), while that is not true for the neighborhoods of v' and v'' in Y.

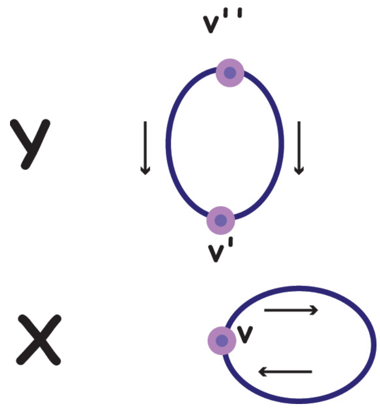


FIGURE 37. This is an **example of an illegal covering map** since a neighborhood of vertex v of X has one edge going in and one going out (once you take $1/3$ of each edge), while that is not true for the neighborhoods of v' and v''.

Construction of a d-Sheeted Covering Y of X.

Construct a covering Y of a connected graph X as follows. First find a spanning tree T in X. For a d-sheeted covering, make d copies of T. This gives the nd vertices of our graph Y. That is, Y can be viewed as the set of points (x,i), $x \in X$, $i = 1, ..., d$. Then lift to Y the edges of X left out of T to get edges of Y. See Figure 38.

We look at the copies of the spanning tree as the **sheets** of the covering Y of X. Thus the cube (see Figure 39) is a 2-sheeted covering of the tetrahedron. We refer to such a 2-sheeted covering as a **quadratic** covering in keeping with the terminology from number fields. Similarly we call a 3-sheeted covering **cubic**. A 4-sheeted covering is **quartic**, and so on.

Conversely, by Proposition 6 below, we see that the spanning tree in X has a unique lift intersecting each vertex in $\pi^{-1}(v)$, for a fixed vertex v of X. This gives the sheets of the cover.

We need to recall a result from topology about uniqueness of liftings of paths C in X to a unique path starting on sheet 1 say in a covering space Y. See Massey [83] pp. 151 and 201, for example. As noted above, we made our definitions of covering involving directed neighborhoods for this result to work even in the presence of loops and multiple edges.

Proposition 6. ***Uniqueness of Lifts of Paths in Covers.*** *Suppose Y is a covering of X. Let C be a path in X. Then C has a unique lift $\widetilde{C}$ to Y once you fix the initial vertex of $\widetilde{C}$.*

Proof. Let π be the covering map from Y onto X. According to our definitions we may assume every edge of Y and X directed and that π preserves directions. Suppose e is a directed edge starting at vertex a in X. Then e has a unique lift to $\widetilde{e}$ in Y such that $\pi(\widetilde{e}) = e$ once you know which vertex in $\pi^{-1}(a)$ is the starting vertex for $\widetilde{e}$. Figure 42 in the next subsection shows many examples of such lifts.

Then to obtain the unique lift of a path $e_1 e_2 \cdots e_s$ in X, you just lift each directed edge e_j, in order, as j goes from 1 to s, completing the proof. □

Exercise 53. *Consider any of the coverings in Figure 42 in the next subsection. Show that Proposition 6 is false if we delete the directions on edges.*

Next we want to define what we mean by a Galois or normal covering. Of course this will be our favorite kind of cover, since our aim is to develop Galois theory for coverings.

Definition 33. *If Y/X is a d-sheeted covering with projection map $\pi : Y \longrightarrow X$, we say that it is a* ***normal or Galois covering*** *when there are d graph automorphisms $\sigma : Y \longrightarrow Y$ such that $\pi \circ \sigma = \pi$. The* ***Galois group*** *$G(Y/X)$ is the set of these maps σ. By* ***"graph automorphism"*** *we mean a 1-1 onto map of vertices and directed edges of Y preserving directions.*

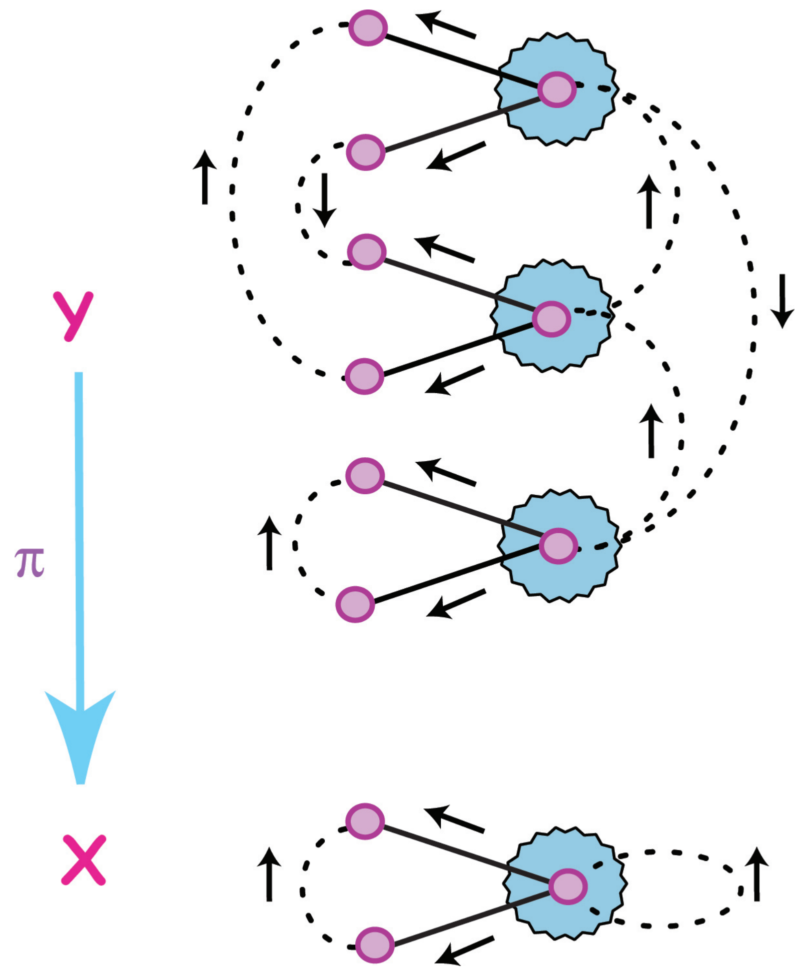


FIGURE 38. **A 3-sheeted covering.** The blue fuzzy area in X is a neighborhood of a selected vertex. The 3 blue fuzzies in Y are all of its inverse images under π.

Later we will see that if we want to make Y a normal cover of X, with Galois group G, we can make use of an appropriate permutation representation π of G to tell us how to lift edges. Let us consider a few basic examples. We will explain how to construct such examples once we have the basic facts about the Galois theory of normal graph coverings.

Example 18. ***The cube is a normal quadratic or 2-sheeted covering of the tetrahedron.*** *See Figure 39, where the edges in a spanning tree for X are shown as pink dotted lines. The edges of the corresponding two sheets of Y are also shown as dotted lines.*

Exercise 54. *Create another 2-cover Y' of K_4 using the same spanning tree as we used in Figure 39, except this time when you lift the 3 non-tree edges of K_4, arrange it so that the lift of only 1 non-tree edge goes from sheet 1 to sheet 2 while the other 2 lifts of non-tree edges do not change sheet. Is Y' normal over K_4?*

Example 19. ***A Non-Normal Cubic Covering of K_4.*** *See Figure 40.*

Exercise 55. *Explain why the 3-sheeted covering in Figure 40 is not a normal covering of the tetrahedron. Hint: a' is adjacent to b' but a'' is not adjacent to b''.*

Proposition 7. *Suppose Y/X is a normal covering. The Galois group $G = G(Y/X)$ acts transitively on the sheets of the covering.*

Proof. By Proposition 6, each spanning tree has a unique lift starting at any point in $\pi^{-1}(v_0)$, where v_0 is a fixed point in X. These d lifts are the sheets of the covering.

An automorphism $\sigma \in G$ that fixes a vertex $\widetilde{v_0} \in \pi^{-1}(v_0)$ is the identity. To see this, suppose $\widetilde{v}$ is any vertex in Y. Then a path $\widetilde{P}$ from $\widetilde{v_0}$ to $\widetilde{v}$ in Y projects under π to a path from v_0 to $v = \pi(\widetilde{v})$ in X. So $\sigma(\widetilde{P})$ is also a lift of P starting at $\widetilde{v_0}$. Thus, by Proposition $\widetilde{v_0}$, we see that $\sigma(\widetilde{P}) = \widetilde{P}$. It follows, since $\widetilde{v}$ was arbitrary, that σ must be the identity.

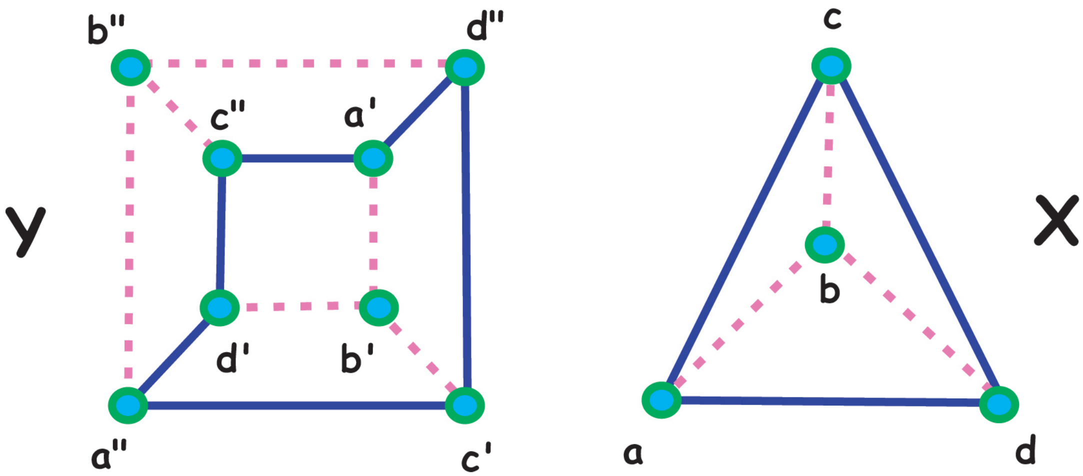


FIGURE 39. **The cube is a normal quadratic covering of the tetrahedron.** The 2 sheets of Y are copies of the spanning tree in X pictured with pink dashed lines.

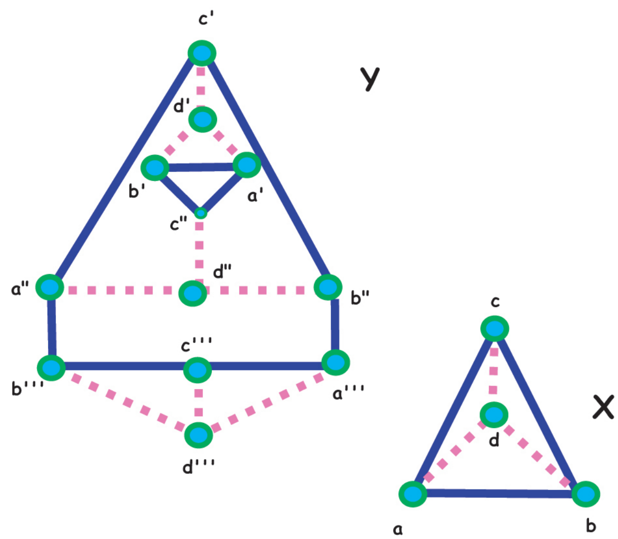


FIGURE 40. **A non-normal cubic (3-sheeted) covering of the tetrahedron.** The spanning tree in $X = K_4$ is shown with dashed pink lines. The sheets of the covering Y are similarly colored.

So each distinct $\sigma \in G$ takes $\widetilde{v_0}$ to a different point and there are only d different points in Y above v_0. It follows that the action of G is transitive. Otherwise two different automorphisms would take v_0 to the same point and we just showed that is impossible. □

Notation 3. ***Our notation for vertices and sheets of a normal cover.*** *Suppose Y/X is normal with Galois group G. We choose one of the sheets of Y and call it sheet 1. The image of sheet 1 under an element g in G will be called* **sheet** *g. Any vertex $\widetilde{x}$ on Y can then be uniquely denoted $\widetilde{x} = (x, g)$, where $x = \pi(\widetilde{x})$ and g is the sheet containing $\widetilde{x}$.*

Definition 34. ***Action of the Galois Group*** *The Galois group $G(Y/X)$ moves sheets of Y via $g \circ (\text{sheet } h) = \text{sheet}(gh)$:*

$$g \circ (x, h) = (x, gh), \quad \text{for } x \in X, g, h \in G$$

It follows that g moves a path in Y as follows:

$$g \circ (\text{path from } (a, h) \text{ to } (b, j)) = \text{path from } (a, gh) \text{ to } (b, gj). \tag{13.1}$$

Even non-normal coverings have the nice property that the inverse Ihara zeta below divides the inverse Ihara zeta above. The analogous fact is only conjectured for Dedekind zetas of extensions of number fields.

Proposition 8. ***Divisibility Properties of Zeta Functions of Covers.***
Suppose Y is a d-sheeted (possibly non-normal) covering of X. Then $\zeta(u,X)^{-1}$ divides $\zeta(u,Y)^{-1}$.

Proof. Start with the Ihara formula $\zeta(u,Y)^{-1} = (1-u^2)^{r_Y-1}\det\left(I_Y - A_Y u + Q_Y u^2\right)$. Note that $r_Y - 1 = |E_Y| - |V_Y| = d\left(|E_X| - |V_X|\right)$. Thus $(1-u^2)^{r_X-1}$ divides $(1-u^2)^{r_Y-1}$.

Now order the vertices of Y in blocks corresponding to the sheets of the cover, so that A_Y consists of blocks $\widetilde{A}_{ij}$, with $1 \le i, j \le d$ such that $\sum_j \widetilde{A}_{ij} = A_X$.

The same ordering puts Q_Y in block diagonal form with d copies of Q_X down the diagonal. Similarly I_Y has block diagonal form consisting of d copies of I_X down the diagonal.

Consider $I_Y - A_Y u + Q_Y u^2$. Without changing the determinant, we can add the right $d-1$ block columns to the first block column. The new first column is

$$\begin{pmatrix} I_X - A_X u + Q_X u^2 \\ \vdots \\ I_X - A_X u + Q_X u^2 \end{pmatrix}.$$

Then subtract the first block row from all the rest of the block rows. Then the first block column becomes:

$$\begin{pmatrix} I_X - A_X u + Q_X u^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The proposition follows. □

13.2. **Examples of Coverings.** This subsection should provide enough examples to clarify the definitions.

Example 20. ***An n-cycle is a normal n-fold covering of a loop with cyclic Galois group.*** *See Figure 41 for this example.*

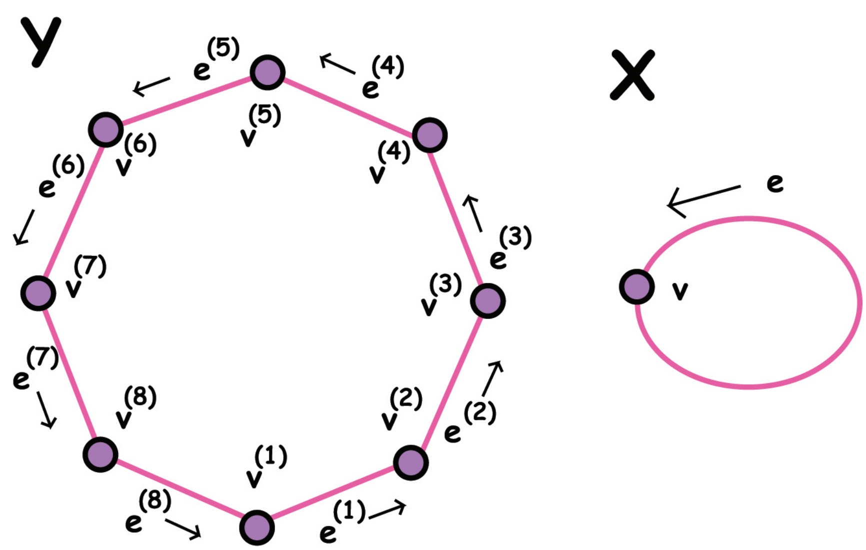


FIGURE 41. **An n-cycle is a normal n-fold covering of a loop with cyclic Galois group.**

The Ihara zeta function of the loop X in Figure 41 is $\zeta_X(u) = (1-u)^{-2}$, and the zeta function of the n-cycle is $\zeta_Y(u) = (1-u^n)^{-2}$. Thus

$$\zeta_Y(u) = (1-u^n)^{-2} = \prod_{j=0}^{n-1} (1-w^j u)^{-2}, \quad \text{where} \quad w = e^{2\pi i/n}.$$

This factorization of $\zeta_Y(u)$ will later be seen as a special case of the factorization of the Ihara zeta function of Y into a product of Artin L-functions associated to representations of the Galois group of Y/X. See Corollary 5 in Section 18 of Part 4.

Question. **Suppose a graph Y has a large symmetry group S and G is a subgroup of S. Is there a graph X such that Y is a normal cover of X with group G?**

Answer. Not always. For example, the cube has S_4 symmetry group - a group of order 24. But if G is a subgroup of order d such that d does not divide 4, G cannot be the Galois group $G(Y/X)$. Why? If Y/X were a Galois cover with d sheets, it would follow that d divides the number of vertices (and edges) of Y. But the cube has 8 vertices and 12 edges. Therefore d must divide $r_Y - 1 = |E| - |V| = 12 - 8 = 4$.

The next figure shows why our definitions were so messy.

Example 21. ***A Klein 4-Group ($\mathbb{Z}_2 \times \mathbb{Z}_2$)$-$ cover of the Dumbbell,*** *illustrating the need for our Definition 32 and our later definition of intermediate cover in order to get the fundamental theorem of Galois theory. See Figure 42 for this example. It is not hard to check that Z is a Galois cover of the dumbbell X with Galois group $\mathbb{Z}_2 \times \mathbb{Z}_2$. In the next section we will see that according to our definitions of intermediate cover, we have 3 intermediate 2-covers Y, Y", Y'''. The last 2 are isomorphic if you ignore directions. That would mean that ignoring directions invalidates the fundamental theorem of Galois theory.*

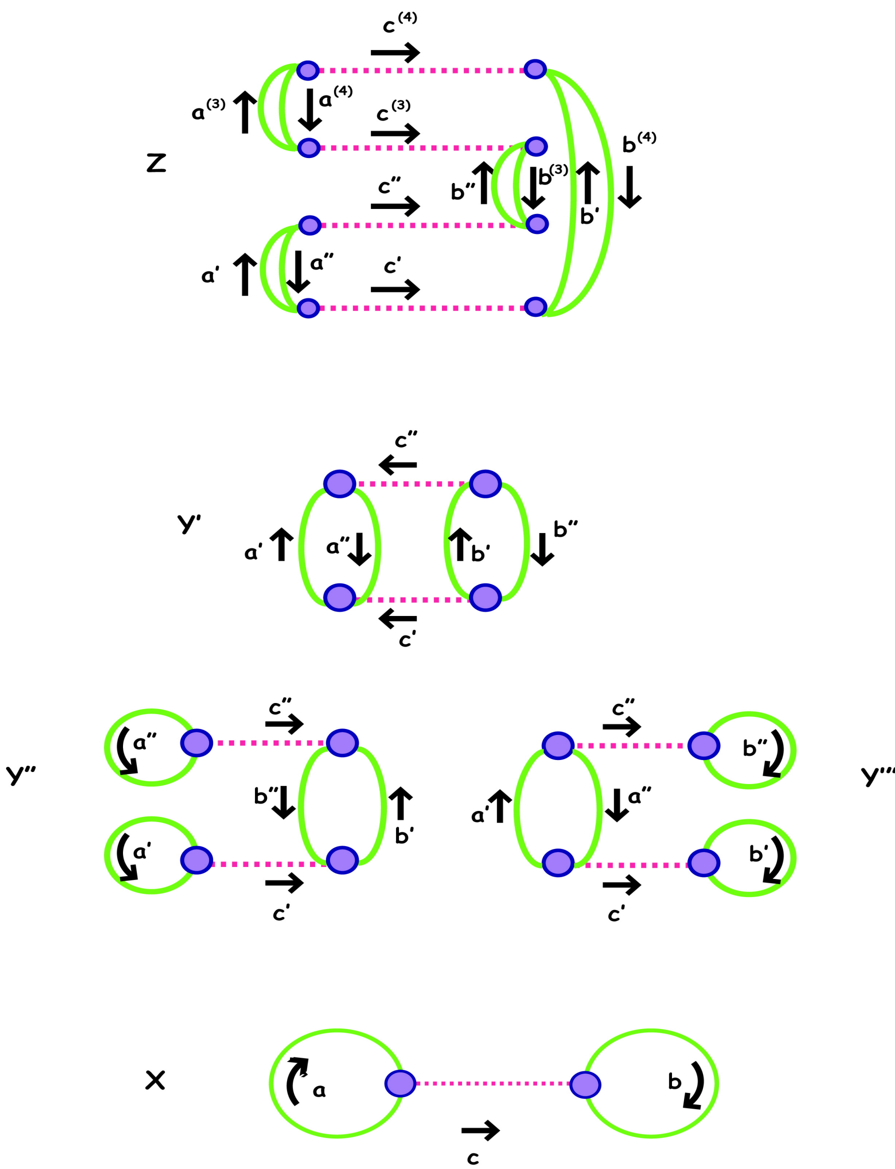


FIGURE 42. Z **is a normal Klein 4 group covering of the dumbbell** X. We show 3 intermediate 2-covers of X named Y, Y' and Y'''. Note that the last 2 are isomorphic as undirected graphs. The spanning tree of X and the sheets of the covers are indicated by dotted pink lines.

Example 22. ***Two graphs with the cube as a normal cover.*** *See Figures 43 and 44 for these examples. Let Y be the cube. Then $|V| = 8$, $|E| = 12$ and $|G|$ divides $g.c.d.(8, 12) = 4$. Figure 43 is a normal covering Y/X where the cube=Y such that $G = G(Y/X)$ is a cyclic group of order 4. Figure 44 is another such covering Y/X in where $G = G(Y/X)$ is the Klein 4-group. We include in the figures an intermediate quadratic cover in each case. The concept of intermediate cover will be discussed in the next section.*

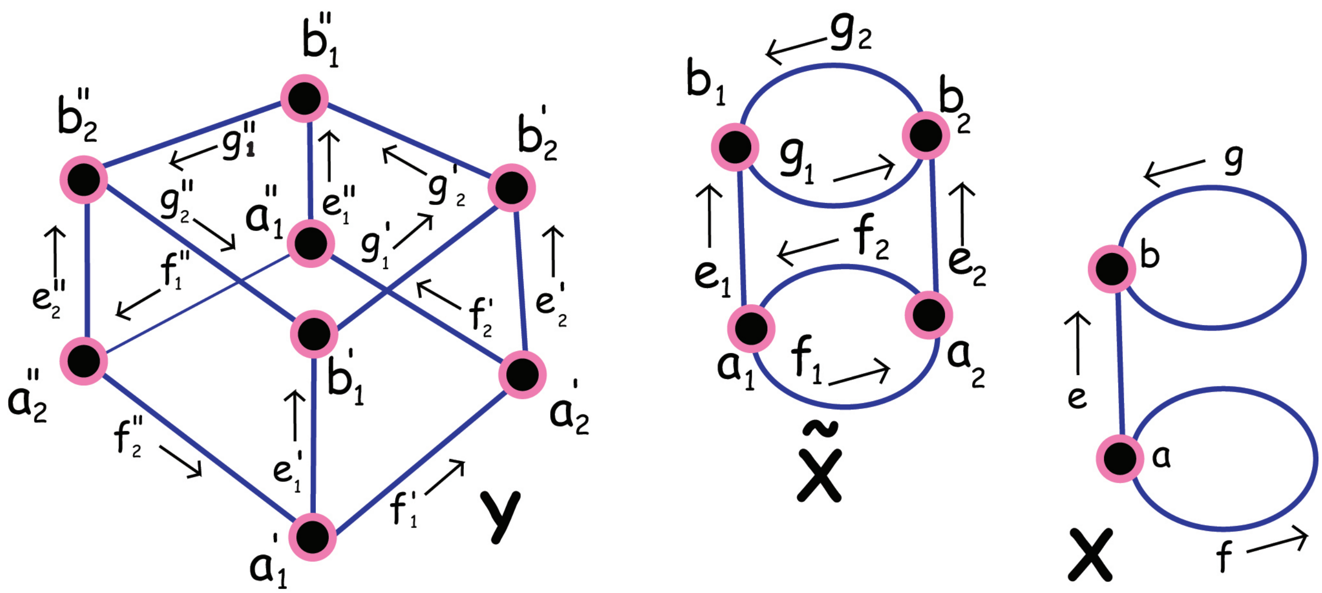


FIGURE 43. **An order 4 cyclic cover Y/X, where Y is the cube.** Included is the intermediate quadratic cover $\widetilde{X}$. The notation makes clear the covering projections $\pi : Y \longrightarrow X$, $\pi_2 : Y \longrightarrow \widetilde{X}$, $\pi_1 : \widetilde{X} \longrightarrow X$,

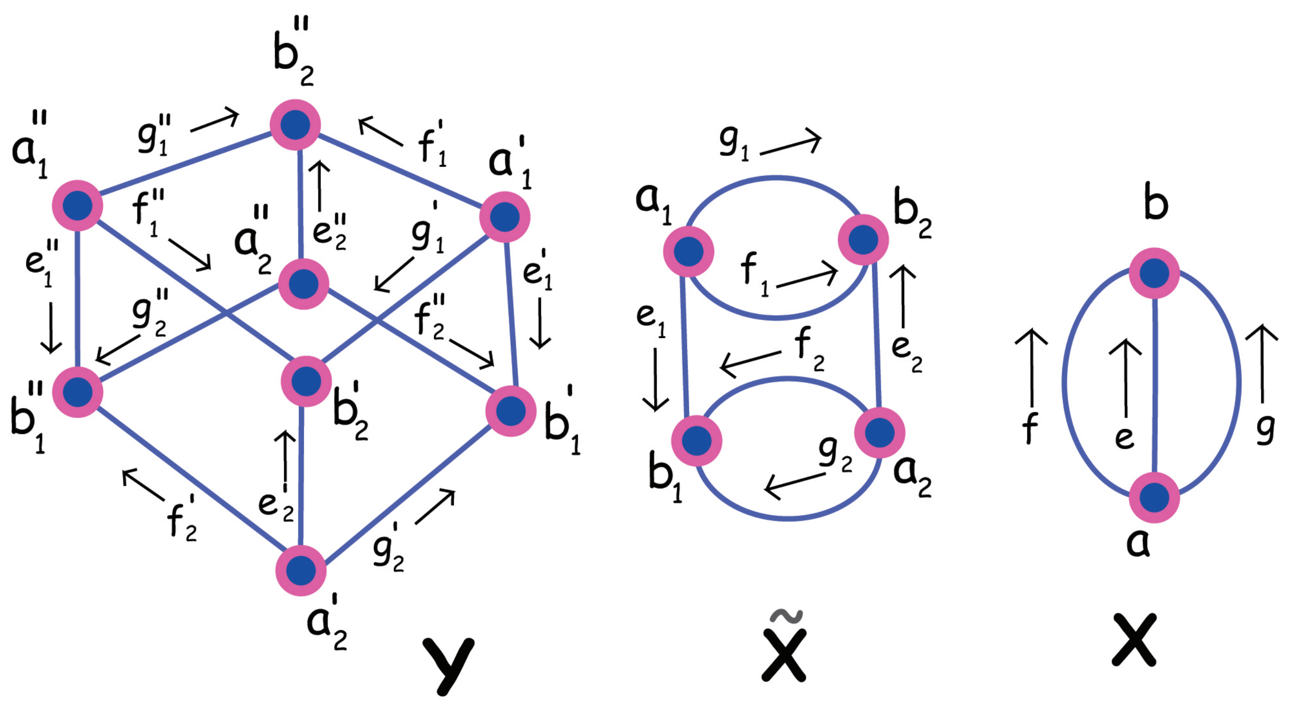


FIGURE 44. **A Klein 4-group Cover** Y/X, where Y = the cube. Included is one of the 3 intermediate quadratic covers.

Example 23. ***The Octahedron as a Cyclic 6-fold Cover of 2 Loops.*** *The octahedron has* $|V| = 6$, $|E| = 12$ *which implies that a Galois group of order 6 is possible for the octahedron as a covering of a bouquet of 2 loops. See Figure 45 for an example.*

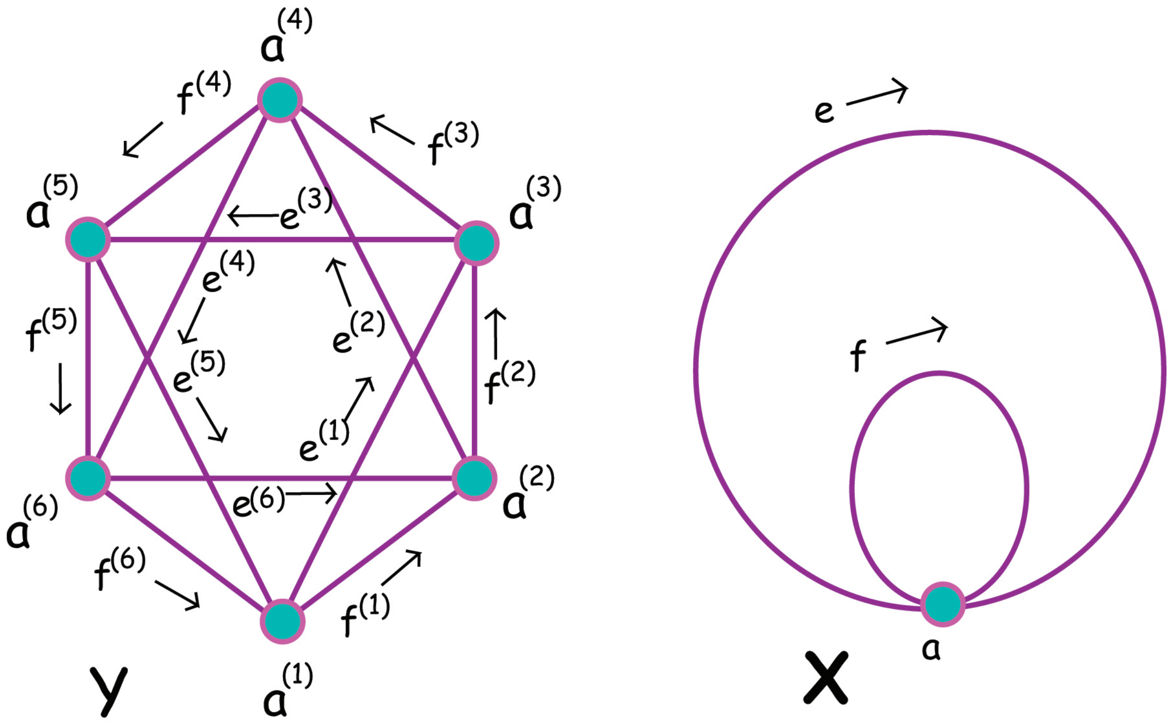


FIGURE 45. **A Cyclic 6-fold cover** Y/X, where Y is the octahedron.

13.3. **Some Ramification Experiments.** The word "ramified" comes from the theory of extensions of algebraic number fields or function fields over finite fields. In particular, there is a **conjecture of Dedekind** saying that if K is an extension of the number field F, then the Dedekind zeta function $\zeta_K(s)$ divides $\zeta_K(s)$, even when there is ramification. One can also view the theory of graph coverings as analogous to coverings of Riemann surfaces or topological manifolds. A ramified surface would be a branched surface such as the Riemann surface of $\sqrt{z}$ or $\Gamma\backslash H$ with a discrete group Γ acting with fixed points such the modular group $SL(2,\mathbb{Z})$.I have tried some experiments on ramifying vertices and edges of coverings. See the examples below. References are Malmskog and Manes [81] plus Baker and Norine [8].

Ramified Example 1. The zeta function of a graph L_n consisting of 1 vertex and n loops can be found from the Ihara determinant formula in Theorem 1. Here the adjacency matrix is $1\times 1:$ $A = 2n$. The matrix Q, also 1×1, is $Q = 2n-1$. The rank of the fundamental group is $r = n$. The Ihara formula says

$$\begin{aligned}\zeta(u,L_n)^{-1} &= (1-u^2)^{r_X-1}\det\left(I - A_X u + Q_X u^2\right)\\ &= (1-u^2)^{n-1}\left(1-2nu+(2n-1)u^2\right)\\ &= (1-u^2)^{n-1}(1-u)(1-(2n-1)u)\end{aligned}$$

If we view L_n as a ramified covering of L_1, we are happy since $\zeta(u,L_1)^{-1}$ divides $\zeta(u,L_n)^{-1}$. However $\zeta(u,L_2)^{-1}$ does not divide $\zeta(u,L_{2n})^{-1}$. The good thing is that there is only one bad factor and it is linear.

It is easy to turn this example into a bouquet of n triangles T_n covering a triangle T_1. Since each path in L_n is tripled in length, we see that $\zeta(u,T_n) = \zeta(u^3,L_n)$. The good thing is still that only one factor is bad. See Beth Malmskog and Michelle Manes [81] for a more general result.

Ramified Example 2. Next we obtain a graph Z from 2 copies of K_4 and identifying (fusing) an edge. See Figure 46. Here I was looking for an analog of the Riemann surface of $\sqrt{z}$.

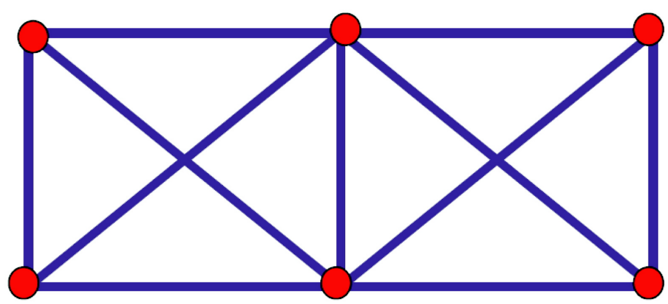


FIGURE 46. **Edge ramified cover Z** of K_4 obtained by taking 2 copies of K_4 and identifying an edge.

We compute the zeta function of Z in Figure 46 to be

$$\zeta(u,Z)^{-1} = (1-u^2)^5(u-1)(4u^2+u+1)\left(2u^2-u+1\right)\left(8u^3+2u^2+u-1\right)\left(2u^2+u+1\right)^2.$$

The only factor of $\zeta(u,K_4)^{-1}$ that does not divide $\zeta(u,Y)^{-1}$ is $1-2u$. This looks like the result of Beth Malmskog and Michelle Manes [81].

Exercise 56. *Compute the zeta when you identify or fuse an edge on n copies of K_4. Consider the divisibility properties of the corresponding zetas.*

14. Fundamental Theorem of Galois Theory

Question. What does it mean to say that $\widetilde{X}$ is intermediate to Y/X? Our goal is to prove the fundamental theorem of graph Galois theory; e.g. the existence of a 1-1 correspondence between subgroups H of the Galois group G of Y/X and intermediate graphs $\widetilde{X}$ to Y/X. For this, we need a definition which is stronger than just saying $Y/\widetilde{X}$ is a covering and $\widetilde{X}/X$ is a covering. To see this, consider Figures 42 and 44. These examples would contradict the fundamental theorem of Galois theory for graph coverings if we make definitions that are too simple. To avoid this problem, we make the following perhaps annoyingly complicated definition.

Exercise 57. *Draw the other 2 intermediate graphs for Figure 44.*

Definition 35. *Suppose that Y is a covering of X with projection map π. A graph $\widetilde{X}$ is an* ***intermediate covering*** *to Y/X if $Y/\widetilde{X}$ is a covering and $\widetilde{X}/X$ is a covering and the projection maps $\pi_1 : \widetilde{X} \longrightarrow X$ and $\pi_2 : Y \longrightarrow \widetilde{X}$ have the property that $\pi = \pi_1 \circ \pi_2$.*

See Figure 47. Technically, it is the triple $(\widetilde{X}, \pi_1, \pi_2)$ that gives the intermediate covering.

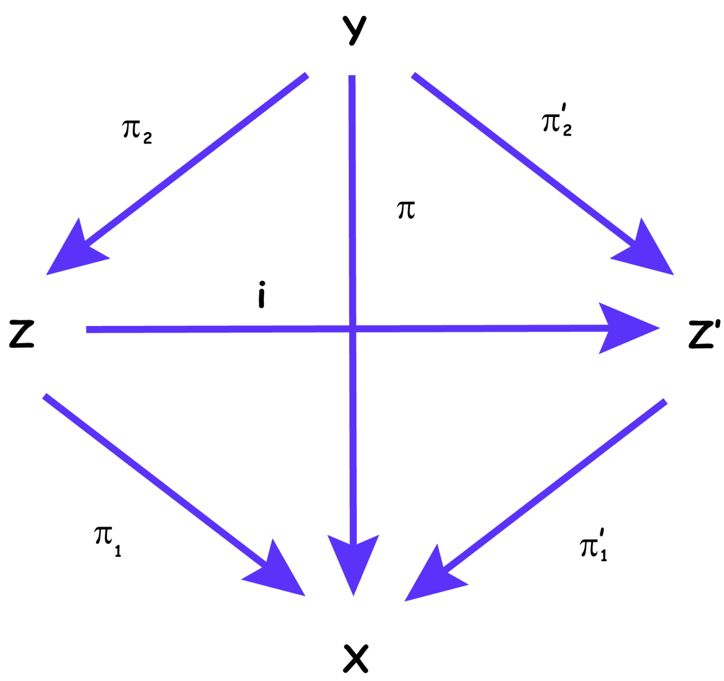


FIGURE 47. **A covering isomorphism** i of intermediate graphs.

A second definition (which may also make the reader's hair stand on end) tells us when two intermediate graphs are to be considered the same or equal. Again, please remember our goal to prove the fundamental theorems of Galois theory.

Definition 36. *Let $\widetilde{X}$ and $\widetilde{X}'$ be intermediate to Y/X with projection maps as in Figure 47. We assume all graphs have edges which have been assigned directions that are consistent with the projection maps. Suppose i is a graph isomorphism between $\widetilde{X}$ and $\widetilde{X}'$ (meaning it is 1-1, onto on vertices and directed edges). If the notation is as in Figure 47, and we have $\pi_1 = \pi_1' \circ i$, then we say i is a* ***covering isomorphism***. *We say that $\widetilde{X}$ and $\widetilde{X}'$ are* ***covering isomorphic***. *If, in addition, we have $i \circ \pi_2 = \pi_2'$, we say that $\widetilde{X}$ and $\widetilde{X}'$ are* ***the same*** *or* ***equal***.

Later in Theorem 18 we will see that covering isomorphic intermediate graphs to the Galois cover Y/X with Galois group G correspond to conjugate subgroups of G. This means covering isomorphic intermediate graphs are analogous to number fields like $\mathbb{Q}(\sqrt[3]{2})$ and $\mathbb{Q}(e^{2\pi i/3}\sqrt[3]{2})$.

Now we can prove the fundamental theorem. Note that most of these proofs are based on the uniqueness of lifts from Proposition 6.

Theorem 17. ***Fundamental Theorem of Galois Theory.*** *Suppose Y/X is an unramified normal covering with Galois group $G = G(Y/X)$.*

1) Given a subgroup H of G, there exists a graph $\widetilde{X}$ intermediate to Y/X such that $H = G(Y/\widetilde{X})$. Write $\widetilde{X} = \widetilde{X}(H)$.

2) Suppose $\widetilde{X}$ is intermediate to Y/X. Then there is a subgroup $H = H(\widetilde{X})$ of G which is $G(Y/\widetilde{X})$.

3) Two intermediate graphs $\widetilde{X}$ and $\widetilde{X}'$ are equal (as in Definition 36) if and only if $H(\widetilde{X}) = H(\widetilde{X}')$.

4) We have $H(\widetilde{X}(H)) = H$ and $\widetilde{X}(H(\widetilde{X})) = \widetilde{X}$. So we write $\widetilde{X} \leftrightarrow H$ for the correspondence between intermediate graphs $\widetilde{X}$ to Y/X and subgroups H of the Galois group $G = G(Y/X)$.

5) If $\widetilde{X_1} \leftrightarrow H_1$ and $\widetilde{X_2} \leftrightarrow H_2$ then $\widetilde{X_1}$ is intermediate to $Y/\widetilde{X_2}$ iff $H_1 \subset H_2$.

Proof. **Part 1)** Let H be a subgroup of G. Vertices of Y are of the form (x, g), with $x \in X$ and $g \in G$. Define the vertices of $\widetilde{X}$ to be (x, Hg) for $x \in X$, and coset $Hg \in H\backslash G$. Put an edge from (a, Hr) to (b, Hs), for $a, b \in X$ and $r, s \in G$ iff there are $h, h' \in H$ such that there is an edge from (a, hr) to $(b, h's)$ in Y.

The edge between (a, Hr) and (b, Hs) in $\widetilde{X}$ is given the label and direction of the projected edge between a and b in X.

Exercise 58. *Show that $\widetilde{X}$ is well-defined, intermediate to Y/X and connected.*

Part 2) Let $\widetilde{X}$ be intermediate to Y/X, with projections $\pi : Y \longrightarrow X$, $\pi_2 : Y \longrightarrow \widetilde{X}$, $\pi_1 : \widetilde{X} \longrightarrow X$. Fix a vertex $v_0 \in X$ with $\widetilde{\widetilde{v_0}} \in \pi^{-1}(v_0)$ on sheet 1 of Y. That is, $\widetilde{\widetilde{v_0}} = (v_0, 1)$ using our labeling of sheets of Y. Let $\widetilde{v_0} = \pi_2(\widetilde{\widetilde{v_0}}) \in \widetilde{X}$. Define

$$H = \left\{ h \in G \;\middle|\; h(\widetilde{\widetilde{v_0}}) \in \pi_2^{-1}(\widetilde{v_0}) \right\} \tag{14.1}$$
$$= \{ h \in G \mid \pi_2(v_0, h) = \pi_2(v_0, 1) \}.$$

To see that H is a subgroup of G, we need only show that H is closed under multiplication. Let h_1 and h_2 be elements of H. Then, by the definition of H, the vertices $\pi_2(v_0, h_1) = \pi_2(v_0, h_2) = \pi_2(v_0, 1) = \widetilde{v_0}$.

Let $\widetilde{\widetilde{p_1}}$ and $\widetilde{\widetilde{p_2}}$ be paths on Y from $(v_0, 1)$ to the vertices (v_0, h_1) and (v_0, h_2), respectively. Then $\widetilde{\widetilde{p_1}}$ and $\widetilde{\widetilde{p_2}}$ project under π_2 to closed paths $\widetilde{p_1}$ and $\widetilde{p_2}$ in $\widetilde{X}$ beginning and ending at $\widetilde{v_0}$. And $\widetilde{\widetilde{p_1}}$ and $\widetilde{\widetilde{p_2}}$ project under $\pi = \pi_1 \circ \pi_2$ to closed paths p_1 and p_2 in X beginning and ending at v_0.

By formula (13.1), $h_1 \circ \widetilde{\widetilde{p_2}}$ starts at (v_0, h_1) and ends at (v_0, h_1h_2). Thus the lift of $\widetilde{p_1}\widetilde{p_2}$ from $\widetilde{X}$ to Y beginning at $(v_0, 1)$, which is the same as the lift of p_1p_2 from X to Y beginning at $(v_0, 1)$, ends at (v_0, h_1h_2). It follows that h_1h_2 is in H and H is a subgroup of G.

Question. How does H depend on v_0?

Part 4). Let $\widetilde{X}$ be intermediate to Y/X. We want to prove that $\widetilde{X}(H(\widetilde{X})) = \widetilde{X}$, with the definitions from the proofs of Parts 1 and 2 as well as Definition 36.

Before attempting to prove the equality, we need to prove a characterization of $\pi_2^{-1}(\widetilde{v})$ for any vertex $\widetilde{v}$ of $\widetilde{X}$. This says that there is an element $g_v \in G$ such that if $H(\widetilde{X}) = H$,

$$\pi_2^{-1}(\widetilde{v}) = \{ (v, hg_v) \mid h \in H \}. \tag{14.2}$$

Let v_0 be the fixed vertex of X from the definition of H in the proof of Part 2. Let $\widetilde{q}$ be a path in $\widetilde{X}$ from $\widetilde{v_0}$ to $\widetilde{v}$. There is also a lift $\widetilde{\widetilde{q}}$ of $\widetilde{q}$ to Y starting at $(v_0, 1)$ and ending at (v, g_v). Write $\widetilde{\widetilde{v}} = (v, g_v)$ in Y with $\widetilde{v} = \pi_2(\widetilde{\widetilde{v}}) \in \widetilde{X}$ and $\pi(\widetilde{\widetilde{v}}) = v$. Projected down to X, we get the path q from v_0 to v.

Look at Figure 48. For $g \in G$, by equation (13.1), the path $\widetilde{q}$ in $\widetilde{X}$ lifts to a path $g \circ \widetilde{\widetilde{q}}$ from (v_0, g) to (v_0, gg_v) in Y. Thus, by the uniqueness of lifts, starting on a given sheet, we must have $\pi_2 \circ g \circ \widetilde{\widetilde{q}} = \widetilde{q}$ if and only if the initial sheet of the lift of q is that of $\widetilde{v_0}$. That is, $\pi_2 \circ g \circ \widetilde{\widetilde{q}} = \widetilde{q}$ iff $g \in H$. This proves formula (14.2).

Now we seek to show that $\widetilde{X}' = \widetilde{X}(H(\widetilde{X})) = \widetilde{X}$ in the sense of Definition 36. Recall that $\widetilde{X}' = \widetilde{X}(H(\widetilde{X}))$ has vertices (x, Hg) and projections $\pi_2'(x, g) = (x, Hg)$ and $\pi_1'(x, Hg) = x$. Define $i : \widetilde{X} \to \widetilde{X}'$ by $i(\widetilde{v}) = (v, Hg_v)$ using the element $g_v \in G$ from formula (14.2).

Exercise 59. *a) Why do $\widetilde{X}' = \widetilde{X}(H(\widetilde{X}))$ and $\widetilde{X}$ have the same number of vertices?*

b) Prove that i is a graph isomorphism (i.e., 1-1, onto between vertices and directed edges) and $i \circ \pi_2 = \pi_2'$, $\pi_1' \circ i = \pi_1$.

To complete the proof of Part 4), we must show that $H(\widetilde{X}(H)) = H$. By our definitions made in the proof of Parts 1 and 2, we have

$H(\widetilde{X}(H)) = \{ g \in G \mid \pi_2(v_0, g) = \pi_2(v_0, 1) \} = \{ g \in G \mid (v_0, Hg) = (v_0, H) \} = H.$

Part 5). Suppose $\pi_2 : Y \to \widetilde{X_1}$ and $\pi_1 : \widetilde{X_1} \to \widetilde{X_2}$ with $\pi_3 = \pi_1 \circ \pi_2 : Y \to \widetilde{X_2}$. Then by the proof of Part 2),

$$H(\widetilde{X_1}) = H_1 = \{ h \in G \mid \pi_2(v_0, h) = \pi_2(v_0, 1) \},$$
$$H(\widetilde{X_2}) = H_2 = \{ h \in G \mid \pi_3(v_0, h) = \pi_3(v_0, 1) \}.$$

Since $\pi_3 = \pi_1 \circ \pi_2$, it follows that $H_1 \subset H_2$.

For the converse, suppose that $H_1 \subset H_2$. Then we have the intermediate graphs $\widetilde{X_i}$ with vertices $(x, H_i\sigma)$ for $x \in X$, and coset $H_i\sigma \in H_i\backslash G$. There is an edge between $(a, H_i\sigma)$ and $(b, H_i\tau)$, for $a, b \in X$ and $\sigma, \tau \in G$ iff there are $h, h' \in H_i$ such that $(a, h\sigma)$ and $(b, h'\tau)$ have an edge in Y. We need to show $\exists\, \pi_2 : Y \to \widetilde{X_1}$ and $\pi_1 : \widetilde{X_1} \to \widetilde{X_2}$ with $\pi_3 = \pi_1 \circ \pi_2 : Y \to \widetilde{X_2}$. Here $\pi_2(v, g) = (v, H_1g)$ and $\pi_3(v, g) = (v, H_2g)$, for $v \in X, g \in G$. Then since $H_1 \subset H_2$, we see that $\pi_1(v, H_1g) = (v, H_2g)$ makes sense as $H_1a = H_1b$ iff $ab^{-1} \in H_1$. Since $H_1 \subset H_2$ this implies $H_2a = H_2b$.

Part 3). Suppose we have 2 intermediate graphs $\widetilde{X}$ and $\widetilde{X}'$ to Y/X with the projections $\pi_2 : Y \longrightarrow \widetilde{X}$, $\pi_1 : \widetilde{X} \longrightarrow X$ and $\pi_2' : Y \longrightarrow \widetilde{X}'$, $\pi_1' : \widetilde{X}' \longrightarrow X$. Set $H = H(\widetilde{X})$ and $H' = H(\widetilde{X}')$. Suppose $\widetilde{X} = \widetilde{X}'$. Then there is a graph isomorphism $i : \widetilde{X} \to \widetilde{X}'$ as in Definition 36 such that $i \circ \pi_2 = \pi_2'$, $\pi_1' \circ i = \pi_1$.

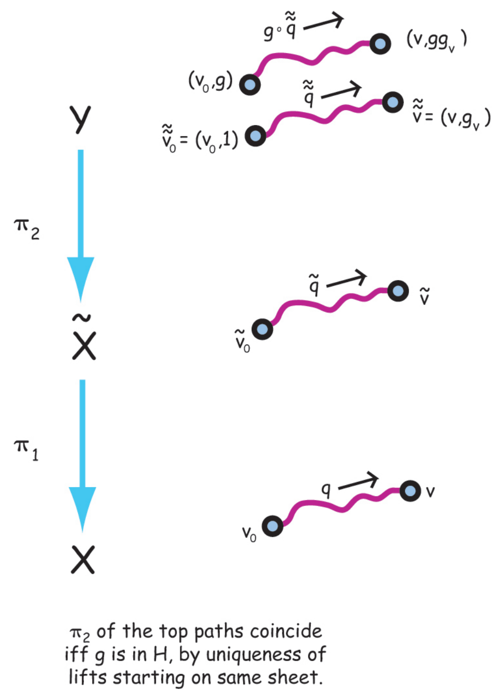


FIGURE 48. **Part of the proof of part 4 of Theorem 17** showing that $\{(v, hg_0) \mid h \in H\} = \pi_2^{-1}(\tilde{v})$. The dashed lines are the projections maps π_1 and π_2.

Then

$$H = \{h \in G \mid \pi_2(v_0, h) = \pi_2(v_0, 1)\},$$
$$H' = \left\{h \in G \mid \pi_2'(v_0, h) = \pi_2'(v_0, 1)\right\}.$$

Since $i \circ \pi_2 = \pi_2'$ and i is 1-1, we find that $H = H'$.

To go the other way, suppose that $H(\tilde{X}) = H(\tilde{X}')$. Then we need to show that there is a graph isomorphism $i : \tilde{X} \to \tilde{X}'$ as in Definition 36 such that $i \circ \pi_2 = \pi_2'$, $\pi_1' \circ i = \pi_1$. Note first that $\tilde{X}$ and $\tilde{X}'$ have the same number of vertices. We know from formula (14.2) that

$$\pi_2^{-1}(\tilde{v}) = \{(v, hg_v) \mid h \in H\},$$
$$\pi_2'^{-1}(\tilde{v}') = \left\{(v', hg_{v'}) \mid h \in H\right\}.$$

Define $i(\tilde{v}) = \pi_2'(v, hg_v)$.

Exercise 60. *Show that i is a graph isomorphism such that $i \circ \pi_2 = \pi_2'$, $\pi_1' \circ i = \pi_1$.*

□

Next we need to think about conjugate subgroups of the Galois group and their corresponding intermediate graphs.

Definition 37. *Suppose we have the following correspondences between intermediate graphs and subgroups of G:*

$$\tilde{X} \longleftrightarrow H \subset G$$
$$\tilde{X}' \longleftrightarrow gHg^{-1} \subset G, \text{ for some } g \in G.$$

*We say $\tilde{X}$ and $\tilde{X}'$ are **conjugate**.*

This definition turns out to be equivalent to part of Definition 36.

Theorem 18. *Intermediate graphs $\tilde{X}$ and $\tilde{X}'$ to the normal cover Y/X with Galois group G are conjugate in the sense of Definition 37 if and only if they are covering isomorphic in the sense of Definition 36.*

Proof. Suppose that H and $H' = g_0 H g_0^{-1}$ are conjugate subgroups of G, where $g_0 \in G$. We want to show that the corresponding intermediate graphs $\widetilde{X} = \widetilde{X}(H)$ and $\widetilde{X}' = \widetilde{X}(H')$ (using the notation of Theorem 17) are covering isomorphic in the sense of Definition 36. We have the disjoint coset decompositions

$$G = \bigcup_{j=1}^{d} Hg_j \quad \text{and} \quad G = \bigcup_{j=1}^{d} H' g_0 g_j .$$

This means that the graphs $\widetilde{X}$ and $\widetilde{X}'$ have vertices $\{(v, Hg_j) \,|\, v \in X,\ 1 \le j \le d\}$ and $\{(v, H'g_0 g_j) \,|\, v \in X,\ 1 \le j \le d\}$, respectively. The isomorphism $i : \widetilde{X} \longrightarrow \widetilde{X}'$ is defined by $i(v, Hg) = (v, H'g_0 g)$.

Exercise 61. *Prove that $i : \widetilde{X} \longrightarrow \widetilde{X}'$ is a covering isomorphism in the sense of Definition 36.*

For the converse, suppose that $\widetilde{X}$ and $\widetilde{X}'$ are covering isomorphic intermediate graphs. We must show that the corresponding subgroups $H = H(\widetilde{X})$ and $H' = H(\widetilde{X}')$ (in the notation of Theorem 17) are conjugate. By Definition 36, there is an isomorphism $i : \widetilde{X} \longrightarrow \widetilde{X}'$ such that $\pi_1 = \pi_1' \circ i$. Fix vertex $v_0 \in X$ and let $\widetilde{\widetilde{v_0}} = (v_0, 1)$ be on sheet 1 of Y, and $\widetilde{v_0} = \pi_2(\widetilde{\widetilde{v_0}})$ in $\widetilde{X}$. For any $\widetilde{v} \in \widetilde{X}$, suppose it projects to $v \in X$ under π_1 and suppose $\widetilde{\widetilde{v}} = (v, g_v) \in Y$ projects to $\widetilde{v}$ under π_2. See Figure 49. The set $\{g \in G \,|\, \pi_2(v, g) = \widetilde{v}\} = Hg_v$ by formula (14.2). Let $\widetilde{\widetilde{p}}$ be a path on Y from $\widetilde{\widetilde{v_0}}$ to $\widetilde{\widetilde{v}}$. It projects via π_2 to a path $\widetilde{p}$ in $\widetilde{X}$ from $\widetilde{v_0}$ to $\widetilde{v}$ and to a path p in X from v_0 to v.

The path $i(\widetilde{p})$ in $\widetilde{X}'$ from $i(\widetilde{v_0})$ to $i(\widetilde{v})$ also projects under π_1' to p. As $i(\widetilde{v_0})$ projects under π_1' to v_0, there is a $g_0 \in G$ such that $(v_0, g_0) \in Y$ projects via π_2' to $i(\widetilde{v_0})$. Now $\pi\left(g_0 \circ \widetilde{\widetilde{p}}\right) = \pi\left(\widetilde{\widetilde{p}}\right) = p$. Since $\pi = \pi_1' \circ \pi_2'$, it follows that the path $\pi_2'\left(g_0 \circ \widetilde{\widetilde{p}}\right)$ in $\widetilde{X}'$ has initial vertex $i(\widetilde{v_0})$ and projects to p in X. By the uniqueness of lifts, then $i(\widetilde{p}) = \pi_2'\left(g_0 \circ \widetilde{\widetilde{p}}\right)$. However, $g_0 \circ \widetilde{\widetilde{p}}$ ends at $(v, g_0 g)$. Therefore π_2' takes $(v, g_0 g)$ to $i(\widetilde{v})$. In particular, the set of all such $g_0 g$ is $g_0 H g_v = (g_0 H g_0^{-1}) g_0 g_v$. Therefore $H' = g_0 H g_0^{-1}$. □

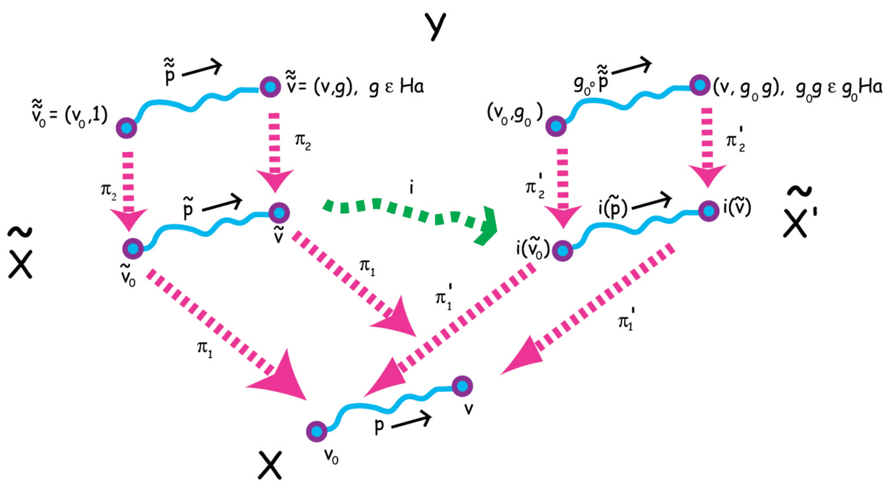


FIGURE 49. **Part of the proof of Theorem 18.**

Remark 1. *The previous proof showed that the effect of the isomorphism i can be achieved by the element $g_0 \in G$. In fact, g_0 may be replaced by any element of the right coset $(g_0 H g_0^{-1}) g_0 = g_0 H$, a left coset of H. This gives a* ***1-1 correspondence between left cosets*** *$g_0 H$* ***of*** *H* ***and all possible "embeddings" of*** *$\widetilde{X}$* ***in*** *Y/X.*

Theorem 19. *Suppose Y/X is a normal covering with Galois group G and $\widetilde{X}$ is an intermediate covering corresponding to the subgroup H of G. Then $\widetilde{X}$ is itself a normal covering of X if and only if H is a normal subgroup of G and then $G(\widetilde{X}/X) \cong H\backslash G$.*

Proof. Recall the proof of Theorem 18. View $\widetilde{X}$ as $\widetilde{X}(H)$ (using the notation of Theorem 17), with vertex set

$$\{(v, Hg_j) \,|\, v \in X, 1 \le j \le d\},$$

where the g_j are right coset representatives for $H\backslash G$.

Suppose H is a normal subgroup of G. A coset Hg acts on $\widetilde{X}(H)$ by sending (v, Hg_j) to (v, Hgg_j). This action preserves edges and is transitive on the cosets Hg.

Exercise 62. *Prove this last statement. You need to use the normality of H to see that the action preserves edges.*

This gives $d = |G/H|$ automorphisms of $\widetilde{X}(H)$ showing that $\widetilde{X}(H)$ is normal over X with Galois group G/H.

For the converse, suppose $\widetilde{X}/X$ is normal and i is an automorphism of $\widetilde{X}$ in $G(\widetilde{X}/X)$. Apply Theorem 18 with $\widetilde{X}' = \widetilde{X}$ and $\pi_1 = \pi_1'$ and $\pi_2 = \pi_2'$. Although i is not the map that makes $\widetilde{X}' = \widetilde{X}$ (that map is the identity map), nevertheless, i is an isomorphism between $\widetilde{X}$ and $\widetilde{X}'$ and it is a conjugation map since $\pi_1' \circ i = \pi_1 \circ i = \pi_1$. Thus Theorem 18 says there is a $g_0 \in G$ such that the intermediate graph $\widetilde{X}'$ corresponds to the subgroup $g_0 H g_0^{-1}$. Since $\widetilde{X}' = \widetilde{X}$, we have $g_0 H g_0^{-1} = H$.

Moreover choosing $\widetilde{v_0} \in \widetilde{X}$ as in the proof of Theorem 18, we have $\pi_2((v_0, g_0)) = \pi_2'((v_0, g_0)) = i(\widetilde{v_0})$. As i runs through the d elements of $G(\widetilde{X}/X)$, the $i(\widetilde{v_0})$ run through the d lifts of v_0 to $\widetilde{X}$. Thus the corresponding d different g_0's run through all d left cosets of H in G, and we have $g_0 H g_0^{-1} = H$ for all of these which says H is normal in G. □

The reader should now go back and reconsider the examples in Figures 43 and 44. As an **exercise**, write down all the intermediate graphs. Next let's consider a new example given in Figure 50.

Example 24. ***An S_3 cover of $K_4 - e$ with 2 intermediate covers.*** *Figure 50 shows a normal covering Y_6 of $X = K_4 - edge$ with Galois group $G(Y_6/X) = S_3$, the symmetric group of permutations of 3 things. Here we shall use the disjoint cycle notation for permutations, so that, for example,* (123) *means the permutation which sends 1 to 2, 2 to 3 and 3 to 1.*

The intermediate graph Y_3 corresponding to the subgroup $H = \{(1), (23)\}$ is also in the figure. An explanation of the method used to create these graphs is given in the following paragraphs. We then leave it to the reader to see how the intermediate graph Y_2 is created.

Figure 50 should be compared with 51 below.

The top graph Y_6 in Figure 50 is a normal 6-fold cover of the bottom graph X with Galois group S_3, the symmetric group of permutations of 3 objects. We make the identifications

$$a' = (a, (1)), a" = (a, (13)), a^{(3)} = (a, (132)), a^{(4)} = (a, (23)), a^{(5)} = (a, (123)), a^{(6)} = (a, (12)).$$

One way to construct this example is obtained by using a permutation representation of S_3. See the Exercise below. A spanning tree in the bottom graph X is given in green. The edges in X left out of the spanning tree (drawn with red dashed lines) generate the fundamental group of X. Call the directed edge from vertex 2 to vertex 4 edge c. Call the directed edge from vertex 4 to vertex 3 edge d.

We get the top graph Y_6 as follows Take 6 copies of the spanning tree for X. This gives all the vertices of Y_6 but some edges are missing. Label the vertex on the ith sheet of Y_6 projecting to vertex x in X by $x^{(i)}$. We lift the edge c to 6 edges in the top graph using the permutation $\sigma(c) = (14)(23)(56)$.[3] This permutation comes to us (not out of the blue but) from the second Exercise below. It tells us to connect vertex $2^{(1)}$ with vertex $4^{(4)}$ in Y_6 and then connect vertex $2^{(4)}$ and $4^{(1)}$. After that, connect vertex $2^{(2)}$ with $4^{(3)}$ and vertex $2^{(3)}$ with $4^{(2)}$. Finally connect vertex $2^{(5)}$ with vertex $4^{(6)}$ and vertex $2^{(6)}$ with vertex $4^{(5)}$. Do a similar construction with $\sigma(d) = (12)(36)(45)$ to obtain the remaining 6 edges of Y_6. The permutations $\sigma(c)$ and $\sigma(d)$ have order 2 and they generate a subgroup of S_6 isomorphic to S_3. In Section 17 below, we will have more to say about this construction.

We can also identify S_3 with the dihedral group D_3 of rigid motions of an equilateral triangle. Let R be a 120° rotation of an equilateral triangle and F a flip. The we have $D_3 = \{I, R, R^2, F, FR, FR^2\}$, with $R^3 = I, FR = R^2 F$. We identify

$$a' = (a, I), a" = (a, FR^2), a^{(3)} = (a, R^2), a^{(4)} = (a, FR), a^{(5)} = (a, R), a^{(6)} = (a, F).$$

We can identify $\sigma(c) = FR$ and $\sigma(d) = FR^2$.

Next we want to construct the intermediate graph Y_3 corresponding to the subgroup $H = \{I, FR\}$. First we need the appropriate permutation representation of G on the 3 cosets Hg_i, $i = 1, 2, 3$, with $g_1 = I, g_2 = FR^2, g_3 = F$. Since $\sigma(c) = FR$, we have $Hg_1\sigma(c) = Hg_1, Hg_2\sigma(c) = Hg_3, Hg_3\sigma(c) = Hg_2$. Thus the cycle decomposition of the permutation corresponding to $\sigma(c)$ acting on cosets of H is $(1)(23)$. Similarly the permutation of cosets of H corresponding to $\sigma(d) = FR^2$ is $(12)(3)$.

We now construct the intermediate graph Y_3 to Y_6/X corresponding to H. First take 3 copies of the spanning tree in X. This gives all the vertices of Y_3 but some edges are missing. We label the sheets of Y_3 with $', '', '''$. The permutation $(1)(23)$ tells us how to lift edge c in X to 3 edges in Y_3. We obtain edges in Y_3 from vertex $2'$ to $4'$, from $2''$ to $4'''$, and from $2'''$ to $4''$. Similarly the permutation $(12)(3)$ tells us how to lift edge d in X to 3 edges in Y_3. We get the edges from $4'$ to $3''$, from $4''$ to $3'$, and from $4'''$ to $3'''$.

The 3 two element subgroups of S_3 are all conjugate to H. Each will lead (by Theorem 18) to a graph isomorphic to Y_3. This is because we have not given the projections from Y_6 to Y_3. Without knowing this projection, the 3 conjugate intermediate cubic covers to Y_6/X are all isomorphic with the isomorphism preserving projections to X.

[3] Here $\sigma(c)$ denotes the normalized Frobenius automorphism corresponding to edge e, which we define later. See Definition 41.

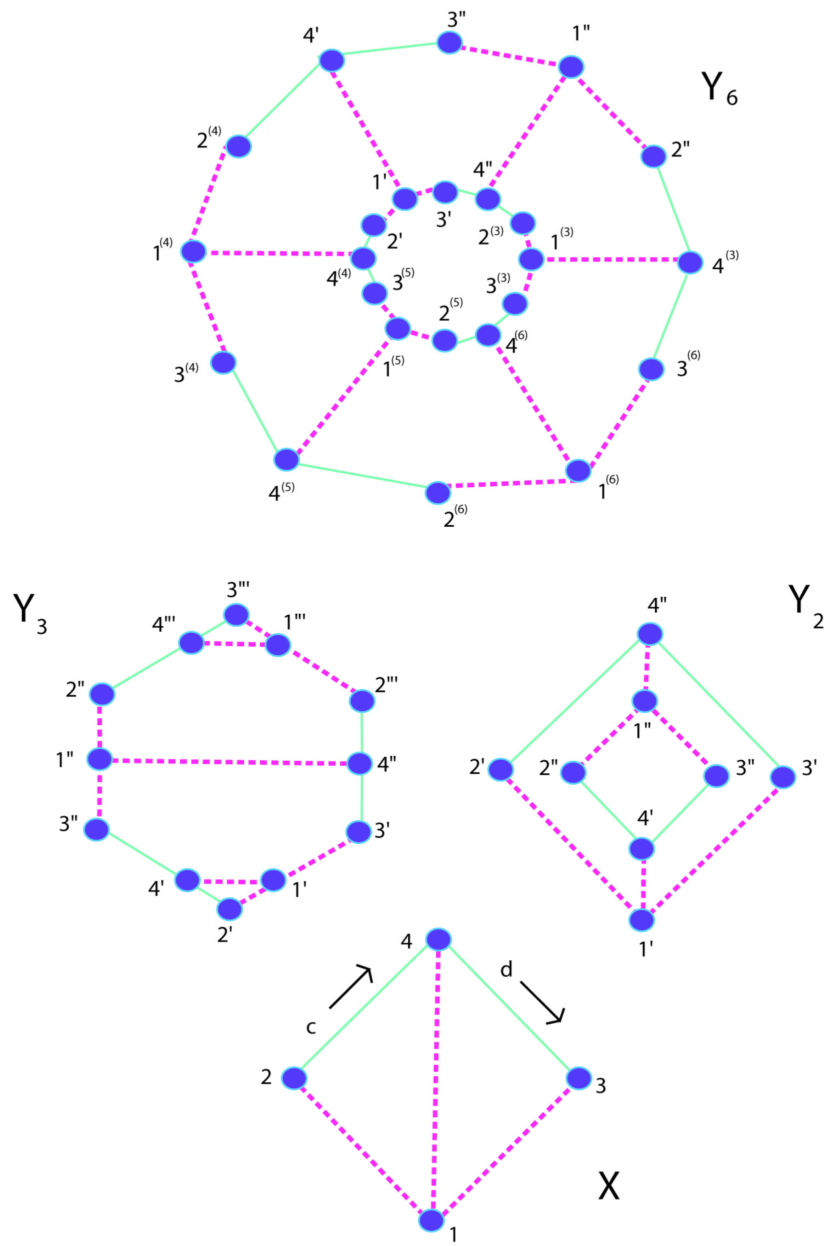


FIGURE 50. **A 6-sheeted normal cover** Y_6 of X with a non-normal cubic intermediate cover Y_3 as well as a quadratic intermediate cover Y_2. Here the Galois group is the symmetric group $G = G(Y/X) = S_6$ and the subgroup $H = \{(1), (23)\}$ fixes Y_3. We write $a^{(1)} = (a, (1)), a^{(2)} = (a, (13)), a^{(3)} = (a, (132)), a^{(4)} = (a, (23)), a^{(5)} = (a, (123)), a^{(6)} = (a, (12))$, using standard cycle notation for elements of the symmetric group. A spanning tree in the base graph is a green solid line. The sheets of the covers are also drawn this way.

Exercise 63. *Construct the intermediate graph Y_2 to Y_3/X in a similar way to that used to construct Y_3 above.*

Exercise 64. *Suppose we list the elements of S_3, using cycle notation, as*

$$g_1 = (1), g_2 = (12), g_3 = (123), g_4 = (23), g_5 = (132), g_6 = (13).$$

Then write $g_i g = g_{\mu(g)}$, where $\mu(g) \in S_6$. Show that $\mu(23) = (14)(23)(56)$ and $\mu(12) = (12)(36)(45)$.

Exercise 65. *Create a covering of the cube graph which is normal with Galois group a cyclic group of order 3.*

It is possible to define coverings of weighted graphs. See Chung and Yau [29] or Osborne and Severini [96]. The second paper applies the idea combined with that of quantum walks on graphs.

We do not consider infinite graphs here but it is possible to extend the Galois theory to that situation. Then the Galois group of the universal covering tree T of a finite graph X can be identified with the fundamental group of X. This may make more sense after we have discussed Frobenius automorphisms in section 16 below.

15. Behavior of Primes in Coverings

We seek analogs of the laws governing the behavior of prime ideals in extensions of algebraic number fields. Figure 6 shows what happens in a quadratic extension of the rationals. Figure 51 shows a non-normal cubic extension of the rationals. See Stark [116] for more information on these examples.

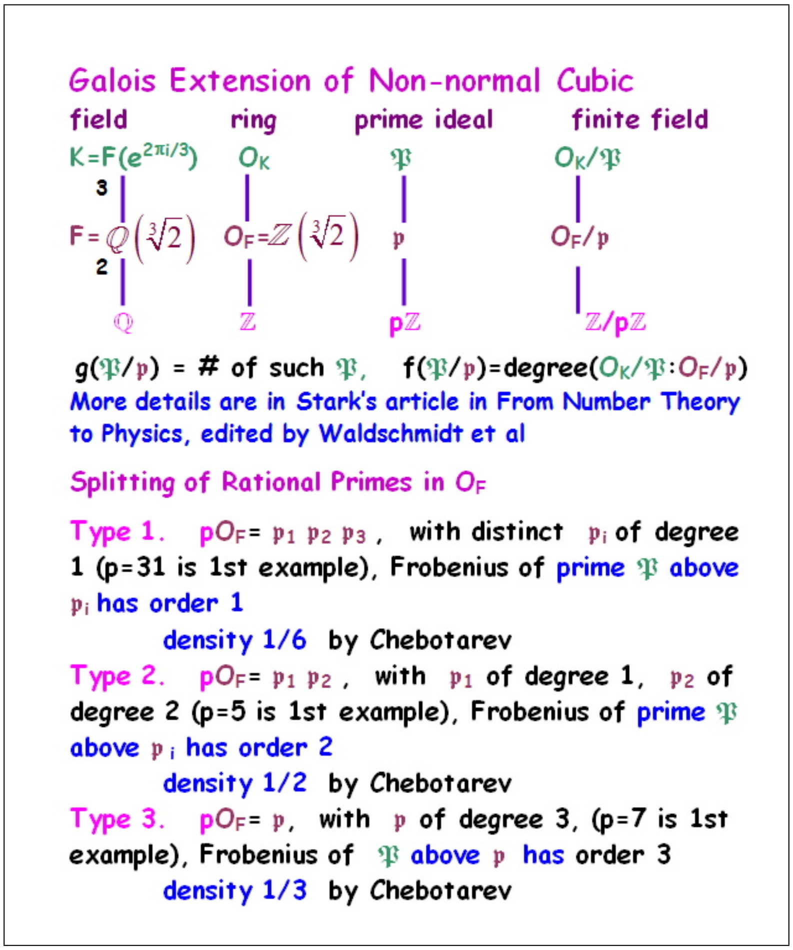


FIGURE 51. **Example of splitting of unramified primes in non-normal cubic extension of the rationals.**

The graph theory analog of the example in Figure 51 is found in Figure 50 above and Example 24 below. Figure 54 gives examples of primes that split in various ways in the non-normal cubic intermediate field.

So now let us consider the graph theory analog. The field extension is replaced by a graph covering Y/X, with projection map π. Suppose $[D]$ is a prime in Y. Then $\pi(D)$ is a closed, backtrackless, tailless path in X but it may not be primitive. There will, however, be a prime $[C]$ in X and an integer f such that $\pi(D) = C^f$. The integer f is independent of the choice of D in $[D]$.

Definition 38. *If $[D]$ is a prime in a covering Y/X with projection map π and $\pi(D) = C^f$, where $[C]$ is a prime of X, we will say that $[D]$ is a **prime of Y above $[C]$**, or more loosely, that D is a **prime above C** and write $D|C$ and $f = f(D, Y/X) =$ the **residual degree** of D with respect to Y/X.*

If Y/X is normal, for a prime C of X and a given integer j, either every lift of C^j is closed in Y or no lift is closed. Thus the residual degree of $[D]$ above C is the same for all $[D]$ above C. This will not always be the case for non-normal extensions.

Definition 39. *Let $g = g(D, Y/X)$ be the **number of primes** $[D]$ above $[C]$.*

Since our covers are unramified, the analog of the ramification index is $e = e(D, Y/X) = 1$ and we will be able to prove the familiar formula from algebraic number theory for normal covers:

$$efg = d = \text{number of sheets of the cover.} \tag{15.1}$$

See part 6) of Proposition 9 below.

Example 25. ***Primes in the cube Y over primes in the tetrahedron X.***

In Figure 52 we show a prime $[C]$ of length 3 in the tetrahedron X defined by $C = \langle a, d, c, a \rangle$. Here we list the vertices through which the path passes within <> . The prime $[D]$ in the cube Y, with $D = \left\langle a', d'', c', a'', d', c'', a' \right\rangle$, has length 6 and is over $[C]$ in X. Let the Galois group be $G = G(Y/X) = \{1, \sigma\}$. We are using the notation $x' = (x, 1)$ and $x'' = (x, \sigma)$ in Y, for $x \in X$. Then $D = C_1\ (\sigma \circ C_1)$, where $C_1 = \left\langle a', d'', c', a'' \right\rangle$. Here $\nu(D) = 2\nu(C) = 6$. In this example $f = 2$ and $g = 1$.

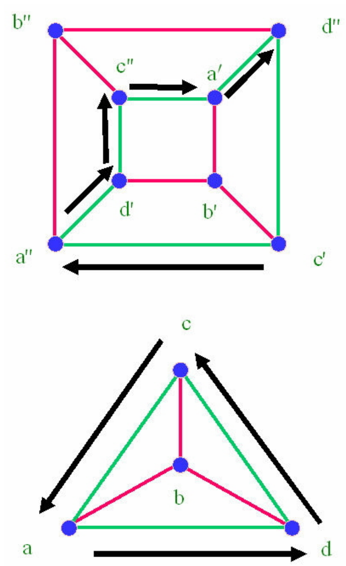


FIGURE 52. **Picture of splitting of prime** C with $f = 2, g = 1, e = 1$. There is 1 prime cycle D above C and D is the lift of C^2.

A second example in Y/X is shown in Figure 53. In this case, the prime $[D]$ of Y is represented by $D = \left\langle a'', c', d'', b'', a'' \right\rangle$. Then $D|C$ with the prime $[C]$ represented by $C = \langle a, c, d, b, a \rangle$ in X. Here $\nu(D) = \nu(C) = 4$, $f = 1$, and $g = 2$ since there is another prime D' in Y over C, also shown in Figure 53.

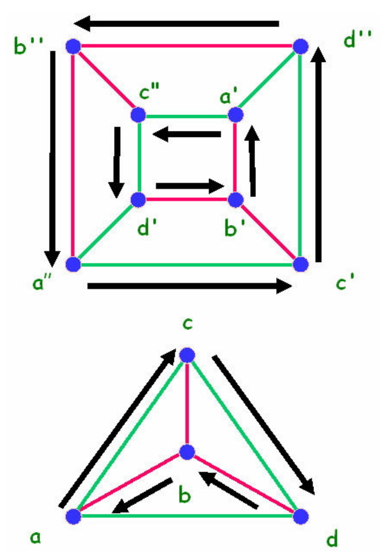


FIGURE 53. **Picture of a prime C which splits completely**; i.e., $f = 1, g = 2, e = 1$. There are 2 prime cycles D, D' in the cube above C, each with the same length as C below in the tetrahedron.

Example 26. ***Primes in a non-normal cubic cover*** *Y_3 of $X = K_4 - edge$ pictured in Figure 50. Figure 54 below gives examples of primes in X with various sorts of splittings in the non-normal cubic cover Y_3 of $X = K_4 - edge$ from Example 24. The densities of the primes in various classes come from the Chebotarev density theorem explained in section 22 below.*

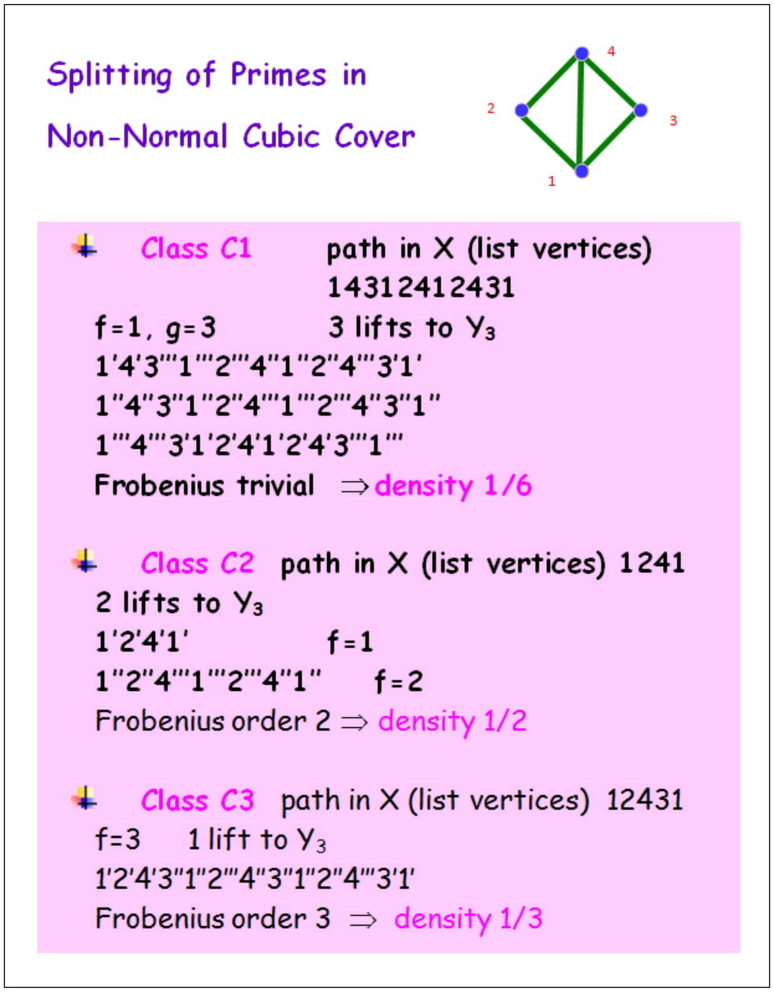


FIGURE 54. **Splitting of primes in the non-normal cubic cover** Y_3 of $K_4 - e$ pictured in Figure 50. This should be compared with Figure 51 which shows the splitting of primes in a non-normal cubic extension of the rational numbers.

Exercise 66. *Look at Figure 54 which shows the splitting of primes in a non-normal cubic cover of $K_4 - edge$. The prime in class 1 has length* 10. *Is it a prime of minimal length with $f = 1$ and $g = 3$?*

Definition 40. *If Y/X is normal and $[D]$ is a prime of Y over $[C]$ in X and σ is in $G(Y/X)$, we refer to $[\sigma \circ D]$ as a* ***conjugate prime*** *of Y over C.*

We then have $f(\sigma \circ D, Y/X) = f(D, Y/X)$. If $f = f(D, Y/X)$, then as σ runs through $G(Y/X)$, $\sigma \circ D$ runs through all possible lifts of C^f from X to Y and thus the conjugates of $[D]$ account for all the primes of Y above $[C]$. That is, there are $d = |G(Y/X)|$ lifts of C^f starting on different sheets of Y, but only g of these lifts give rise to different primes of Y.

Exercise 67. *Show that when the cover Y/X is not normal, formula (15.1) becomes*

$$\sum_{i=1}^{g} f_i = d,$$

the number of sheets of the cover. Here the f_i denote the residual degrees of the primes of Y above some fixed prime of X.

Given Y/X a (finite unramified) graph covering and suppose $\widetilde{X}$ is intermediate to Y/X. Suppose $\pi_1 : \widetilde{X} \longrightarrow X$ and $\pi_2 : Y \longrightarrow \widetilde{X}$, $\pi : Y \longrightarrow X$ are the covering maps, with $\pi = \pi_1 \circ \pi_2$. Let E be a prime of Y over the prime C of X and let $\pi_2(E) = D^{f_2}$, where D is a prime of $\widetilde{X}$ and $f_2 = f(E, Y/\widetilde{X})$. Then we have the **transitivity property**:

$$f(E, Y/X) = f(E, Y/\widetilde{X})\, f(D, \widetilde{X}/X). \tag{15.2}$$

This is the graph theoretic analog of a result about the behavior of residual degrees of primes in extensions of algebraic number fields.

Exercise 68. *Prove Formula (15.2).*

Hint. *Note that* $\pi(E) = C^{f(E,Y/X)}$ *and* $\pi_2(E) = D^{f(E,Y/\widetilde{X})}$.

16. Frobenius Automorphisms

Before defining Artin L-functions of normal graph coverings, we should perhaps recall what Artin L-functions are and do for algebraic number fields. References for Artin L-functions of number fields are Lang [73] and Stark [116]. Figures 55 and 56 summarize some of the facts.

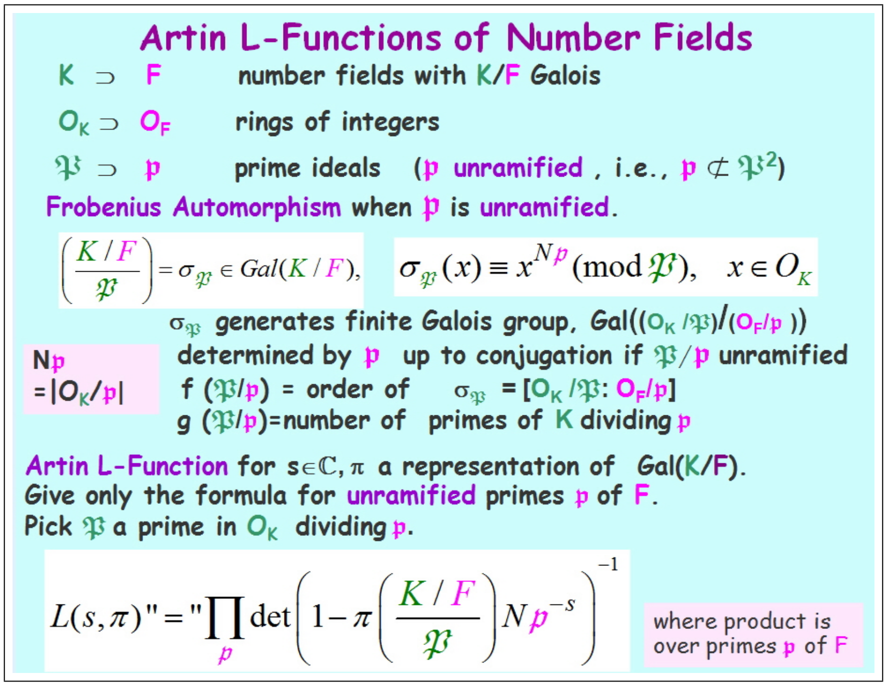


FIGURE 55. **Definition of Frobenius symbol and Artin L-function of Galois extension of number fields.**

We want to find an analog of the Frobenius automorphism in number theory. References for the number theory version are Lang [73], and Stark [116].

Assume that Y is a normal cover of the graph X with Galois group G. We want to define the Frobenius automorphism $[Y/X, [D]]$ for a prime $[D]$ in Y over the prime $[C]$ in X. First we can define the normalized Frobenius automorphism $\sigma(p) \in G = G(Y/X)$ associated to a directed path p of X - the existence of which simplifies the graph theory version of things. This normalized Frobenius automorphism should be compared with the voltage assignment map in Gross and Tucker [47]. See Figure 57 for a summary of our definitions.

Definition 41. *Suppose Y/X is normal with Galois group $G = Gal(Y/X)$. For a path p of X, Proposition 6 says there is a unique lifting to a path $\widetilde{p}$ of Y starting on sheet 1, having the same length as p. If $\widetilde{p}$ has its terminal vertex on the sheet labeled with $g \in G$, define the* ***normalized Frobenius automorphism*** *$\sigma(p) \in G$ by $\sigma(p) = g$.*

Exercise 69. *Compute the normalized Frobenius automorphism $\sigma(C)$ for the paths C in the tetrahedron K_4 which are pictured in Figures 52 and 53.*

Exercise 70. *Compute the normalized Frobenius automorphism $\sigma(C)$ for the paths C in $K_4 - e$ which are pictured in Figure 54, when lifted to the top graph Y_6 in Figure 50.*

Lemma 3. *1) Suppose that p_1 and p_2 are two paths on X such that the terminal vertex of p_1 is the initial vertex of p_2. Then $\sigma(p_1 p_2) = \sigma(p_1)\sigma(p_2)$.*

2)If a path $p = e_1 \cdots e_s$, for directed edges $e_1, ..., e_s$; then $\sigma(p) = \sigma(e_1) \cdots \sigma(e_s)$.

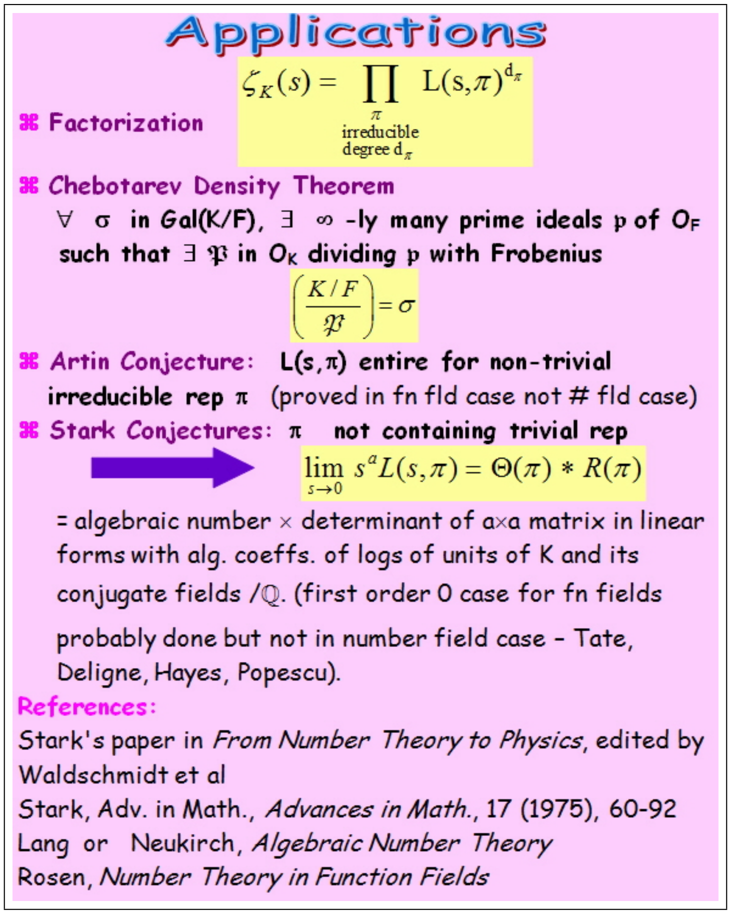


FIGURE 56. **Applications of Artin L-functions of Number Fields.**

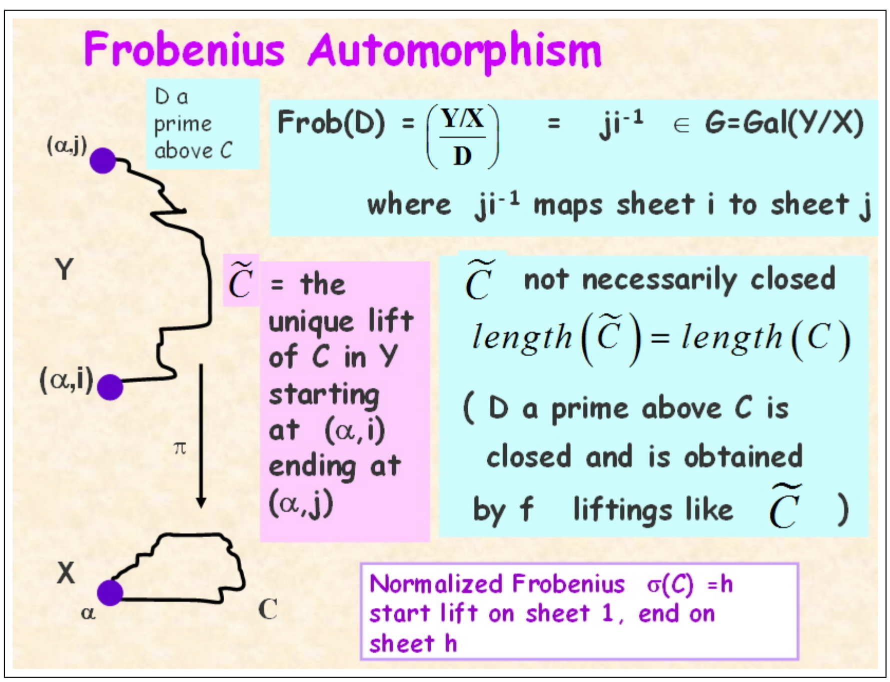


FIGURE 57. **The Frobenius automorphism and the normalized version.**

Proof. 1) If p_1 goes from a to b in X and p_2 goes from b to c in X, then the lift $\widetilde{p_1}$ of p_1 starting on sheet 1 of Y goes from $(a, 1)$ to $(b, \sigma(p_1))$ and the lift $\widetilde{p_2}$ of p_2 starting on sheet 1 of Y goes from $(b, 1)$ to $(c, \sigma(p_2))$. See Figure 58. Therefore the lift of p_2 starting on sheet $\sigma(p_1)$ goes from $(b, \sigma(p_1))$ to $(c, \sigma(p_1)\sigma(p_2))$. This implies that the lift of $p_1 p_2$ beginning on sheet 1 of Y will end on sheet $\sigma(p_1)\sigma(p_2)$.

2) This follows easily from part 1). □

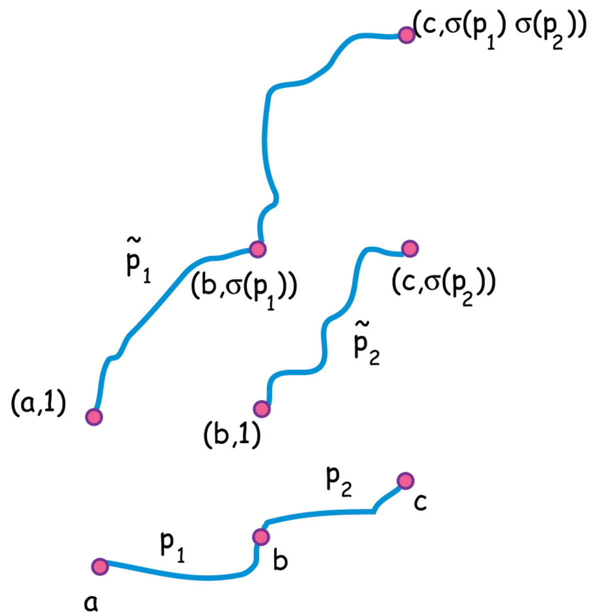


FIGURE 58. The **map σ preserves composition of paths.**

Now we can define the Frobenius automorphisms and decomposition groups. See Figure 57 again.

Definition 42. *Assume Y/X normal with Galois group G. Let $[C]$ be a prime on X, such that C starts and ends at vertex a. Let $[D]$ be a prime of Y over C such that D starts and ends at vertex (a, g) on sheet $g \in G$ of Y. If the residual degree of D over C is f, then D is the lifting of C^f which begins on sheet g. Suppose C itself lifts to a path $\widetilde{C}$ on Y starting on sheet g at (a, g) and ending on sheet h at (a, h). Define the* ***Frobenius automorphism*** *to be*

$$[Y/X, D] = \left(\frac{Y/X}{D}\right) = hg^{-1} \in G.$$

Note that the Frobenius $[Y/X, D] = \left(\frac{Y/X}{D}\right) = hg^{-1}$ maps sheet g of Y to sheet h of Y. To get the normalized version of the Frobenius you have to take $g = 1$, the identity of G.

Our next definition yields a group analogous to one from algebraic number theory. The letter chosen for it corresponds to the German version of the name.

Definition 43. *The* ***decomposition group*** *of D with respect to Y/X is*

$$Z(D) = Z(D, Y/X) = \{\tau \in G \mid [\tau \circ D] = [D]\}.$$

The next proposition gives analogs of the usual properties of the Frobenius automorphism of a normal extension of number fields (as in Lang [73]).

Proposition 9. ***Properties of the Frobenius Automorphism.***

As usual, Y/X is a normal d-sheeted covering with Galois group G.

1) For a prime cycle D in Y over C in X, the Frobenius automorphism is independent of the choice of D in its equivalence class $[D]$. Thus we can define $[Y/X, [D]] = [Y/X, D]$, without ambiguity.

2) The order of $[Y/X, D]$ in G is the residual degree $f = f(D, Y/X)$.

3) If $\tau \in G$, then $[Y/X, \tau \circ D] = \tau[Y/X, D]\tau^{-1}$.

4) If D begins on sheet 1, then $[Y/X, D] = \sigma(C)$, the normalized Frobenius automorphism of Definition 41.

5) The decomposition group $Z(D)$ is the cyclic subgroup of G of order f generated by $[Y/X, D]$. In particular, $Z(D)$ does not depend on the choice of D in its equivalence class $[D]$.

6) For a prime cycle D in Y over C in X, if $f = f(D, Y/X)$ is from Definition 38 and $g = g(D, Y/X)$ is as in Definition 39, then $d = fg$. (Here $e =$ the ramification is assumed to be 1.)

Proof. **Part 4)** is proved by noting that the 2 definitions are the same.

Parts 1) and 3). Suppose C has initial (and terminal) point vertex a in X and D is the lifting of C^f beginning at vertex (a, μ_0) on sheet μ_0. In lifting C^f, we lift C a total of f times consecutively, beginning at (a, μ_0) and ending respectively at $(a, \mu_1), (a, \mu_2), ..., (a, \mu_{f-1}), (a, \mu_f)$, where $\mu_f = \mu_0$, and $\mu_j \neq \mu_0$, for $j = 1, 2, ..., f - 1$. See Figure 59.

Suppose that (b, κ) is another vertex on D, where b is on C. Thus (b, κ) lies on one of the f consecutive lifts of C in Figure 59, say the r^{th}. Vertex b splits C into two paths $C = p_1 p_2$, where b is the ending vertex of p_1 and the starting vertex of p_2. The vertex (b, κ) on Y is the ending vertex of the lift of p_1 to D starting at (a, μ_{r-1}). The lift of the version of C in $[C]$ starting at b, namely $p_2 p_1$ to a path on Y which starts at (b, κ) then ends at a vertex (b, λ) on D which lies on the $(r + 1)^{st}$ consecutive lift of C.

Let $\widetilde{C}'$ be a path on Y from $(a, 1)$ to (a, μ_0) and let C' be the projection of $\widetilde{C}'$ to X. The vertices $(a, \mu_0), (a, \mu_1), (b, \kappa)$, and (b, λ) of Y are then the end points of the lifts on the paths

$$C', C'C, C'C^{r-1}p_1, C'C^r p_1,$$

respectively, to paths on D starting at $(a, 1)$. Therefore, by Lemma 3, we have

$$\mu_0 = \sigma(C'), \quad \mu_1 = \sigma(C'C) = \sigma(C')\sigma(C);$$
$$\kappa = \sigma(C'C^{r-1}p_1) = \sigma(C')\sigma(C)^{r-1}\sigma(p_1); \;\; \lambda = \sigma(C'C^r p_1) = \sigma(C')\sigma(C)^r\sigma(p_1).$$

It follows that $[Y/X, D]$ is the common value of

$$\lambda\kappa^{-1} = \mu_1\mu_0^{-1} = \sigma(C')\sigma(C)\sigma(C')^{-1}.$$

This proves 1). It also proves 3) in the case $\tau = \mu_0^{-1} = \sigma(C')^{-1}$ and this suffices to prove 3) in general.

Part 2). As above, we see that for each j,

$$\mu_j = \sigma(C'C^j) = \sigma(C')\sigma(C)^j$$

and thus

$$\mu_j\mu_0^{-1} = \sigma(C')\sigma(C)^j\sigma(C')^{-1} = [Y/X, D]^j. \tag{16.1}$$

This proves 2).

Part 5). Recall that $\tau \in Z(D)$ means that $\tau \circ D$ is equivalent to D in the sense of the equivalence relation giving $[D]$ in formula (2.2).

Suppose $\tau \circ D$ is equivalent to D. If the picture is as in Figure 59, then since $\tau \circ D$ also starts at a vertex projecting under $\pi : Y \to X$ to a, and we must have $\tau\mu_0 = \mu_j$, for one of the μ_j above. Thus, for some j, $\tau = \mu_j\mu_0^{-1} = [Y/X, D]^j$ by (16.1).

Conversely, any such τ has $[\tau \circ D] = [D]$.

To see that the order of the decomposition group $Z(D)$ is $f = f(D, Y/X)$, note that $1 = \mu_j\mu_0^{-1} = [Y/X, D]^j$ iff f divides j.

Part 6) The Galois group $G = G(Y/X)$ of order d acts transitively on primes $[D]$ above $[C]$. The subgroup $Z(D)$ is the subgroup of G fixing $[D]$. Since $Z(D)$ has order f, it follows that the number g of distinct $[D]$ is $g = |G/Z(D)| = d/f$. □

It remains only to discuss the behavior of the Frobenius automorphism with respect to intermediate coverings.

Theorem 20. ***More Properties of the Frobenius Automorphism***

1) Suppose $\widetilde{X}$ is an intermediate covering to Y/X and corresponds to the subgroup H of $G = G(Y/X)$. Let $[D]$ be an equivalence class of prime cycles in Y such that D lies above $\widetilde{C}$ in $\widetilde{X}$. Let $f = f(D, Y/X) = f_1 f_2$, where $f_2 = f(D, Y/\widetilde{X})$ and $f_1 = f(\widetilde{C}, \widetilde{X}/X)$. Then f_1 is the minimal power of $[Y/X, D]$ which lies in H and we have

$$[Y/X, D]^{f_1} = [Y/\widetilde{X}, D]. \tag{16.2}$$

2) If further $\widetilde{X}$ is normal over X, then as an element of $H\backslash G$, we have

$$[\widetilde{X}/X, \widetilde{C}] = H[Y/X, D].$$

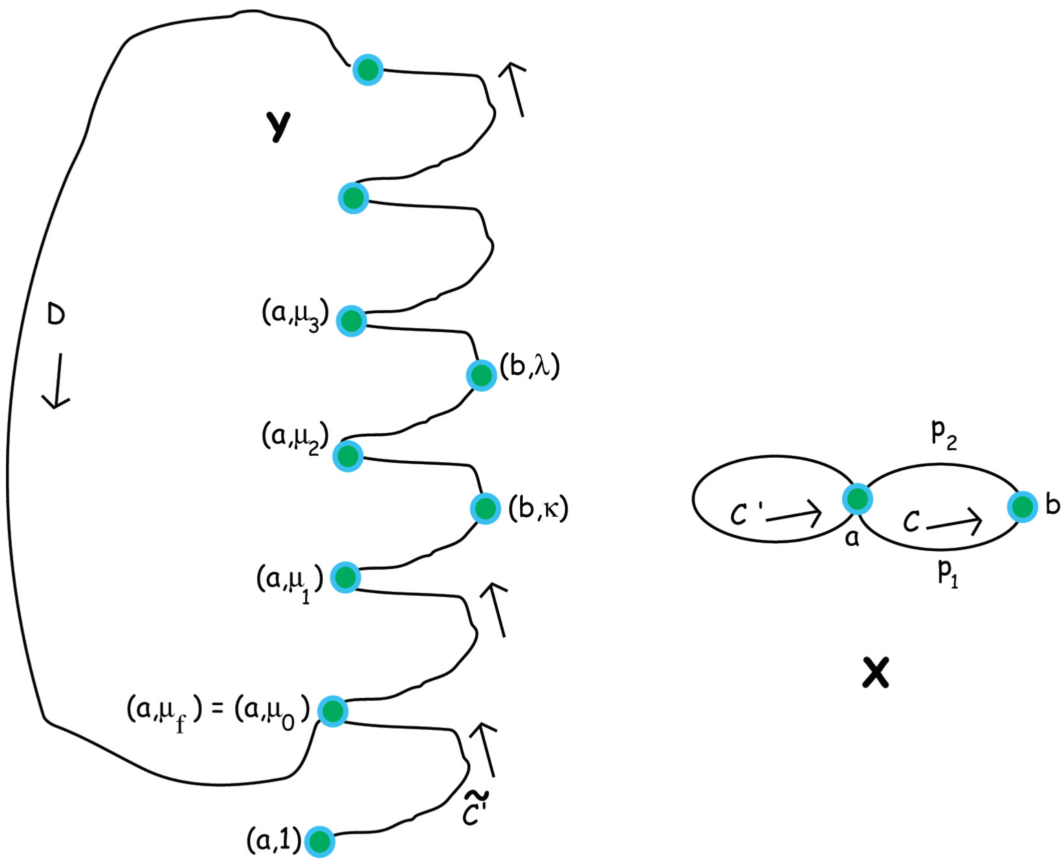


FIGURE 59. **Part of the proof of Proposition 9.** The vertex (b,κ) lies on the r^{th} consecutive lift of C (shown with $r = 2$). The lift to a path in Y starting at (b,κ) of the version of C in $[C]$ starting at b ends at a vertex (b,λ) which arises on the $(r+1)^{st}$ consecutive lift of C.

Proof. **Part 1)** Let C be the prime of X below $\widetilde{C}$. The Frobenius automorphism $[Y/\widetilde{X}, D]$ is found by lifting $\widetilde{C}$ from $\widetilde{X}$ to Y. This is the same as lifting C^{f_1} from X to Y and the analysis in the proof of proposition 9 (equation (16.1) in particular) gives equation (16.2) of 1). The fact that f_1 is the minimal power of $[Y/X, D]$ which lies in H follows from the fact that

$$Z(Y/\widetilde{X}, D) = Z(Y/X, D) \cap H$$

which we know to be cyclic of order f_2. Therefore since $[Y/X, D]$ is of order $f_1 f_2$, we see that $[Y/X, D]^j$ cannot be in H if $j < f_1$.

Part 2) Now let $\widetilde{X}$ be normal over X. View $\widetilde{X}$ as having vertices $(v, H\tau)$, for $v \in X$, $\tau \in G$. Let D start and end at (a, μ_0) in Y and $\widetilde{C}$ start and end at $(a, H\mu_0)$ in $\widetilde{X}$. If C in X lifts to a path in Y starting at (a, μ_0) and ending at (a, μ_1), then C lifts to a path in $\widetilde{X}$ starting at $(a, H\mu_0)$ and ending at $(a, H\mu_1)$. Then 2) follows from Definition 42 of the Frobenius automorphism. □

17. HOW TO CONSTRUCT INTERMEDIATE COVERINGS USING THE FROBENIUS AUTOMORPHISM

Let us now explain how to construct intermediate coverings. First recall Example 24 and Figure 50. The following Lemmas and Theorem give a general construction that we followed in that example. Moreover they allow us to turn the tables and start with the intermediate cover Y_3 and produce its minimal normal cover Z with Y intermediate to Z/X.

Lemma 4. *Suppose Y/X is normal with Galois group G. Fix a spanning tree T of X. Let $e_1, ..., e_r$ be the non-tree edges of X (i.e., those corresponding to generators of the fundamental group) with directions assigned. The r normalized Frobenius automorphisms $\sigma(e_j)$, $j = 1, ..., r$, generate G.*

Proof. Since $\sigma(t) = 1$ for all edges t on the tree of X, for any path p on X, $\sigma(p)$ is a product of the $\sigma(e_j)$ and their inverses, by Lemma 3. Y is connected. Thus we can get to every sheet of Y by lifting paths of X to paths starting on sheet 1 of Y. It follows that any $g \in G$ is a product $\sigma(e_j)$. □

Lemma 5. *Suppose Y/X is normal with Galois group G and $\widetilde{X}$ is an intermediate graph corresponding to the subgroup H of G. Let $H_0 = \bigcap_{g\in G} gHg^{-1}$. Then $H_0 = \{1\}$ if and only if there are no intermediate graphs, other than Y, which are normal over X and intermediate between Y and $\widetilde{X}$.*

Proof. A normal intermediate graph covering $\widetilde{X}$ would correspond to a normal subgroup of G (a subgroup which must be contained in H) and conversely. Any normal subgroup of G contained in H is also contained in every conjugate of H and hence is contained in H_0. Since H_0 is a normal subgroup of G, the result is proved. □

Lemma 6. *Suppose $\widetilde{X}$ is a covering of X and that Y/X is a normal covering of X of minimal degree such that $\widetilde{X}$ is intermediate to Y/X. Let $G = G(Y/X)$ and $H = G(Y/\widetilde{X})$. Let $Hg_1, ..., Hg_n$ be the right cosets of H. We have a 1-1 group anti-homomorphism μ from G into the symmetric group S_n defined by setting $\mu(g)(i) = j$ if $Hg_i g = Hg_j$.*

Proof. By the Exercise below, $\mu(g')\mu(g) = \mu(gg')$. The kernel of μ is the set of $g \in G$ such that $Hg'g = Hg'$, $\forall\ g' \in G$. This means $Hg'gg'^{-1} = H$, $\forall\ g' \in G$, which is equivalent to $g \in g'^{-1}Hg'$, $\forall\ g' \in G$. By Lemma 5, $g = 1$ and μ is 1-1. □

Exercise 71. *Check the claim in the preceding proof that $\mu(g')\mu(g) = \mu(gg')$.*

Now put these three Lemmas together.

Theorem 21. *Let the graphs $Y, \widetilde{X}, X$, the groups G, H, and the representation μ be as in Lemma 6. Let T be a fixed spanning tree of X. Suppose that e is one of the non-tree edges of X. Let $\sigma(e)$ be the corresponding normalized Frobenius automorphism of G. Suppose that v is the starting vertex of e and v' is the terminal vertex of e. If $\mu = \mu(\sigma(e))$ is the permutation of $1, ..., n$ such that $\mu(i) = \mu(\sigma(e))(i) = j$, then the directed edge e lifts to an edge in $\widetilde{X}$ starting at (v, Hg_i) and terminating at (v', Hg_j).*

Proof. By the definition of μ, $Hg_i\sigma(e) = Hg_j$. This means that $g_i\sigma(e) = hg_j$ for some element $h \in H$. By definition of $\sigma(e)$, the edge e lifts to an edge on Y from $(v, 1)$ to $(v', \sigma(e))$. If we apply g_i to this edge, we get an edge on Y starting at (v, g_i) and ending at $(v', g_i\sigma(e)) = (v', hg_j)$. Hence e lifts to a directed edge on $\widetilde{X}$ from (v, Hg_i) to (v', Hg_j). □

This theorem shows us how to create intermediate graphs given a normal cover and it also allows us to construct the minimal normal cover Y of X having a given intermediate covering graph $\widetilde{X}$ of X as well as the Galois group $G(Y/X)$.

Example 27. *Construction of the minimal normal cover Z of the cubic cover Y_3/X in Figure 50.*

Let G be the Galois group $G(Z/X)$ and H the subgroup corresponding to Z. There are 3 cosets Hg_i, $i = 1, 2, 3$. Label the cosets so that Hg_1, Hg_2, Hg_3 correspond to the sheets of Y_3 labeled $', '', '''$. According to Lemma 6, the permutation corresponding to $\sigma(c)$ is $(1)(23)$ and the permutation corresponding to $\sigma(d)$ is $(12)(3)$. By Lemmas 4 and 6, these permutations generate an isomorphic copy of G in S_3. But the permutations (12) and (23) generate S_3. Thus $G = S_3$.

It follows that the minimal normal cover Z has 6 sheets. The vertices of Z are labeled (v, g) with $v \in V(X)$ and $g \in G$. We lift edge c by connecting $(2, g)$ with $(4, g\sigma(c))$. We lift edge d by connecting $(4, g)$ to $(4, g\sigma(d))$. The result is the graph $Z = Y_6$ pictured in Figure 50.

We cannot say which subgroup of S_3 corresponds to Y_3. We can only identify this subgroup up to conjugation. Why? We do not know which coset Hg_1, Hg_2, Hg_3 contains the identity. The 3 choices give 3 embeddings. Equivalently, we can relabel the sheets of Y_3 are the 3 cosets. On S_3, this relabeling is equivalent to a conjugation.

More examples of this theorem can be found in Figures 42, 43 and 44. We will give another series of examples based on the simple group of order 168 later.

Exercise 72. *Check that the intermediate graphs in Figure 42 correspond under the correspondence of the fundamental theorem of Galois theory to the intermediate subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Exercise 73. *Construct your own examples of n-fold cyclic covers of the graph K_4-edge with all possible intermediate covers.*

18. Artin L-Functions

18.1. **Brief Survey on Representations of Finite Groups.** Artin L-functions involve representations of the Galois group. Thus the reader needs to know a bit about representations of finite groups. If this subject is new to you, perhaps it is best to restrict yourself to abelian or even cyclic groups at first.

Let $\mathbb{T}$ denote the multiplicative group of complex number of norm 1. Write $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ to denote the additive (cyclic) group of integers modulo n. A **representation of the cyclic group** $\pi : \mathbb{Z}_n \to \mathbb{T}$ is a group homomorphism. There are n distinct (inequivalent) representations of $\mathbb{Z}_n$ given by $\chi_a(x \bmod n) = \exp(2\pi iax/n)$. Note that χ_a is well defined and does indeed change addition mod n to multiplication of complex numbers on the unit circle. These are the functions on $\mathbb{Z}_n$ that can be used to get a **Fourier transform** $\widehat{f}$ of any $f:\mathbb{Z}_n \to \mathbb{C}$ by writing

$$\widehat{f}(a) = \sum_{x \in G} f(x)\overline{\chi_a(x)}. \tag{18.1}$$

This finite Fourier transform has analogous properties to the usual one on the real line and may be used to approximate the "real" version. In particular, one has **Fourier inversion**

$$nf(-x) = \widehat{\widehat{f(x)}} \ \forall x \in \mathbb{Z}_n.$$

More generally, let G be any finite group. What is an irreducible unitary representation π of G? First π is a group homomorphism $\pi : G \to U(n, \mathbb{C})$, where $U(n, \mathbb{C})$ is the group of $n \times n$ unitary complex matrices under matrix multiplication. A unitary matrix U means ${}^t\overline{U}U = I$. The representation π is **irreducible** means we cannot block upper triangularize the matrices $\pi(g)$ by uniform change of basis. The **degree of the representation** π, denoted $d_\pi = n$, is the size of the matrices $\pi(g)$. If the group is abelian, the only irreducible representations are those of degree 1.

Two representations π_1 and π_2 of G are called **equivalent** (denoted $\cong$) if there is an invertible $n \times n$ complex matrix T such that $T\pi_1(g)T^{-1} = \pi_2(g)$, for all $g \in G$. Then $\widehat{G}$ is the set of all unitary irreducible inequivalent representations of G.

The **trivial representation** denoted by 1 sends every element of G to the 1×1 matrix 1.

Given 2 representations π and ρ, we form the **direct sum** $\pi \oplus \rho$ by creating the block matrix

$$(\pi \oplus \rho)(g) = \begin{pmatrix} \pi(g) & 0 \\ 0 & \rho(g) \end{pmatrix}.$$

Every representation ρ of G is equivalent to a direct sum of irreducible representations:

$$\rho \cong \sum_{\pi \in \widehat{G}}^{\oplus} m_\pi \pi, \quad \text{where the multiplicity } m_\pi \text{ is a non} - \text{negative integer}. \tag{18.2}$$

If the group is not abelian, we get lots of examples of representations (though not necessarily irreducible ones) by a construction called induction. Suppose H is a subgroup of G and ρ is a representation of H. Think of $\rho(g)$ as a linear map $\rho(g) : W \to W$, where W is a vector space over $\mathbb{C}$. Create the **representation induced** by ρ, denoted $\pi = Ind_H^G \rho$, as follows. Define the vector space V to consist of functions on G which transform upon left action by H according to ρ. Then let $\pi(g)$ act on functions in V by right translation. That is define $\pi = Ind_H^G \rho$ by:

$$V = \{ f : G \to W \ \mid \ f(hg) = \rho(h) f(g), \forall h \in H, \forall g \in G \}, \quad (\pi(g) f)(x) = f(xg), \ \forall x, g \in G$$

One then has the important theorem below. We call the representation $Ind_{\{e\}}^G 1$ the **right regular representation.** Our theorem below says the representation $Ind_{\{e\}}^G 1$ (induced from the trivial representation on the trivial subgroup) is the mother of everything in $\widehat{G}$.

Theorem 22. *If e denotes the identity element of G and 1 is the trivial representation, then*

$$\pi = Ind_{\{e\}}^G 1 \cong \sum_{\pi \in \widehat{G}}^{\oplus} d_\pi \pi.$$

The notation means that the right regular representation is equivalent to that obtained from a block diagonal matrix built from all the inequivalent irreducible representations each taken its degree number of times.

Define the **character of the representation** ρ to be $\chi_\rho(g) = Tr(\rho(g))$. For irreducible representations of abelian groups the characters are the representations and we have seen them in earlier parts of the book when we investigated eigenvalues of the adjacency matrix for Cayley graphs like the Paley graph. See subsection 9.2. It turns out that the **character determines the representation up to equivalence**. That is,

$$\chi_\pi = \chi_{\pi'} \quad \text{iff} \quad \pi \cong \pi'.$$

Clearly characters are invariant on **conjugacy classes** $\{g\} = \{xgx^{-1} \mid x \in G\}$. Thus one creates **character tables** indexed by the conjugacy classes $\{g\}$ of G and the inequivalent irreducible unitary representations $\pi \in \widehat{G}$. See Example 28 below for the character table of S_3 and look at [132] for more examples.

There is also an analog of the finite Fourier transform in formula (18.1) and an inversion formula. See [132]. There are many other references for the representation theory of finite groups but few emphasize the connections with Fourier analysis.

We will need the following result of Frobenius. The proof can be found in [132], Chapter 16.

Theorem 23. ***(Frobenius Character Formula).*** *Suppose H is a subgroup of the finite group G. Let σ be a representation of H. Define $\widetilde{\chi_\sigma}(y) = \chi_\sigma(y)$ if $y \in H$ and $\widetilde{\chi_\sigma}(y) = 0$ if $y \notin H$. Then $\pi = Ind_H^G \sigma$ has as character*

$$\chi_\pi(g) = \frac{1}{|H|} \sum_{x \in G} \widetilde{\chi_\sigma}\left(xgx^{-1}\right).$$

The following inner product of functions on G is of great use in representation theory. It can be used to see how many copies of an irreducible representation π of G are contained in an arbitrary representation ρ of G (the **multiplicity** of π in ρ).

Definition 44. *Suppose that $f, g : G \to C$. Define the* ***inner product*** $\langle f, g\rangle_G$ *by* $\langle f, g\rangle_G = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)}$.

One can show that the characters of representations $\pi, \pi' \in \widehat{G}$ satisfy the **orthogonality relations**

$$\langle \chi_\pi, \chi_{\pi'} \rangle_G = \begin{cases} 0, & \text{if } \chi_\pi \neq \chi_{\pi'}, \\ 1, & \text{if } \chi_\pi = \chi_{\pi'}. \end{cases} \tag{18.3}$$

The earlier formula (10.5) was a special case of these orthogonality relations.

It follows from the orthogonality relations above that the **multiplicity** m_π of $\pi \in \widehat{G}$ in a representation ρ of G from formula (18.2) can be computed from

$$m_\pi = \langle \chi_\pi, \chi_\rho \rangle_G .$$

This explains how the character determines the representation up to equivalence.

We will find that a second theorem of Frobenius is very useful.

Theorem 24. ***(Frobenius Reciprocity Law).*** *Under the same hypotheses as the preceding theorem, with $\pi = Ind_H^G \sigma$, we have for any representation ρ of G,*

$$\langle \chi_\rho, \chi_\pi \rangle_G = \langle \chi_{\rho|_H}, \chi_\sigma \rangle_H .$$

Here $\rho|_H$ denotes the restriction of ρ to H.

The following example is a favorite for understanding the zeta functions of the covering in Figure 50.

Example 28. ***The Character Table for S_3.***

We want to consider the character table for the symmetric groups S_3 of permutations of 3 objects. As usual, we employ the disjoint cycle notation. The conjugacy classes then are easily seen to be $\{(1)\}, \{(12)\}$, and $\{(123)\}$. There are 2 obvious 1-dimensional representations, the trivial representation χ_1, and the representation $\chi_1'(\sigma) = (-1)^{sgn(\sigma)}$, where $sgn(\sigma) =$ the number of transpositions needed if we write the permutation σ as a product of transpositions (ab).

The third element of $\widehat{G}$ is a degree 2 representation ρ which can be obtained as follows. For any $\sigma \in G$, define the 3×3 matrix $M(\sigma)$ by

$$M(\sigma) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_{\sigma^{-1}(1)} \\ v_{\sigma^{-1}(2)} \\ v_{\sigma^{-1}(3)} \end{pmatrix}.$$

One sees that M is a representation of G which induces a degree 2 representation ρ on the subspace

$$W = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \middle| v_1 + v_2 + v_3 = 0 \right\}.$$

The ***character table*** *for S_3 is the following.*

$\widehat{G}\backslash\{g\}$	$\{(1)\}$	$\{(12)\}$	$\{(123)\}$
χ_1	1	1	1
χ_1'	1	-1	1
χ_ρ	2	0	-1

The following Exercise will be useful for our understanding of the factorization of zeta functions for normal S_3 covers.

Exercise 74. *a) Define $\pi_1 = Ind_{\{(1)\}}^{S_3} 1$. Show that $\chi_{\pi_1} = \chi_1 + \chi_1' + 2\chi_\pi$.*

b) Define $\pi_2 = Ind_{\{(1),(23)\}}^{S_3} 1$. Show that $\chi_{\pi_2} = \chi_1 + \chi_\pi$.

c) Define $\pi_3 = Ind_{\{(1),(123),(132)\}}^{S_3} 1$. Show that $\chi_{\pi_3} = \chi_1 + \chi_1'$.

18.2. **Definition of Artin-Ihara L-Function.** Suppose that Y is a normal unramified covering of X with Galois group $G = G(Y/X)$.

Definition 45. *If ρ is a representation of G with degree $d = d_\rho$, and u is a complex variable with $|u|$ sufficiently small, define the* ***Artin-Ihara L-function*** *by*

$$L(u, \rho, Y/X) = L(u, \rho) = \prod_{[C]} \det \left(I - \rho\left([Y/X, D]\right) u^{\nu(C)} \right)^{-1},$$

where the product runs over primes $[C]$ of X and $[D]$ is arbitrarily chosen from the primes in Y above C. Here $[Y/X, D]$ is the Frobenius automorphism of Definition 42 and $\nu(C)$ is the length of a path C representing the prime $[C]$.

The Frobenius automorphism is only unique up to conjugacy, but this does not matter, thanks to the determinant. When the representation ρ is trivial $(= 1)$, this is the Ihara zeta function of Definition 2

$$L(u, 1, Y/X) = \zeta_X(u). \tag{18.4}$$

As usual, the Artin L-function is the reciprocal of a polynomial, thanks to analogs of the two determinant formulas. The simplest such formula involves a W_1-type matrix.

Definition 46. *Define the* ***Artin edge adjacency matrix*** *$W_{1,\rho}$ to be a matrix built up out of blocks corresponding to directed edges e and f,*

$$\left(W_{1,\rho}\right)_{e,f} = \rho(\sigma(e)) \left(W_1\right)_{e,f}$$

Here $\sigma(e)$ is the normalized Frobenius automorphism attached to directed edge e from Definition 41, W_1 is the edge adjacency matrix from Definition 8.

Theorem 25. $L(u, \rho, Y/X)^{-1} = \det\left(I - uW_{1,\rho}\right)$.

Proof. We imitate the proof of Formula (4.4). Since $\exp\left(Tr(A)\right) = \det(\exp A)$, for any matrix A (by Exercise 11), we have:

$$\begin{aligned}
\log L(u, \rho, Y/X) &= -\sum_{\substack{[P] \\ \text{prime}}} \log\det\left(I - \rho(\sigma(P) u^{\nu(P)}\right) = -\sum_{[P]} Tr\left(\log\left(I - \rho(\sigma(P)) u^{\nu(P)}\right)\right) \\
&= Tr\left(\sum_{[P]} \sum_{j \geq 1} \frac{1}{j} \rho\left(\sigma(P)\right)^j u^{j\nu(P)}\right) = Tr\left(\sum_{P} \sum_{j \geq 1} \frac{1}{\nu(P^j)} \rho\left(\sigma(P^j)\right) u^{\nu(P^j)}\right) \\
&= \sum_{\substack{C \\ \text{closed} \\ \text{no bktrck or tail}}} \frac{1}{\nu(C)} \chi_\rho\left(\sigma(C)\right) u^{\nu(C)}.
\end{aligned}$$

We also use the fact that the equivalence class $[P]$ has $\nu(P)$ primitive paths. And we need to know that the normalized Frobenius $\sigma(C)$ is multiplicative, as well as the fact that any closed path C without backtracking and tail has the form $C = P^j$, for some prime $[P]$ and some positive integer j. Next set $B = W_{1,\rho}$ and note that

$$\begin{aligned}
Tr(B^m) &= Tr \sum_{e_1,\ldots,e_m} b_{e_1e_2} b_{e_2e_3} \cdots b_{e_{m-1}e_m} b_{e_m e_1} = Tr \sum_{e_1,\ldots,e_m} \rho\left(\sigma(e_1)\right) \rho\left(\sigma(e_2)\right) \cdots \rho\left(\sigma(e_{m-1})\right) \rho\left(\sigma(e_m)\right) \\
&= \sum_{\substack{C,\ \nu(C)=m \\ \text{closed} \\ \text{no bktrck or tail}}} Tr\left(\rho(\sigma(C))\right).
\end{aligned}$$

Therefore using $\exp\left(Tr(A)\right) = \det(\exp A)$ from Exercise 11 again, we obtain

$$\begin{aligned}
\log L(u, \rho, Y/X) &= \sum_{m \geq 1} \frac{1}{m} \sum_{\nu(C)=m} \chi_\rho\left(\sigma(C)\right) u^m = \sum_{m \geq 1} \frac{1}{m} \sum_{\nu(C)=m} Tr(\rho(\sigma(C))) u^m \\
&= Tr\left(\sum_{m \geq 1} \frac{1}{m} W_{1,\rho}^m u^m\right) = Tr\left(\log\left(I - uW_{1,\rho}\right)^{-1}\right) = \log\det\left(I - uW_{1,\rho}\right)
\end{aligned}$$

and the proof is complete. $\square$

Example 29. ***Klein 4 Group Cover of Dumbbell and Intermediate Covers***

Recall Figure 42 and Example 21. First look at the 3 quadratic intermediate covers and their Artin L-functions. I am using Scientific Workplace on my PC to compute the 4×4 determinants. We need the edge adjacency matrix W_1 for the base graph, the dumbbell X, where the edges of X are ordered $a, b, c, a^{-1}, b^{-1}, c^{-1}$:

$$W_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

The zeta function of the dumbbell is

$$\begin{aligned} \zeta_X(u)^{-1} = \det(I - uW_1) = \det & \begin{pmatrix} 1-u & 0 & -u & 0 & 0 & 0 \\ 0 & 1-u & 0 & 0 & 0 & -u \\ 0 & -u & 1 & 0 & -u & 0 \\ 0 & 0 & -u & 1-u & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-u & -u \\ -u & 0 & 0 & -u & 0 & 1 \end{pmatrix} \\ &= -4u^6 + 8u^5 - 3u^4 - 4u^3 + 6u^2 - 4u + 1 \\ &= -(u+1)(u-1)^2(2u-1)\left(2u^2 - u + 1\right). \end{aligned}$$

Then we need the normalized Frobenius automorphisms for the 3 intermediate quadratic covers. Write the Galois group as $G = \mathbb{Z}_2 = \{0, 1 \pmod 2\}$.

$$\begin{aligned} Y' &: \ \sigma(a) = 1(\operatorname{mod} 2), \sigma(b) = 1(\operatorname{mod} 1), \sigma(c) = 0(\operatorname{mod} 2). \\ Y'' &: \ \sigma(a) = 0(\operatorname{mod} 2), \sigma(b) = 1(\operatorname{mod} 1), \sigma(c) = 0(\operatorname{mod} 2). \\ Y''' &: \ \sigma(a) = 1(\operatorname{mod} 2), \sigma(b) = 0(\operatorname{mod} 1), \sigma(c) = 0(\operatorname{mod} 2). \end{aligned}$$

Let the 2 representations of G be denoted 1 for the trivial representation and ρ, where $\rho(0(\operatorname{mod} 2)) = 1$ and $\rho(1(\operatorname{mod} 1)) = -1$. Then the matrices W_σ are found below for each of the 3 intermediate quadratic covers.

For Y', we get the Artin L-function

$$\begin{aligned} L(u, \rho, Y'/X)^{-1} = \det(I - uW_\rho) = \det & \begin{pmatrix} 1+u & 0 & u & 0 & 0 & 0 \\ 0 & 1+u & 0 & 0 & 0 & u \\ 0 & -u & 1 & 0 & -u & 0 \\ 0 & 0 & u & 1+u & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+u & u \\ -u & 0 & 0 & -u & 0 & 1 \end{pmatrix} \\ &= -4u^6 - 8u^5 - 3u^4 + 4u^3 + 6u^2 + 4u + 1 \\ &= -(u+1)^2(u-1)(2u+1)\left(2u^2 + u + 1\right). \end{aligned}$$

For Y'', we get the Artin L-function

$$\begin{aligned} L(u, \rho, Y''/X)^{-1} = \det(I - uW_\rho) = \det & \begin{pmatrix} 1-u & 0 & -u & 0 & 0 & 0 \\ 0 & 1+u & 0 & 0 & 0 & u \\ 0 & -u & 1 & 0 & -u & 0 \\ 0 & 0 & -u & 1-u & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+u & u \\ -u & 0 & 0 & -u & 0 & 1 \end{pmatrix} \\ &= -4u^6 + 5u^4 - 2u^2 + 1 = (u-1)(u+1)\left(-4u^4 + u^2 - 1\right). \end{aligned}$$

For Y''', we get the Artin L-function

$$L(u,\rho,Y'''/X)^{-1} = \det(I-uW_\rho) = \det\begin{pmatrix} 1+u & 0 & u & 0 & 0 & 0 \\ 0 & 1-u & 0 & 0 & 0 & -u \\ 0 & -u & 1 & 0 & -u & 0 \\ 0 & 0 & u & 1+u & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-u & -u \\ -u & 0 & 0 & -u & 0 & 1 \end{pmatrix}$$

$$= -4u^6+5u^4-2u^2+1 = (u-1)(u+1)\left(-4u^4+u^2-1\right).$$

So we find that the zeta functions of the 3 intermediate quadratic covers are

$$\zeta_{Y'}(u)^{-1} = \zeta_X(u)^{-1}L(u,\rho,Y'/X)^{-1} = (u+1)^3(u-1)^3(2u+1)(2u-1)\left(2u^2+u+1\right)\left(2u^2-u+1\right);$$

$$\zeta_{Y''}(u)^{-1} = \zeta_X(u)^{-1}L(u,\rho,Y''/X)^{-1} = (u+1)^2(u-1)^3(2u-1)\left(2u^2-u+1\right)\left(-4u^4+u^2-1\right);$$

$$\zeta_{Y'''}(u)^{-1} = \zeta_X(u)^{-1}L(u,\rho,Y'''/X)^{-1} = (u+1)^3(u-1)^2(2u+1)\left(2u^2+u+1\right)\left(-4u^4+u^2-1\right).$$

Now consider the top graph Z. The Galois group is $\mathbb{Z}_2\times\mathbb{Z}_2 = \{(a,b)\,|a,b\in\mathbb{Z}_2\}$. The representations are: χ_c, where $c=(c_1,c_2)\in\mathbb{Z}_2\times\mathbb{Z}_2$, and for $x=(x_1,x_2)$:

$$\chi_c(x) = e^{2\pi i(c_1x_1+c_2x_2)/2} = (-1)^{c_1x_1+c_2x_2}.$$

We identify sheet 1 (the lowest sheet in Figure) with the element (0, 0) of the Galois group. Then we identify sheet 2 (the next highest sheet) with element (0, 1), sheet 3 (the next highest sheet) with (1, 0) and sheet 4 (the top sheet) with (1, 1). The normalized Frobenius automorphisms are

$$\sigma(a) = (0,1), \sigma(b) = (1,1), \sigma(c) = (0,0).$$

Then

$$\chi_{(0,1)}(\sigma(a)) = (-1)^1 = -1,\ \chi_{(0,1)}(\sigma(b)) = (-1)^1 = -1, \chi_{(0,1)}(\sigma(c)) = (-1)^0 = 1;$$

$$\chi_{(1,0)}(\sigma(a)) = (-1)^0 = 1,\ \chi_{(1,0)}(\sigma(b)) = (-1)^1 = -1, \chi_{(1,0)}(\sigma(c)) = (-1)^0 = 1;$$

$$\chi_{(1,1)}(\sigma(a)) = (-1)^1 = -1,\ \chi_{(1,1)}(\sigma(b)) = (-1)^2 = 1, \chi_{(1,1)}(\sigma(c)) = (-1)^0 = 1.$$

We know that

$$L(u,\chi_{(0,0)},Z/X) = \zeta_X(u) = -(u+1)(u-1)^2(2u-1)\left(2u^2-u+1\right).$$

The 3 new Artin L-functions for Z/X from the non-trivial representations are found easily now. First note that $\chi_{(0,1)}(\sigma(e))$ $=\rho\,(\sigma(e))$ for the graph Y'. This means

$$L(u,\chi_{(0,1)},Z/X)^{-1} = L(u,\rho,Y'/X)^{-1} = -(u-1)(u+1)^2(2u+1)\left(2u^2+u+1\right).$$

Then we see that $\chi_{(1,0)}(\sigma(e))$ $=\rho\,(\sigma(e))$ for the graph Y'', which says

$$L(u,\chi_{(1,0)},Z/X)^{-1} = L(u,\rho,Y''/X)^{-1} = (u-1)(u+1)\left(-4u^4+u^2-1\right).$$

Finally $\chi_{(1,1)}(\sigma(e))$ $=\rho\,(\sigma(e))$ for the graph Y''', implying that

$$L(u,\chi_{(1,1)},Z/X)^{-1} = L(u,\rho,Y'''/X)^{-1} = (u-1)(u+1)\left(-4u^4+u^2-1\right).$$

It is not surprising that the L-functions corresponding to the covers Y'' and Y''' are the same. These graphs are isomorphic as abstract graphs while Y' is not isomorphic to Y''. The L function for Y' comes from that for the trivial representation by replacing u by $-u$.

The reader can now check using the Ihara determinant formula that, as stated in the Corollary to the Proposition below, the zeta function of the top quartic cover Z is the product of all the Artin L-functions of the Galois group:

$$\zeta_Z(u) = L(u,\chi_{(0,0)},Z/X)L(u,\chi_{(0,1)},Z/X)L(u,\chi_{(1,0)},Z/X)L(u,\chi_{(1,1)},Z/X)$$

$$= (u+1)^5(u-1)^5(2u+1)(2u-1)\left(2u^2+u+1\right)\left(2u^2-u+1\right)\left(-4u^4+u^2-1\right)^2.$$

Note that Z is obtained from an 8-cycle by replacing every other edge with a double edge. It follows that

$$\zeta_X(u)^2\zeta_Z(u) = \zeta_{Y'}(u)\zeta_{Y''}(u)\zeta_{Y'''}(u).$$

18.3. **Properties of Artin-Ihara L-Functions.** Now we want to list the properties of Artin L-functions. They are essentially the same as those for the Artin L-functions of Galois extensions of number fields from Lang [73].

Proposition 10. ***Properties of the Artin-Ihara L-Function***
Assume that Y/X is a normal covering with Galois group G.
1) $L(u, \rho_1 \oplus \rho_2) = L(u, \rho_1)L(u, \rho_2)$.
2) Suppose $\widetilde{X}$ is intermediate to Y/X and assume $\widetilde{X}/X$ is normal, $G = Gal(Y/X)$, $H = Gal(Y/\widetilde{X})$. Let ρ be a representation of $G/H \cong Gal(\widetilde{X}/X)$. Thus ρ can be viewed as a representation of G (the ***lift*** *of ρ). Then*

$$L(u, \rho, Y/X) = L(u, \rho, \widetilde{X}/X).$$

3) If $\widetilde{X}$ is an intermediate cover to the normal cover Y/X and ρ is a representation of $H = Gal(Y/\widetilde{X})$, then let $\rho^{\#} = Ind_H^G \rho$, that is, the representation induced by ρ from H up to G. Then

$$L(u, \rho^{\#}, Y/X) = L(u, \rho, Y/\widetilde{X}).$$

Here we do not assume $\widetilde{X}$ normal over X.

Proof. Only property 3) really requires some effort. We postpone the proofs until the next section when we do the more general case of edge L-functions. □

From these properties and Theorem 22, we have the following Corollary.

Corollary 5. ***Factorization of the Ihara Zeta Function of an Unramified Normal Extension of Graphs*** *Suppose that Y/X is normal with Galois group $G = G(Y/X)$. Let $\widehat{G}$ be a complete set of inequivalent irreducible unitary representations of G. Then*

$$\zeta_Y(u) = L(u, 1, Y/Y) = \prod_{\rho \in \widehat{G}} L(u, \rho, Y/X)^{d_\rho}.$$

Proof. Take $Y = \widetilde{X}$ in Part 3) of Proposition 10. The corresponding subgroup H of G is $H = \{e\}$. Then let $\rho = 1$ the trivial representation on H and $\rho^{\#} = Ind_{\{e\}}^G 1$ is the right regular representation. Theorem 22 says $\rho^{\#} = Ind_{\{e\}}^G 1 \cong \sum_{\pi \in \widehat{G}}^{\oplus} d_\pi \pi$. Use formula (18.4), as well as Parts 1) and 3) of Proposition 10, to see that

$$\begin{aligned} \zeta_Y(u) &= L(u, 1, Y/Y) = L(u, \rho^{\#}, Y/X) \\ &= L(u, \sum_{\pi \in \widehat{G}}^{\oplus} d_\pi \pi, Y/X) = \prod_{\rho \in \widehat{G}} L(u, \rho, Y/X)^{d_\rho}. \end{aligned}$$

□

We define some matrices associated to a representation ρ of $G(Y/X)$, where Y/X is a finite unramified normal covering of graphs.

Definition 47. *For $\sigma, \tau \in G$ and vertices $a, b \in X$, define the* ***$A(\sigma, \tau)$ matrix*** *to be the $n \times n$ matrix given by setting the entry $A(\sigma, \tau)_{a,b}$ = the number of directed edges in Y from (a, σ) to (b, τ). Here every undirected edge of Y has been given both directions.*

Except when (a, σ) and (b, τ) are the same vertex on Y (i.e., $a = b$ and $\sigma = \tau$), and even then if there is no loop at $(a, \sigma) = (b, \tau)$, $A(\sigma, \tau)_{a,b}$ is simply the number of undirected edges on Y connecting (a, σ) to (b, τ). However if there is a loop at $(a, \sigma) = (b, \tau)$, then it is counted in both directions and thus the undirected loop is counted twice. It follows from the Exercise below that we can write

$$A(\sigma, \tau) = A(1, \sigma^{-1}\tau) = A(\sigma^{-1}\tau). \tag{18.5}$$

Exercise 75. *Show that*

$$A(\sigma, \tau) = A(1, \sigma^{-1}\tau).$$

Definition 48. *If ρ is a representation of $G(Y/X)$ and $A(\sigma, \tau)$ is given by Definition 47 and Formula (18.5), define the* ***A_ρ matrix*** *by*

$$A_\rho = \sum_{\sigma \in G} A(\sigma) \otimes \rho(\sigma).$$

Also set

$$Q_\rho = Q \otimes I_d,$$

where $Q =$ *the* $|X| \times |X|$ *diagonal matrix with diagonal entry corresponding to* $a \in X$ *given by* $q_a =$ *(degree* a*)* $- 1$ *and* d *is the degree of* ρ.

Theorem 26. ***Block Diagonalization of the Adjacency Matrix of a Normal Cover.*** *Suppose that* Y/X *is normal with Galois group* $G = G(Y/X)$. *Let* $\widehat{G}$ *be a complete set of inequivalent irreducible unitary representations of* G. *Then one can block diagonalize the adjacency matrix of* Y *as with diagonal blocks* A_ρ, *each listed* d_ρ *times as* ρ *runs through* $\widehat{G}$.

Proof. The adjacency operator on Y may be viewed as coming from the representation $Ind_{\{e\}}^G 1$ with the decomposition in Theorem 22. List the vertices of Y as (x, τ), $x \in X$, $\tau \in G$. This decomposes A_Y into $n \times n$ blocks, where $n = |X|$, with blocks given by Definition 47 $A(\sigma, \tau) = A(\sigma^{-1}\tau)$, using formula (18.5) for $\sigma, \tau \in G$. This means $\sigma \in G$ is acting on the function $A : G \to \mathbb{R}$ via $\lambda(\sigma)A(\tau) = A(\sigma^{-1}\tau)$, with $\sigma, \tau \in G$. Then λ is the left regular representation of G. This is equivalent to $Ind_{\{e\}}^G 1$. It follows from Theorem 22 that A_Y has block decomposition into blocks A_ρ corresponding to $\rho \in \widehat{G}$, each listed d_ρ times. □

Now we can generalize Theorem 1.

Theorem 27. ***Ihara Theorem for Artin L-Function.***
With the hypotheses and definitions above, we have

$$L(u, \rho, Y/X)^{-1} = (1 - u^2)^{(r-1)d} \det(I - A_\rho\ u + Q_\rho u^2).$$

Here r *is the rank of the fundamental group of* X.

Proof. We postpone the proof until the next section where we give the L-function version of Bass's proof of Theorem 1. For this we will need edge Artin L-functions. □

18.4. Examples of Factorizations of Artin-Ihara L-Functions.

Example 30. ***The Cube over the Tetrahedron.***

See Figure 39, where the action of the group $G = G(Y/X) = \{1, \sigma\}$ on Y is denoted with primes; i.e., $x' = (x, 1)$ and $x'' = (x, \sigma)$, for $x \in X$. In this case the representations of G are the trivial representation $\rho_0 = 1$ and the representation ρ defined by $\rho(1) = 1, \rho(\sigma) = -1$. So $Q_\rho = 2I_4$. There are two cases.

Case 1. The representation $\rho_0 = 1$.
Here $A_1 = A(1) + A(\sigma) = A$, where

$$A(1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad A(\sigma) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

and A is the adjacency matrix of X.

Case 2. The representation ρ.
Here we find

$$A_\rho = A(1) - A(\sigma) = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{pmatrix}.$$

Now we proceed to check our formulas for this case. We know by the Corollary to Proposition 10 that

$$\zeta_Y(u) = L(u, 1, Y/Y) = L(u, 1, Y/X)L(u, \rho, Y/X) = \zeta_X(u)L(u, \rho, Y/X). \tag{18.6}$$

Ihara's Theorem 1 implies that

$$\zeta_X(u)^{-1} = (1 - u^2)^2(1 - u)(1 - 2u)(1 + u + 2u^2)^3$$

and

$$\zeta_Y(u)^{-1} = (1 - u^2)^2(1 + u)(1 + 2u)(1 - u + 2u^2)^3\zeta_X(u)^{-1}.$$

Then Theorem 27 implies (since $r = 3$)

$$\begin{aligned} L(u, \rho, Y/X)^{-1} &= (1 - u^2)^2 \det(I_4 - A'_\rho u + 2u^2 I_4) \\ &= (1 - u^2)^2(1 + u)(1 + 2u)(1 - u + 2u^2)^3. \end{aligned}$$

Note that $L(u,\rho,Y/X)=\zeta_X(-u)$, although this is not obvious from the determinant formula where $-A\neq A_\rho$.

Note. Theorem 27 implies that equation (18.6) is a factorization of an 8×8 determinant as a product of 4×4 determinants:

$$det(I_8-A_Y u+2I_8u^2)=det(I_4-A_X u+2I_4u^2)\cdot det(I_4-A'_\rho u+2I_4u^2).$$

Exercise 76. *Compute the spectra of the adjacency matrices of the cube and the tetrahedron. Are these graphs Ramanujan?*

It is perhaps worthwhile to state the case of 2-coverings separately. We will also give an example of this proposition below (the cube over the tetrahedron).

Proposition 11. *If Y/X is a 2-covering, then the adjacency matrix A_Y has block decomposition with 2 blocks: one block being A_X (the adjacency matrix of X) and the other block being A_-. The matrix A_- is defined by having entry corresponding to two vertices a,b of X given by:*

$$(A_-)_{a,b}=\begin{cases}+1, & a \text{ and } b \text{ joined by edge } e \text{ in } X \text{ which lifts to edge of } Y \text{ that does not change sheets;}\\ -1 & a \text{ and } b \text{ joined by edge } e \text{ in } X \text{ which lifts to edge of } Y \text{ that changes sheets;}\\ 0 & a \text{ and } b \text{ not joined by edge } e \text{ in } X.\end{cases}$$

Note the following Conjecture made in Hoory et al [55].

Conjecture 1. *Every d-regular graph X has a 2 covering Y such that if A_Y is the adjacency matrix of Y, then*

$$Spectrum(A_Y)-Spectrum(A_X)\subset[-2\sqrt{d-1},2\sqrt{d-1}].$$

This conjecture would allow one to construct families of Ramanujan graphs of arbitrary degree with number of vertices going to infinity by taking repeated 2-covers. By Proposition 11, this conjecture is a conjecture about the spectrum of the matrix A_-.

Exercise 77. *1) For Example 30 all edges of K_4 not in the chosen spanning tree were lifted to start at sheet 1 and end at sheet 2. What happens if you only lift 1 edge?*
2) Check Conjecture 1 for spectra of 2-covers of K_4.

Exercise 78. *Experiment with Conjecture 1 concerning spectra of 2-covers to see whether any k-regular graph does have a 2-cover such that the spectrum of A_- lies in the interval $[-2\sqrt{k-1},2\sqrt{k-1}]$. For example, you could look at all 2-coverings of the torus graph X obtained by taking a product of a 3-cycle and a 5-cycle. Are any of the 2-covers of X Ramanujan?*
There are commands in Mathematica to do most of this.

Example 31. ***The Cube over the Dumbbell.***

The covering we consider in this example is Y/X in Figure 43. The covering group $G(Y/X)$ is the integers mod 4 denoted $\mathbb{Z}_4=\{0,1,2,3 \pmod 4\}$. We label the sheets as follows:

$$x'_1=(x,0(mod\ 4)),\ \ x'_2=(x,1(mod\ 4)),\ \ x''_1=(x,2(mod\ 4)),\ \ x''_2=(x,3(mod\ 4)).$$

The irreducible representations are all one - dimensional and may be written $\chi_\nu(j)=\exp\left(\frac{2\pi i\nu j}{4}\right)=i^{\nu j}$, for $j,\nu\in\mathbb{Z}_4$. Note that although X has loops, Y does not. It follows that

$$A(0)=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ A(1)=A(3)=I_2,\ \ A(2)=0.$$

Thus

$$A_{\chi_0}=\begin{pmatrix}2&1\\1&2\end{pmatrix},\ \ A_{\chi_1}=\begin{pmatrix}0&1\\1&0\end{pmatrix}=\ A_{\chi_3},\ \ A_{\chi_2}=\begin{pmatrix}-2&1\\1&-2\end{pmatrix}.$$

The corresponding L-functions are

$$\begin{aligned}L(u,\chi_0,Y/X)^{-1}&=(1-u^2)\det\begin{pmatrix}1-2u+2u^2&-u\\-u&1-2u+2u^2\end{pmatrix}\\&=(1-u^2)(1-u)(1-2u)(1-u+2u^2);\end{aligned}$$

$$\begin{aligned}L(u,\chi_1,Y/X)^{-1}=L(u,\chi_3,Y/X)&=(1-u^2)\det\begin{pmatrix}1+2u^2&-u\\-u&1+2u^2\end{pmatrix}\\&=(1-u^2)(1+u+2u^2)(1-u+2u^2)\end{aligned}$$

$$L(u,\chi_2,Y/X)^{-1} = (1-u^2)\det\begin{pmatrix} 1+2u+2u^2 & -u \\ -u & 1+2u+2u^2 \end{pmatrix}$$
$$= (1-u^2)(1+u)(1+2u)(1+u+2u^2).$$

One sees again that as in the Corollary to Proposition 10

$$\zeta_Y(u)^{-1} = L(u,\chi_0,Y/X)L(u,\chi_1,Y/X)L(u,\chi_2,Y/X)L(u,\chi_3,Y/X).$$

Note. Again you can view the preceding equality as a factorization of the determinant of an 8×8 matrix as a product of 4 determinants of 2×2 matrices.

Example 32. ***An S_3 Cover.***

Now consider the example in Figure 50. Here view the group S_3 as the dihedral group D_3. Thus it consists of motions of a regular triangle and is generated by F a flip and R a rotation.

$$A(I) = \begin{pmatrix} 0&1&1&1\\ 1&0&0&0\\ 1&0&0&0\\ 1&0&0&0 \end{pmatrix}, \quad A(FR^2) = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix},$$

$$A(FR) = \begin{pmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&1&0&0 \end{pmatrix}, \quad A(R^2)=0, \quad A(R)=0, \quad A(F)=0.$$

Next we need to know the representations of S_3. See Example above. The non-trivial 1-dimensional representation of S_3 has the values $\chi_1(FR)=-1$ and $\chi_1(FR^2)=-1$. The 2-dimensional representation ρ has the values

$$\rho(FR) = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, \text{ and } \rho\left(FR^2\right) = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}, \text{ where } \omega = e^{2\pi i/3}.$$

Now we can compute the matrices in our L-functions:

$$A_{\chi_0} = A, \; A_{\chi_1} = \begin{pmatrix} 0&1&1&1\\ 1&0&0&-1\\ 1&0&0&-1\\ 1&-1&-1&0 \end{pmatrix},$$

$$A_\rho = A_1(I)\otimes\rho(I) + A_1(FR)\otimes\rho(FR) + A_1(FR^2)\otimes\rho(FR^2)$$
$$= \begin{pmatrix} 0&1&1&1&0&0&0&0\\ 1&0&0&0&0&0&0&\omega^2\\ 1&0&0&0&0&0&0&\omega\\ 1&0&0&0&0&\omega^2&\omega&0\\ 0&0&0&0&0&1&1&1\\ 0&0&0&\omega&1&0&0&0\\ 0&0&0&\omega^2&1&0&0&0\\ 0&\omega&\omega^2&0&1&0&0&0 \end{pmatrix}.$$

It follows that

$$L(u,\chi_0,Y_6/X)^{-1} = (1-u^2)\det\begin{pmatrix} 1+2u^2 & -u & -u & -u\\ -u & 1+u^2 & 0 & -u\\ -u & 0 & 1+u^2 & -u\\ -u & -u & -u & 1+2u^2 \end{pmatrix}$$
$$= (1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3);$$

$$L(u,\chi_1,Y_6/X)^{-1} = (1-u^2)\det\begin{pmatrix} 1+2u^2 & -u & -u & -u\\ -u & 1+u^2 & 0 & u\\ -u & 0 & 1+u^2 & u\\ -u & u & u & 1+2u^2 \end{pmatrix}$$
$$= (1-u^2)(1+u)(1+u^2)(1-u+2u^2)(1-u^2+2u^3);$$

$$
\begin{aligned}
L(u,\rho,Y_6/X)^{-1} &= (1-u^2)^2\det(I_8 - A_\rho u + u^2 Q_\rho).\\
&= (1-u^2)^2(1+u+2u^2+u^3+2u^4)(1+u+u^3+2u^4)\\
&\times (1-u+2u^2-u^3+2u^4)(1-u-u^3+2u^4).
\end{aligned}
$$

Putting all our results together, using Theorem 27, Exercise 74, and Proposition 10 , we have:

$$
\zeta_X(u)^{-1} = L(u,\chi_0,Y_6/X)^{-1} = (1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3);
$$

$$
\begin{aligned}
\zeta_{Y_2}(u)^{-1}\zeta_X(u) &= L(u,\chi_1,Y_2/X)^{-1} = L(u,\chi_1,Y_6/X)^{-1}\\
&= (1-u^2)(1+u)(1+u^2)(1-u+2u^2)(1-u^2+2u^3);
\end{aligned}
$$

$$
\begin{aligned}
\zeta_{Y_3}(u)^{-1}\zeta_X(u) &= L(u,\rho,Y_6/X)^{-1}\\
&= (1-u^2)^2(1+u+2u^2+u^3+2u^4)(1+u+u^3+2u^4)\\
&\times (1-u+2u^2-u^3+2u^4)(1-u-u^3+2u^4);
\end{aligned}
$$

and

$$
\zeta_{Y_6}(u) = L(u,\chi_0,Y_6/X)L(u,\chi_1,Y_6/X)L(u,\rho,Y_6/X)^2 = \zeta_X(u)\frac{\zeta_{Y_2}(u)}{\zeta_X(u)}\left[\frac{\zeta_{Y_3}(u)}{\zeta_X(u)}\right]^2.
$$

As a consequence, we find that

$$
\zeta_X(u)^2\zeta_{Y_6}(u) = \zeta_{Y_2}(u)\zeta_{Y_3}(u)^2.
$$

This is analogous to an example of zeta functions of pure cubic extensions of number fields that goes back to Dedekind. See Stark [116].

Note. Again the last equality can be viewed as two different factorizations of determinants involving polynomials in u.

Example 33. ***A Klein 4-Group Cover* $\mathbf{Y/X}$ *from Figure 44.***

Here we can identify the Galois group $G = G(Y/X)$ with $\mathbb{Z}_2^2$. The identification is given by: $x_1' = (x,(1,0))$, $x_1'' = (x,(1,1))$, $x_2' = (x,(0,0))$, $x_2'' = (x,(0,1))$.

The representations of G are $\chi_{r,s}(u,v) = (-1)^{ru+sv}$, for $r,s,u,v \in \mathbb{Z}_2$. We find that

$$
\begin{aligned}
L(u,\chi_{0,0},Y/X)^{-1} &= (1-u^2)\det\begin{pmatrix} 1+2u^2 & -3u\\ -3u & 1+2u^2\end{pmatrix}\\
&= Z_X(u)^{-1} = (1-u^2)(1-u)(1+u)(1-2u)(1+2u).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
L(u,\chi_{0,1},Y/X)^{-1} &= (1-u^2)\det\begin{pmatrix} 1+2u^2 & -u\\ -u & 1+2u^2\end{pmatrix} = L(u,\chi_{1,1},Y/X)^{-1}\\
&= Z_X(u)^{-1} = (1-u^2)(1-u+2u^2)(1+u+2u^2).
\end{aligned}
$$

Also

$$
\begin{aligned}
L(u,\chi_{1,0}Y/X)^{-1} &= (1-u^2)\det\begin{pmatrix} 1+2u^2 & u\\ u & 1+2u^2\end{pmatrix}\\
&= (1-u^2)(1-u+2u^2)(1+u+2u^2).
\end{aligned}
$$

Thus all 3 L-functions with non-trivial representations are equal. This happens here because all 3 intermediate quadratic covers of X are isomorphic as abstract graphs and so they have equal zeta functions. Each intermediate zeta function is of the form $\zeta_{\widetilde{X}}(u) = \zeta_X(u)L(u,\chi,Y/X)$, where χ runs through the 3 non-trivial representations of G as $\widetilde{X}$ runs through the 3 intermediate quadratic covers of X. For $\zeta_Y(u)$ we have

$$
\begin{aligned}
\zeta_Y(u)^{-1} &= \prod_{\chi\in\widehat{G}} L(u,\chi,Y/X)\\
&= (1-u^2)^4(1-u)(1+u)(1-2u)(1+2u)(1-u+2u^2)^3(1+u+2u^2)^3.
\end{aligned}
$$

We also have

$$
\zeta_X^2(u)\zeta_Y(u) = \zeta_{\widetilde{X}}(u)^3
$$

which holds for all 3 intermediate quadratic covers $\widetilde{X}$ of X.

Example 34. ***A Cyclic 6-Fold Cover* $\mathbf{Y/X}$ *from Figure 45.***

The covering group $G = G(Y/X) \cong \mathbb{Z}_6 = \{1, 2, 3, 4, 5, 6\ (mod\ 6)\}$, with identity element 6 (mod 6). Let $\omega = e^{2\pi i/6}$. The representations are $\chi_a(b) = \omega^{ab}$, for $a, b \in \mathbb{Z}_6$. Here the matrices $A(\tau)$ are 1×1. We obtain

$$A(6) = A(3) = 0, \quad A(1) = A(2) = A(4) = A(5) = 1.$$

We find that

$$\begin{aligned} A_{\chi_0} &= 4 = A = \text{adjacency matrix of } X; \\ A_{\chi_j} &= 0, \text{ for } j = 1, 3, 5; \\ A_{\chi_j} &= -2, \text{ for } j = 2, 4. \end{aligned}$$

Then

$$\begin{aligned} L(u, \chi_0, Y/X)^{-1} &= \zeta_X(u)^{-1} = (1-u^2)(1-u)(1-3u); \\ L(u, \chi_j, Y/X)^{-1} &= (1-u^2)(1+3u^2), \text{ for } j = 1, 3, 5; \\ L(u, \chi_j, Y/X)^{-1} &= Z_X(u)^{-1} = (1-u^2)(1+2u+3u^2), \text{ for } j = 2, 4. \end{aligned}$$

Set

$$m = \begin{pmatrix} 1+3u^2 & -u & -u & 0 & -u & -u \\ -u & 1+3u^2 & -u & -u & 0 & -u \\ -u & -u & 1+3u^2 & -u & -u & 0 \\ 0 & -u & -u & 1+3u^2 & -u & -u \\ -u & 0 & -u & -u & 1+3u^2 & -u \\ -u & -u & 0 & -u & -u & 1+3u^2 \end{pmatrix}.$$

By Ihara's formula

$$\begin{aligned} \zeta_Y(u)^{-1} &= (1-u^2)^6 \det(m) \\ &= (1-u^2)^6 (3u-1)(u-1)\left(3u^2+2u+1\right)^2 \left(1+3u^2\right)^3, \end{aligned}$$

which agrees with the product

$$\zeta_Y(u)^{-1} = \prod_{\chi \in \widehat{G}} L(u, \chi, Y/X).$$

Exercise 79. *Check whether the Ihara zetas of the preceding graphs and covering graphs satisfy the Riemann hypothesis.*

19. Edge Artin L-Functions

19.1. **Definition and Properties of Edge Artin L-Function.** Suppose that Y/X is a normal graph covering and recall Definition 25 of the edge matrix, Definition 26 of the edge zeta function, Definition 42 of the Frobenius automorphism. We use these definitions to define the edge Artin L-function imitating the definitions from algebraic number theory.

Definition 49. *Given a path C in X, which is written as a product of oriented edges $C = a_1 a_2 \cdots a_s$, the* ***edge norm*** *of C is*

$$N_E(C) = w_{a_1 a_2} w_{a_2 a_3} \cdots w_{a_{s-1} a_s} w_{a_s a_1}.$$

The ***edge Artin L- function*** *associated to a representation ρ of the Galois group $G(Y/X)$ and the edge matrix W is*

$$L(W, \rho) = L_E(W, \rho, Y/X) = \prod_{[C]} \det\left(I - \rho\left(\frac{Y/X}{D}\right) N_E(C)\right)^{-1},$$

where the product is over primes $[C]$ in X and $[D]$ is arbitrarily chosen from the primes in Y over C. Here W is the edge matrix of Definition 25 with variables $|w_{ef}|$ assumed sufficiently small, and $\left(\frac{Y/X}{D}\right)$ is the Frobenius automorphism of Definition 42.

Exercise 80. *Show that the determinant in the definition of the edge zeta function does not depend on the choice of D over C in Definition 49.*

Hint. *The various Frobenii $\left(\frac{Y/X}{D}\right)$ are conjugate to each other.*

For the factorization of edge zeta functions, we need a specialization of W matrices.

Definition 50. *Suppose that $\widetilde{X}$ is an unramified covering of X and that $\widetilde{W}$ and W are the corresponding edge matrices. Suppose that $\widetilde{e}$ and $\widetilde{f}$ are two edges of $\widetilde{X}$ with projections e and f in X using the covering map $\pi : \widetilde{X} \longrightarrow X$. If $\widetilde{e}$ feeds into $\widetilde{f}$ and $\widetilde{e} \neq \widetilde{f}^{-1}$, then e feeds into f and $e \neq f^{-1}$. Thus we can set the variable $\widetilde{w}_{\widetilde{e}\widetilde{f}} = w_{ef}$. When we do this for all the variables of $\widetilde{W}$, we call the* ***X-specialized edge matrix*** *$\widetilde{W}_{spec}$.*

Theorem 28. ***Main Properties of Edge Artin L-Functions.***
Assume $\widetilde{X}$ is a normal (unramified) cover of X.
1) The edge L-function at the trivial representation is the usual edge zeta function:

$$L_E(W, 1, \widetilde{X}/X) = \zeta_E(W, X),$$

2) The edge zeta function of $\widetilde{X}$, with X-specialized edge matrix from Definition 50, factors as a product of edge L-functions:

$$\zeta_E(\widetilde{W}_{spec}, \widetilde{X}) = \prod_{\rho\in\widehat{G}} L_E(W, \rho)^{d_\rho}.$$

Here the product is over all inequivalent irreducible unitary representations of the Galois group $Gal(\widetilde{X}/X)$.
3)Let $m = |E|$ be the number of unoriented edges of X. If the representation ρ of G has degree d, define a $2dm \times 2dm$ ***Artin edge matrix*** *W_ρ with block form*

$$W_\rho = \left(w_{ef}\,\rho(\sigma(e))\right),$$

where $\sigma(e)$ denotes the normalized Frobenius element of Definition 41 corresponding to edge e. Then

$$L_E(W, \rho, Y/X) = \det(I - W_\rho)^{-1}.$$

Part 1) follows from the definitions. Part 2) is proved using Theorem 22 and parts 2 and 4 of the next Theorem as in the proof of Corollary 5 above. We will prove part 3) below in Subsection 19.2.

Theorem 29. ***More Properties of Edge Artin L-Functions.*** *Assume that Y/X is an (unramified) normal cover with Galois group G.*
1) If you specialize the non-zero w_{ij} to be u, then the Artin edge L-function $L_E(W, \rho)$ specializes to the Artin-Ihara L-function $L(u, \rho)$.
2) $L_E(W, \rho_1 \oplus \rho_2) = L_E(W, \rho_1)L_E(W, \rho_2)$.
3) If $\widetilde{X}$ is intermediate to Y/X, $G = Gal(Y/X)$ and $H = Gal(Y/\widetilde{X})$. Assume that $\widetilde{X}/X$ is normal. Let ρ be a representation of $G/H \cong Gal(\widetilde{X}/X)$. Then ρ can be viewed as a representation of G, (the ***lift*** *of ρ). Then*

$$L_E(W, \rho, Y/X) = L_E(W, \rho, \widetilde{X}/X).$$

4) Suppose H is any subgroup of $G = Gal(Y/X)$. Let $\widetilde{X}$ be the intermediate cover to Y/X corresponding to H by Theorem 17. Now we do not assume that H is a normal subgroup of G. Let ρ be a representation of H and let $\rho^\#$ denote the ***representation of*** G ***induced by*** *ρ. Then, using Definition 50 of $\widetilde{W}_{spec}$,*

$$L_E(\widetilde{W}_{spec}, \rho, Y/\widetilde{X}) = L_E(W, \rho^\#, Y/X).$$

Proof. Part 1) follows from the definitions. Part 2) is easily proved by rewriting the logarithm of the L-function as a sum involving traces of the representations since $Tr\,(\rho_1 \oplus \rho_2) = Tr\rho_1 + Tr\rho_2$. Part 3) follows from the definitions. Part 4) will be proved later in subsection 19.3.

□

Example 35. ***The edge L-function of a Cube covering a Dumbbell.***

The edge L-functions for the representations of the Galois group of Y/X, which is $\mathbb{Z}_4$, require the matrix W which has entries w_{ij}, when edge e_i feeds into edge e_j. For the labeling of the edges of the dumbbell, see Figure 60. We find that the matrix W is:

$$W = \begin{pmatrix} w_{11} & w_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & w_{23} & 0 & 0 & w_{26} \\ 0 & 0 & w_{33} & 0 & w_{35} & 0 \\ 0 & w_{42} & 0 & w_{44} & 0 & 0 \\ w_{51} & 0 & 0 & w_{54} & 0 & 0 \\ 0 & 0 & 0 & 0 & w_{65} & w_{66} \end{pmatrix}.$$

Next we need to compute $\sigma(e_i)$ for each edge e_i where $\sigma(C)$ denotes the normalized Frobenius automorphism of Definition 41. We will write the Galois group $G(Y/X) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, where $(x, \sigma_j) = x^{(j)}$, for $x \in X$. The identification of $G(Y/X)$ with $\mathbb{Z}_4$ sends σ_j to $(j - 1(mod\ 4))$. Then compute the Galois group elements associated to the edges: $\sigma(e_1) = \sigma_2$, $\sigma(e_2) = \sigma_1$, $\sigma(e_3) = \sigma_2$. The representations of our group are 1-dimensional, given by $\chi_a(\sigma_b) = i^{a(b-1)}$, for $a, b \in \mathbb{Z}_4$.

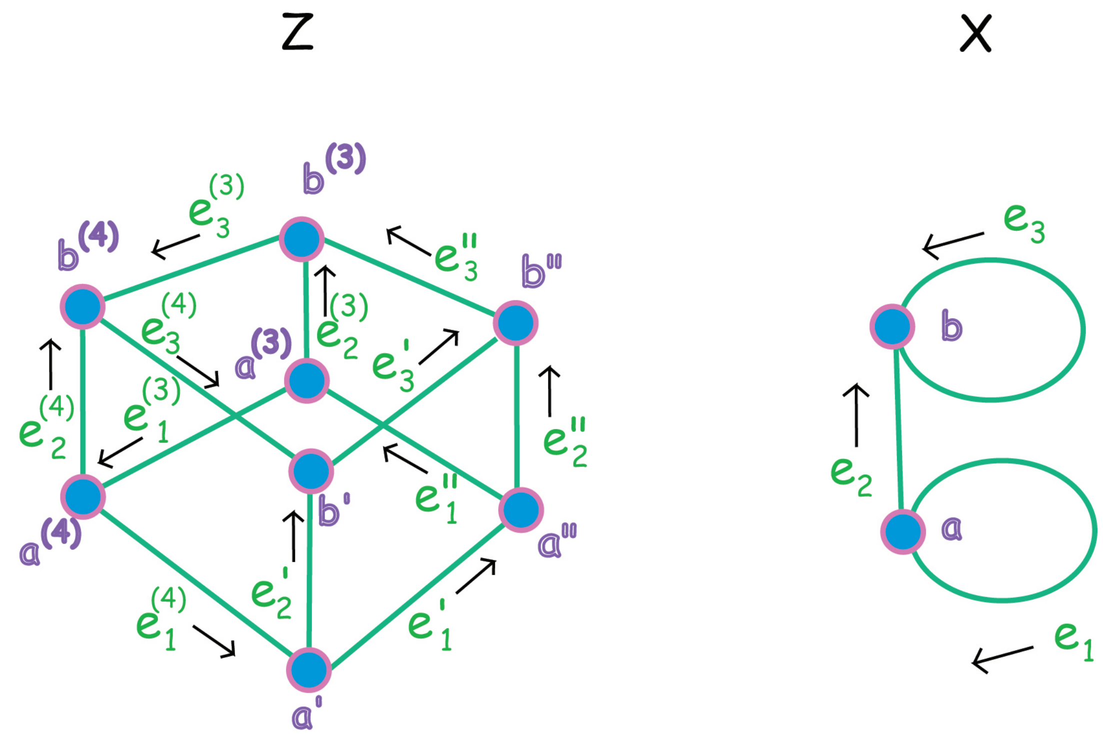


FIGURE 60. **Edge Labelings for the Cube as a $\mathbb{Z}_4$ Covering of the Dumbbell.**

So we obtain

$$L_E(W,\chi_0,Y/X)^{-1} = \zeta_E(W,X)^{-1} = det\begin{pmatrix} w_{11}-1 & w_{12} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{23} & 0 & 0 & w_{26} \\ 0 & 0 & w_{33}-1 & 0 & w_{35} & 0 \\ 0 & w_{42} & 0 & w_{44}-1 & 0 & 0 \\ w_{51} & 0 & 0 & w_{54} & -1 & 0 \\ 0 & 0 & 0 & 0 & w_{65} & w_{66}-1 \end{pmatrix};$$

$$L_E(W,\chi_1,Y/X)^{-1} = det(I-W_{\chi_1}) = det\begin{pmatrix} iw_{11}-1 & iw_{12} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{23} & 0 & 0 & w_{26} \\ 0 & 0 & iw_{33}-1 & 0 & iw_{35} & 0 \\ 0 & -iw_{42} & 0 & -iw_{44}-1 & 0 & 0 \\ w_{51} & 0 & 0 & w_{54} & -1 & 0 \\ 0 & 0 & 0 & 0 & -iw_{65} & -iw_{66}-1 \end{pmatrix};$$

$$L_E(W,\chi_2,Y/X)^{-1} = det(I-W_{\chi_2}) = det\begin{pmatrix} -w_{11}-1 & -w_{12} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{23} & 0 & 0 & w_{26} \\ 0 & 0 & -w_{33}-1 & 0 & -w_{35} & 0 \\ 0 & -w_{42} & 0 & -w_{44}-1 & 0 & 0 \\ w_{51} & 0 & 0 & w_{54} & -1 & 0 \\ 0 & 0 & 0 & 0 & -w_{65} & -w_{66}-1 \end{pmatrix};$$

$$L_E(W,\chi_3,Y/X)^{-1} = det(I-W_{\chi_3}) = det\begin{pmatrix} -iw_{11}-1 & -iw_{12} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{23} & 0 & 0 & w_{26} \\ 0 & 0 & -iw_{33}-1 & 0 & -iw_{35} & 0 \\ 0 & iw_{42} & 0 & iw_{44}-1 & 0 & 0 \\ w_{51} & 0 & 0 & w_{54} & -1 & 0 \\ 0 & 0 & 0 & 0 & iw_{65} & iw_{66}-1 \end{pmatrix}.$$

Note that, by part 2 of Theorem 28, the product of the preceding four 6×6 determinants has to be a 24×24 determinant:

$$det(I - \widetilde{W}_{spec}) = \prod_{i=0}^{3} det(I - W_{\chi_i}),$$

where $\widetilde{W}_{spec}$ is given by specializing the edge matrix of the cube as follows. For the edge variables of the dumbbell, we write

$$a = w_{11}, b = w_{12}, c = w_{23}, d = w_{26}, e = w_{33}, f = w_{35},$$
$$g = w_{42}, h = w_{44}, j = w_{51}, k = w_{54}, m = w_{65}, n = w_{66}.$$

Using the natural ordering of the edges of Y given by

$$e_1', e_1'', e_1^{(3)}, e_1^{(4)}, e_2', e_2'', e_2^{(3)}, e_2^{(4)}, e_3', e_3'', e_3^{(3)}, e_3^{(4)}, \text{ and the inverses in the same order,}$$

the matrix $\widetilde{W}_{spec}$ is the following

$$\begin{pmatrix}
0&a&0&0&0&b&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&a&0&0&0&b&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&a&0&0&0&b&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
a&0&0&0&b&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&c&0&0&0&0&0&0&0&0&0&0&0&0&0&0&d\\
0&0&0&0&0&0&0&0&0&c&0&0&0&0&0&0&0&0&0&0&d&0&0&0\\
0&0&0&0&0&0&0&0&0&0&c&0&0&0&0&0&0&0&0&0&0&d&0&0\\
0&0&0&0&0&0&0&0&0&0&0&c&0&0&0&0&0&0&0&0&0&0&d&0\\
0&0&0&0&0&0&0&0&0&e&0&0&0&0&0&0&0&f&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&e&0&0&0&0&0&0&0&f&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&e&0&0&0&0&0&0&0&f&0&0&0&0\\
0&0&0&0&0&0&0&0&e&0&0&0&0&0&0&0&f&0&0&0&0&0&0&0\\
0&0&0&0&g&0&0&0&0&0&0&0&0&0&0&h&0&0&0&0&0&0&0&0\\
0&0&0&0&0&g&0&0&0&0&0&0&h&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&g&0&0&0&0&0&0&h&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&g&0&0&0&0&0&0&h&0&0&0&0&0&0&0&0&0\\
j&0&0&0&0&0&0&0&0&0&0&0&0&0&0&k&0&0&0&0&0&0&0&0\\
0&j&0&0&0&0&0&0&0&0&0&0&k&0&0&0&0&0&0&0&0&0&0&0\\
0&0&j&0&0&0&0&0&0&0&0&0&0&k&0&0&0&0&0&0&0&0&0&0\\
0&0&0&j&0&0&0&0&0&0&0&0&0&0&k&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&m&0&0&0&0&0&0&n\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&m&0&0&n&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&m&0&0&n&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&m&0&0&n&0
\end{pmatrix}.$$

19.2. **Proofs of Determinant Formulas for Edge Artin L-Functions.** One can prove part 3) of Theorem 28 using the same method as we gave for the analogous result for the edge Ihara zeta function.

Proof. of part 3) of Theorem 28.

We are trying to prove that

$$L_E(W, \rho, Y/X) = \det(I - W_\rho)^{-1}.$$

For this, we imitate the proof of Theorem 25 with $\chi = Tr(\rho)$. So we start by taking the logarithm of the L-function, obtaining

$$\begin{aligned} log(L_E(W, \rho, Y/X)) &= \sum_{[P]} \sum_{j=1}^{\infty} \frac{1}{j} \chi(\sigma(P)^j) N_E(P^j) \\ &= \sum_{P} \sum_{j=1}^{\infty} \frac{1}{j\nu(P)} \chi(\sigma(P^j)) N_E(P^j). \end{aligned}$$

Now the sum is over paths P rather than classes $[P]$.

The block i_1, i_{n+1} entry of W_ρ^n is (as we noted in many earlier proofs)

$$\sum_{i_2,\ldots,i_n} w(i_1, i_2) \cdots w(i_n, i_{n+1}) \rho(\sigma(i_1)) \cdots \rho(\sigma(i_n)) = \sum_{\substack{C = i_1 \cdots i_n \\ \nu(C) = n}} w(i_1, i_2) \cdots w(i_n, i_{n+1}) \rho(\sigma(C)),$$

where the sum is over all paths C on X of length n with leading edge i_1.

The only non-zero entries in the last sum may be restricted to those paths C whose initial edge is i_1 and whose terminal edge i_n feeds into i_{n+1} with the additional condition that i_{n+1} is not the inverse to i_n. Thus when taking the trace, we have $i_{n+1} = i_1$, and we are talking about closed backtrackless, tailless paths of length n.

Therefore using Exercise 11, as usual, we see that

$$\begin{aligned}\log(L_E(W, \rho, Y/X)) &= \sum_C \frac{Tr(\rho(C))}{\nu(C)} N_E(C) \\ &= \sum_{m\geq 1} \frac{1}{m} Tr(W_\rho^m) = Tr\left(\log(I - W_\rho)^{-1}\right) \\ &= log\left(\det\left((I - W_\rho)^{-1}\right)\right).\end{aligned}$$

This completes the proof of part 3) of Theorem 28. □

Bass Proof of Ihara Theorem for Artin L-Functions.

Next we give the Bass proof of the Ihara Theorem 27 for Artin L-functions. We must first generalize the S, T matrices in Proposition 4. For a representation ρ of the Galois group $G(Y/X)$, let d_ρ be its degree (i.e., the size of the matrices $\rho(g)$). When we write the tensor product $B \otimes C$ for the $p \times p$ matrix B and the $r \times r$ matrix C, we mean the $pr \times pr$ matrix with block decomposition

$$B \otimes C = \begin{pmatrix} b_{11}C & \cdots & b_{1p}C \\ \vdots & \ddots & \vdots \\ b_{p1}C & \cdots & b_{pp}C \end{pmatrix}.$$

Definition 51. *With the definitions of the start, terminal and J matrices S, T, J as in Proposition 4, define the* ***Artinized start, terminal and J matrices*** *by*

$$S_\rho = S \otimes I_{d_\rho}, T_\rho = T \otimes I_{d_\rho}, J_\rho = J \otimes I_{d_\rho}.$$

Definition 52. *We will also define the $2md_\rho \times 2md_\rho$ block diagonal* ***R-matrix*** *R_ρ to be*

$$R_\rho = \begin{pmatrix} \rho(\sigma(e_1)) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho(\sigma(e_{2m})) \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix}. \tag{19.1}$$

Here $e_1, ..., e_{2m}$ is our list of oriented edges of X, ordered in our usual way via formula (2.1). The last equality comes from the property of the normalized Frobenius saying $\sigma(e) = \sigma(e^{-1})$.

Recall Definition 46 of $W_{1,\rho}$.

Finally recall that A_ρ, the adjacency matrix associated to ρ is as in Definition 48. With all the preceding definitions, we have the following proposition relating all the matrices.

Proposition 12. ***Formulas Involving $\rho, Q, W, A, R, S, T, J$.***

1)$W_{1,\rho} = R_\rho\,(W_1 \otimes I_d)$.
2) $A_\rho = S_\rho R_\rho\, {}^tT_\rho$.
3) $S_\rho J_\rho = T_\rho,\ T_\rho J_\rho = S_\rho,\ Q_\rho + I_{nd_\rho} = S_\rho\, {}^tS_\rho = T_\rho\, {}^tT_\rho$.
4) $W_{1,\rho} + R_\rho J_\rho = R_\rho\, {}^tT_\rho\, S_\rho$.
5) $(R_\rho J_\rho)^2 = I_{2|E|d}$.

Proof. 1) To see this, just multiply matrices in block form, setting $W_1 = B$ with entries $b_{a,b}$:

$$\left(W_{1,\rho}\right)_{e,f} = \rho(\sigma(e))(W_1)_{ef} = \left(\begin{pmatrix} \rho(\sigma(e_1)) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho(\sigma(e_{2m})) \end{pmatrix}\begin{pmatrix} b_{e_1,e_1}I_{d_\rho} & \cdots & b_{e_1,e_{2m}}I_{d_\rho} \\ \vdots & \ddots & \vdots \\ b_{e_{2m},e_1}I_{d_\rho} & \cdots & b_{e_{2m},e_{2m}}I_{d_\rho} \end{pmatrix}\right)_{e,f}.$$

2) Set $d = d_\rho$. Then we have

$$S_\rho R_\rho\, {}^tT_\rho = (S \otimes I_d) \begin{pmatrix} \rho(\sigma(e_1)) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho(\sigma(e_{2m})) \end{pmatrix} {}^t(T \otimes I_d)$$

$$= \begin{pmatrix} s_{11}I_d & \cdots & s_{1\ 2m}I_d \\ \vdots & \ddots & \vdots \\ s_{n1}I_d & \cdots & s_{n\ 2m}I_d \end{pmatrix} \begin{pmatrix} \rho(\sigma(e_1)) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho(\sigma(e_{2m})) \end{pmatrix} \begin{pmatrix} t_{11}I_d & \cdots & t_{n1}I_d \\ \vdots & \ddots & \vdots \\ t_{1\ 2m}I_d & \cdots & t_{n\ 2m}I_d \end{pmatrix}.$$

Then look at the block corresponding to vertices a, b of X and obtain

$$\left(S_\rho R_\rho\, {}^tT_\rho\right)_{a,b} = \sum_e s_{a,e}\rho(\sigma(e))t_{b,e} = \sum_{g \in G} \rho(g) \sum_{e,\sigma(e)=g} s_{a,e}t_{b,e}$$
$$= \sum_{g \in G} (A(g))_{a,b}\rho(g) = (A_\rho)_{a,b}.$$

Here the last equality uses Definition 48 of A_ρ. The result in part 2) follows.

3) The proof proceeds by the following computation:

$$\left(S_\rho J_\rho\right)_{v,e} = \left(\begin{pmatrix} s_{11}I_d & \cdots & s_{1\ 2m}I_d \\ \vdots & \ddots & \vdots \\ s_{n1}I_d & \cdots & s_{n\ 2m}I_d \end{pmatrix} \begin{pmatrix} 0 & I_m \otimes I_d \\ I_m \otimes I_d & 0 \end{pmatrix}\right)_{v,e} = (T)_{v,e}.$$

$$\left(S_\rho\, {}^tS_\rho\right)_{a,b} = \sum_e s_{a,e}I_d s_{b,e}I_d = (\#\text{ edges out of } a)\,\delta_{a,b}I_d = (Q+I)_{a,b}I_d$$

4)

$$\left(R_\rho\, {}^tT_\rho S_\rho\right)_{e,f} = \left(\begin{pmatrix} \rho(\sigma(e_1)) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho(\sigma(e_{2m})) \end{pmatrix} \begin{pmatrix} t_{11}I_d & \cdots & t_{n1}I_d \\ \vdots & \ddots & \vdots \\ t_{12m}I_d & \cdots & t_{n\ 2m}I_d \end{pmatrix} \begin{pmatrix} s_{11}I_d & \cdots & s_{1\ 2m}I_d \\ \vdots & \ddots & \vdots \\ s_{n1}I_d & \cdots & s_{n\ 2m}I_d \end{pmatrix}\right)_{e,f}$$
$$= \sum_{\substack{v \\ \xrightarrow{e} v \xrightarrow{f}}} \rho(\sigma(e))t_{ev}s_{vf}I_d = \rho(\sigma(e))\,(W_1)_{e,f} + \rho(\sigma(e))J_{e,f}.$$

The last term is for the case that $f = e^{-1}$, when $(W_1)_{e,f} = 0$.

5) To prove this, just note that:

$$\left(\begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right)^2 = \begin{pmatrix} 0 & U \\ U^{-1} & 0 \end{pmatrix}^2 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

□

Next we prove the main formulas in the Bass proof.

Proposition 13. *Main Formulas in Bass Proof of Ihara's Theorem for Artin L-Functions.*

1)

$$\begin{pmatrix} I_{nd} & 0 \\ R_\rho\, {}^tT_\rho & I_{2md} \end{pmatrix}\begin{pmatrix} I_{nd}\left(1-u^2\right) & S_\rho u \\ 0 & I_{2md} - uW_{1,\rho} \end{pmatrix}$$
$$= \begin{pmatrix} I_{nd} - A_\rho u + Q_\rho u^2 & S_\rho u \\ 0 & I_{2md} + R_\rho J_\rho u \end{pmatrix}\begin{pmatrix} I_{nd} & 0 \\ R_\rho\, {}^tT_\rho - {}^tS_\rho u & I_{2md} \end{pmatrix}.$$

2)

$$I_{2md} + R_\rho J_\rho u = \begin{pmatrix} I_{md} & Uu \\ U^{-1}u & I_{md} \end{pmatrix}.$$

3)

$$\begin{pmatrix} I_{md} & 0 \\ -U^{-1}u & I_{md} \end{pmatrix}\left(I_{2md} + R_\rho J_\rho u\right) = \begin{pmatrix} I_{md} & Uu \\ 0 & I_{md}\left(1-u^2\right) \end{pmatrix}.$$

Proof. The proofs are an **Exercise** in block multiplication of matrices using Proposition 12. □

Proof. of Theorem 27.

We are trying to show that the Artin-Ihara L-function has a 3-term determinant formula:

$$L\left(u,\rho,Y/X\right)^{-1}=(1-u^2)^{(r-1)d}\det\left(I_{nd}-A_\rho u+Q_\rho u^2\right).$$

First recall Theorem 25:

$$L\left(u,\rho,Y/X\right)^{-1}=\det\left(I-uW_{1,\rho}\right)$$

Thus we need to show that

$$\det\left(I_{2md}-uW_{1,\rho}\right)=\left(1-u^2\right)^{(r-1)d}\det\left(I_{nd}-A_\rho u+Q_\rho u^2\right).$$

We see, upon taking determinants of the formula in part 1) of the last Proposition, that

$$\left(1-u^2\right)^{nd}\det\left(I_{2md}-uW_{1,\rho}\right)=\det\left(I_{nd}-A_\rho u+Q_\rho u^2\right)\det\left(I_{2md}+R_\rho J_\rho u\right).$$

Then parts 2) and 3) of the last Proposition imply that

$$\det\left(I_{2md}+R_\rho J_\rho u\right)=\left(1-u^2\right)^{md}.$$

Theorem 27 follows from the fact that $m-n=r-1$. □

19.3. **Proof of the Induction Property.** The induction property of the edge Artin L-function (part 4) of Theorem 29 is the next thing for us to prove. To imitate the proof of the analogous number theory fact, one needs the following Lemma.

Lemma 7. *Suppose Y/X is normal with Galois group G and H is the subgroup of G corresponding to an intermediate covering $\widetilde{X}$. Let $\chi=Tr\rho$ be a character of a representation of H and $\chi^{\#}=Tr\left(Ind_H^G\rho\right)$ be the corresponding induced character of G. For any prime $[C]$ of X, we have*

$$\sum_{j=1}^{\infty}\frac{1}{j}\chi^{\#}\left(\sigma(C^j)\right)N_E(C)^j=\sum_{[\widetilde{C}]|[C]}\sum_{j=1}^{\infty}\frac{1}{j}\chi\left(\widetilde{\sigma}\left(\widetilde{C}\right)^j\right)N_E\left(\widetilde{C}\right)^j_{spec}. \tag{19.2}$$

Here $\sigma(C)\in G$ is the normalized Frobenius automorphism for C in X and $\widetilde{\sigma}(\widetilde{C})\in G$ is the normalized Frobenius corresponding to $\widetilde{C}$ in $\widetilde{X}$. The X-specialized edge matrix in the norm on the right is from Definition 50.

Proof. Let D_1 be the prime of Y above C starting on sheet 1. Then $\sigma(C)=[Y/X,D_1]$. Using the Frobenius formula for the induced character (Theorem 23), we have

$$\sum_{j=1}^{\infty}\frac{1}{j}\chi^{\#}\left(\sigma(C)^j\right)N_E(C)^j=\sum_{j=1}^{\infty}\sum_{\substack{g\in G\\ \left(g\sigma(C)g^{-1}\right)^j\in H}}\frac{1}{j|H|}\chi\left(\left(g\sigma(C)g^{-1}\right)^j\right)N_E(C)^j.$$

Each distinct prime $[D]$ of Y above C has the form $D=g\circ D_1$ and occurs for $f=f(D,Y/X)$ elements of G, where f is the residual degree of Definition 38. From Proposition 9 we see that

$$\sum_{j=1}^{\infty}\frac{1}{j|H|}\sum_{\substack{g\in G\\ \left(g\sigma(C)g^{-1}\right)^j\in H}}\chi\left((g\sigma(C)g^{-1})^j\right)N(C)^j=\sum_{[D]|[C]}\sum_{\substack{j\geq 1\\ [Y/X,D]^j\in H}}\frac{f}{j|H|}\chi\left([Y/X,D]^j\right)N(C)^j.$$

Group the various D over C into those over a fixed $\widetilde{C}$ and then sum over the $\widetilde{C}$. For a fixed $\widetilde{C}$, all D dividing $\widetilde{C}$ have the same minimal power $j=f_1=f(\widetilde{C},\widetilde{X}/X)$ such that $[Y/X,D]^j\in H$. This power gives the Frobenius automorphism of D with respect to $Y/\widetilde{X}$ by Theorem 20. Thus the last double sum is

$$\sum_{[\widetilde{C}]|[C]}\sum_{[D]|[\widetilde{C}]}\sum_{j\geq 1}\frac{f}{f_1 j|H|}\chi\left([Y/\widetilde{X},D]^j\right)N(C)^{f_1 j}.$$

For all $[D]|[\widetilde{C}]$, the $[Y/\widetilde{X},D]$ are conjugate to each other in H and there are g_2 such D where $g_2 f_2 = |H|$. Here $f_2 = f(D, Y/\widetilde{X})$ and $g_2 = g(D, Y/\widetilde{X})$. If we pick one fixed D above $\widetilde{C}$, we therefore get

$$\sum_{[D]|[\widetilde{C}]} \sum_{j\geq 1} \frac{f}{f_1 j|H|}\chi\left([Y/\widetilde{X},D]^j\right) N_E(C)^{f_1 j} = \sum_{j\geq 1} \frac{fg_2}{f_1 j|H|}\chi\left([Y/\widetilde{X},D]^j\right) N(C)^{f_1 j}$$

$$= \sum_{j\geq 1} \frac{1}{j}\chi\left([Y/\widetilde{X},D]^j\right) N(C)^{f_1 j}.$$

The proof is completed by putting the chain of equalities together, since

$$N(C)^{f_1} = N(\widetilde{C})_{spec}.$$

$\square$

The following Corollary will be needed for our discussion of graphs that are isospectral but not isomorphic.

Corollary 6. *If Y/X is normal with Galois group G and H is the subgroup of G corresponding to an intermediate cover $\widetilde{X}$. Let $\chi_1^{\#}$ be the character of the representation of G induced from the trivial representation 1 of H. Then the number of primes $\left[\widetilde{C}\right]$ of $\widetilde{X}$ above a prime $[C]$ of X with length $\nu\left(\widetilde{C}\right) = \nu(C)$ is $\chi_1^{\#}(\sigma(C))$, where $\sigma(C)$ denotes the normalized Frobenius automorphism of Definition 41. This means that $\chi_1^{\#}(\sigma(C))$ is the number of primes of $\widetilde{X}$ above $[C]$ with residual degree 1.*

Proof. Set $\chi = \chi_1$ in Lemma 7 and set each non-zero edge variable $w_{ij} = u$. This makes $N_E(C) = u^{\nu(C)}$ and $N_E(\widetilde{C})_{spec} = u^{\nu(\widetilde{C})}$. Look at the $u^{\nu(C)}$ term on both sides of equation (19.2). The coefficient of $u^{\nu(C)}$ on the left side comes from the $j = 1$ term and it is $\chi^{\#}(\sigma(C))$. The coefficient of the $u^{\nu(C)}$ term on the right is the number of $[\widetilde{C}]$ above $[C]$ with $\nu(\widetilde{C}) = \nu(C)$. $\square$

Proof. of the Induction Property of Edge L-Functions.

By the definition of the edge L-function for Y/X, we have

$$log(L_E(W,\rho^{\#},Y/X)) = \sum_{[C]}\sum_{j=1}^{\infty} \frac{1}{j}\chi^{\#}(\sigma(C)^j)N(C)^j.$$

Apply Lemma 7 to see that the right side is

$$\sum_{[\widetilde{C}]}\sum_{j=1}^{\infty} \frac{1}{j}\chi\left(\widetilde{\sigma}(\widetilde{C})^j\right) N_E(\widetilde{C})^j_{spec},$$

where the sum is over all primes $\widetilde{C}$ of $\widetilde{X}$ and $\widetilde{\sigma}(\widetilde{C})$ is the corresponding normalized Frobenius automorphism in H. The proof is completed using the definition of the edge L-function for $Y/\widetilde{X}$. $\square$

We could also give a purely combinatorial proof of the induction property - noting that the two determinants arising from part 3) of Theorem 28 are the same size. Using the definition of induced representations, one can see that the two determinants are the same. See Stark and Terras [122].

Exercise 81. *Find a combinatorial proof of the induction property of the edge L-function,*

$$L_E(\widetilde{W}_{spec},\rho,Y/\widetilde{X}) = L_E(W,\rho^{\#},Y/X),$$

by looking at the formulas $L_E(W,\rho,Y/X) = \det(I - W_\rho)^{-1}$ and the analogous result for $L_E(W,\rho^{\#},Y/X)$, with $\rho^{\#} = Ind_H^G\rho$.

Look at Figure 61. You need to split the $\widetilde{W}_\rho$ matrix into $2m \times 2m$ blocks indexed by oriented edges e, f of X. The e block row comes from directed edges of $\widetilde{X}$ projecting to e. Edge e lifts to each sheet of $\widetilde{X}$. These sheets are labeled by cosets Hg_k. Write an edge of $\widetilde{X}$ as $\widetilde{e_s}$ if it projects to e and has initial vertex on sheet Hg_s. Then we claim

$$\left(\widetilde{W}_{\rho,spec}\right)_{e,f} = \rho^{\#}(\sigma(e))\, w_{ef}.$$

Suppose edge $\widetilde{f_t}$ has initial vertex on sheet Hg_t. If $\widetilde{e_s}$ feeds into $\widetilde{f_t}$, we see that e lifts to edge of Y starting on sheet g_s and ending on sheet $h_{st}g_t$ with

$$g_s\sigma(e) = h_{st}g_t = \widetilde{\sigma}(\widetilde{e_s}) \in Gal(Y/\widetilde{X}).$$

So we find

$$\left(\widetilde{W}_{\rho,spec}\right)_{e,f} = \left(\rho\left(g_s\sigma(e)g_t^{-1}\right)\right) w_{ef}$$

by the formula for the matrix entries of an induced representation, setting ρ equal to 0 outside of H.

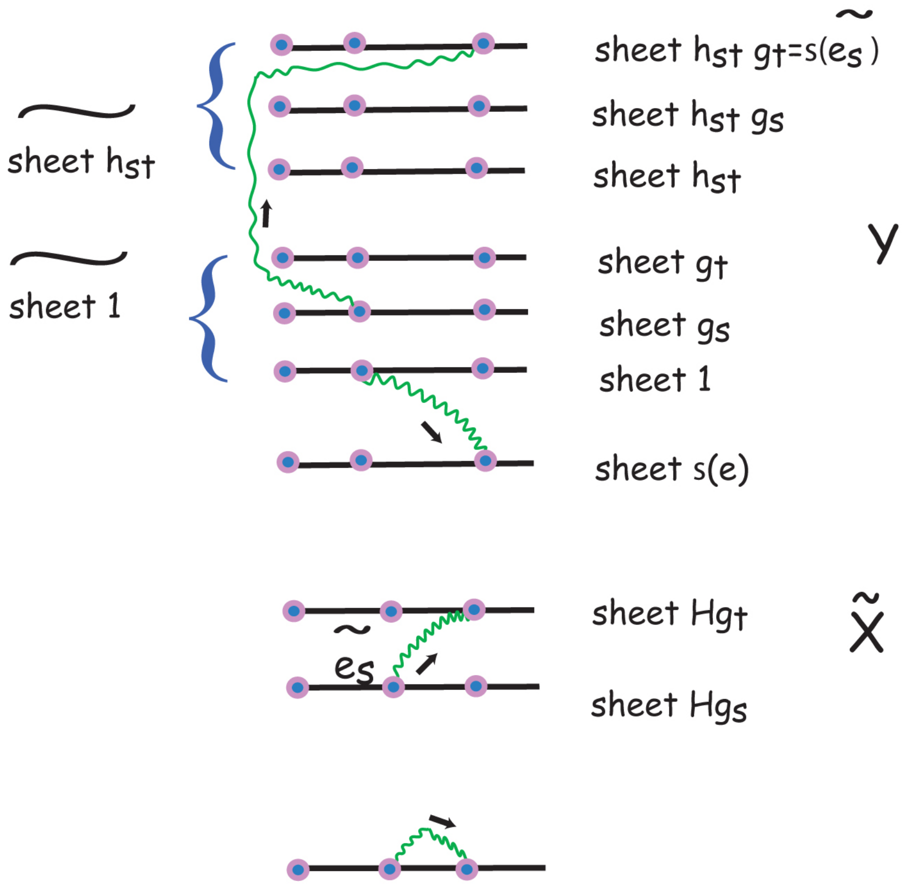


FIGURE 61. **Proving the induced representations property of edge L-functions**

20. PATH ARTIN L-FUNCTIONS.

20.1. Definition and Properties of Path Artin L-Functions.

There is one final kind of Artin L-function - the path L-function invented by Stark which generalizes the path zeta function discussed earlier. Recall Definitions 29 and 30 of the path matrix Z, path norm and path zeta function.

Definition 53. *Assume Y/X normal with Galois group G. Given a representation ρ of G and path matrix Z with $\left|z_{ef}\right|$ sufficiently small, the* ***path Artin L-function*** *is defined by*

$$L_P(Z,\rho)=\prod_{\substack{[C]\ prime\\ in\ X}}\det\left(1-\rho\left(\frac{Y/X}{D}\right)N_P(C)\right)^{-1}.$$

Here $\left(\frac{Y/X}{D}\right)$ is from Definition 42, the path matrix Z is from Definition 29, the path norm $N_P(C)$ is from Definition 30, and the product is over primes $[C]$ of X, with $[D]$ any prime of Y over $[C]$.

The path Artin L-function has analogous properties to the edge L-function. You just have to replace E with P in Theorems 28 and 29.

Proposition 14. ***Some Properties of the Path Artin L-Function.***
1) $L_P(Z,1,Y/X)=\zeta_P(Z,X)$.
2) $L_P(Z,\rho_1\oplus\rho_2,Y/X)=L_P(Z,\rho_1,Y/X)L_P(Z,\rho_2,Y/X)$.
3) Let Y/X be normal with Galois group G, and $\widetilde{X}$ be intermediate to Y/X and normal with Galois group H. Let ρ be a representation of $G/H\cong G(\widetilde{X}/X)$. View ρ as a representation of G (the ***lift*** *of ρ). Then*

$$L_P(Z,\rho,Y/X)=L_P(Z,\rho,\widetilde{X}/X).$$

Theorem 30. ***The path Artin L-Function is the inverse of a polynomial.***
The path L- function satisfies

$$L_P(Z,\rho,Y/X)=\det\left(I-Z_\rho\right)^{-1}.$$

where $Z_\rho=(z_{ef}\,\rho\,(\sigma(e)))$ and I is the $2dr\times 2dr$ identity matrix, where d is the degree of ρ.

Proof. The proof is like that of part 3) of Theorem 28 for the edge L-function. □

Just as with the path zeta functions the variables of the path L-function can be specialized to obtain the edge L-function. This specialization was given in Formula (12.1).

Via this specialization, we find that

$$L_P(Z(W),\rho)=L_E(W,\rho). \tag{20.1}$$

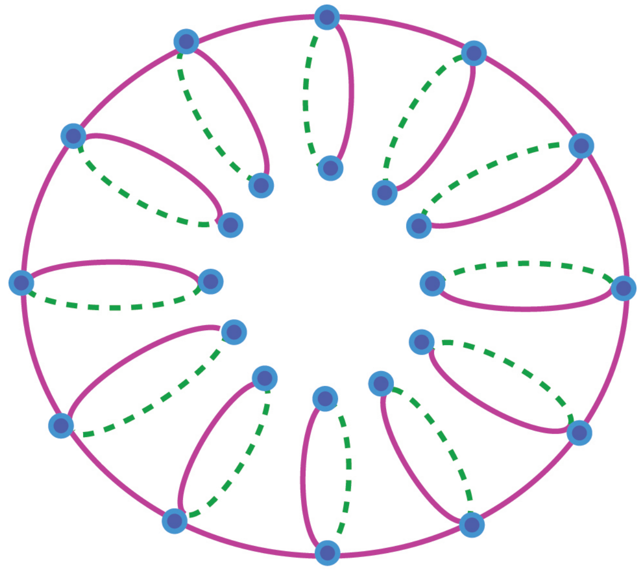
Example 36. ***Path Artin L-functions for Cyclic Cover of 2 Loops with Extra Vertex on 1 Loop.***
Consider the base graph of 2 loops with an extra vertex on 1 loop. Now for our n-cyclic cover we lift edge a up 1 sheet and keep edge b in the same sheet. See Figure 62.

The path matrix of the Artin L-function for an n-cyclic cover of two loops with an extra vertex on one loop as in Figure 62 is 4×4 and we can compute the L-functions for the n-cyclic cover by hand or use a computer. I used Scientific Workplace on my PC. So the L-function for the cyclic n-cover of the graph X consisting of 2 loops with an extra vertex on 1 loop in Figure 62, with $\rho=e^{2\pi ia/n}$ and $s=2\cos(2\pi a/n)$, is

$$L(u,\chi_a)^{-1}=det\begin{pmatrix}\rho u-1 & \rho u & 0 & \rho u^2\\ u^2 & u^2-1 & u^2 & 0\\ 0 & \rho^{-1}u & \rho^{-1}u-1 & \rho^{-1}u^2\\ u & 0 & u & u^2-1\end{pmatrix}$$
$$=\frac{1}{r}(u-1)(u+1)\left(-3ru^4+u^3\left(r^2+1\right)+u\left(r^2+1\right)-r\right).$$

When the character $\chi_a=1$, we have $\rho=1$ and we get the Ihara zeta function of the graph in Figure 62.

We can also compute the Artin L-functions of this cover using the 6×6 $W_{1,\rho}$-matrix. We have decided to plot the eigenvalues of the matrices $W_{1,\rho}$. These eigenvalues are the reciprocals of the poles of the Ihara zeta of the cover. The result is in Figure 63, which implies that the Riemann hypothesis is very false for this graph.



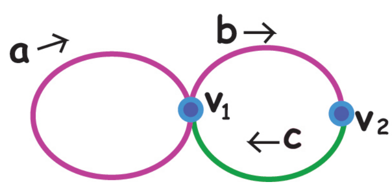


FIGURE 62. **A 12-cyclic cover of the base graph with 2 loops and 2vertices.** The spanning tree in the base graph is green dashed. The sheets of the cover above are also green dashed.

Exercise 82. *Do the of the previous example but instead of keeping the lifts of edge b in the same sheet lift them down 1 sheet.*

Exercise 83. *Do the of the previous examples but replace the base graph with* $K_4 -$ one edge.

It is interesting to compare the spectra for the cyclic covers with those for other abelian covers of the same base graph as well as for random covers. See Figures 82 and 80 in section 26 below.

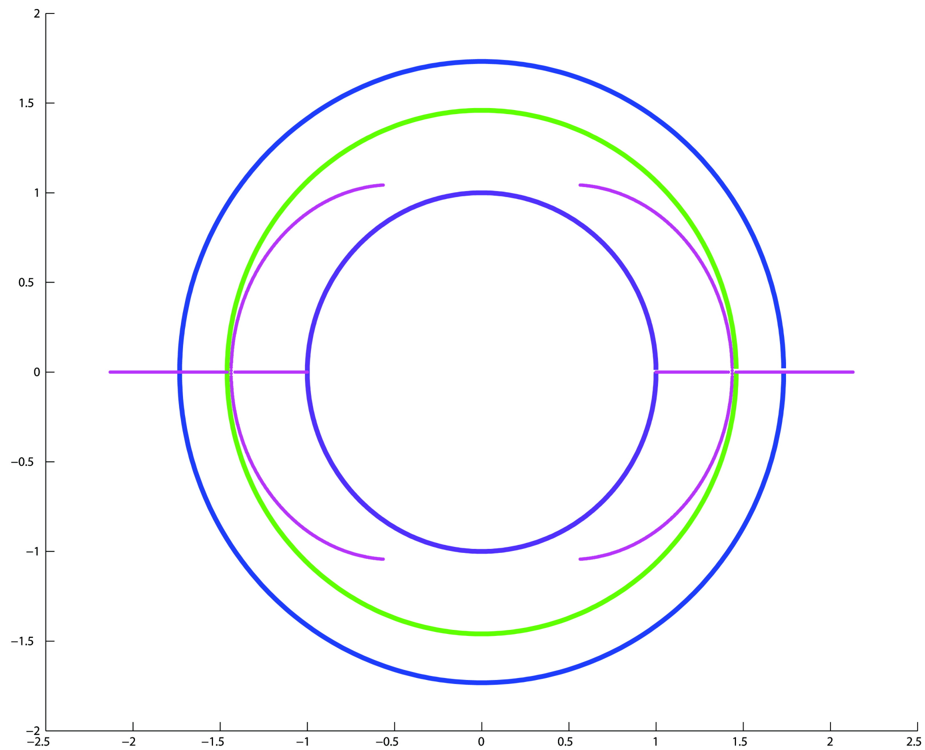


FIGURE 63. **The purple points are the eigenvalues of the edge adjacency matrix** W_1 for the 10,000 cyclic cover of the graph X consisting of 2 loops with an extra vertex on 1 loop analogous to the cover in Figure 62. These are the reciprocals of the poles of the Ihara zeta function for the covering graph. The circles are centered at the origin and have radii $\sqrt{p}$, $1/\sqrt{R}$, $\sqrt{q}$. Here $p = 1, 1/R \cong 2.1304, q = 3$. The Riemann hypothesis is very false.

20.2. **Induction Property.** Next we want to discuss the induction property of the path L-functions. For this, if $\widetilde{X}$ is a covering of X, we need to specialize the path matrix $\widetilde{Z}$ of $\widetilde{X}$ to the variables in the path matrix Z of X. This must be done in such a way that if $\widetilde{C}$ is a reduced cycle in its conjugacy class of the fundamental group of $\widetilde{X}$, then under the specialization, $N_P(\widetilde{C})$ becomes $N_P(C)$ where C is the projected cycle of $\widetilde{C}$ in X.

Specialization Rule for Induction Property of Path Artin L-Functions.

First we need a contraction rule. In X, we contract the spanning tree T in the base graph X to a point. See Figure 64. This gives a graph $B(X)$ which is a bouquet of loops, pictured on the right in the Figure. Graphs X and $B(X)$ have the same fundamental group. The path and edge zeta functions of $B(X)$ are the same. In the cover $\widetilde{X}$, we also contract each sheet (the connected inverse images of T) to a point. This gives a graph we call $C\left(\widetilde{X}\right)$ pictured on the right at the top of Figure 64. The lifts of the r generating paths for the fundamental group of X to $\widetilde{X}$ give the edges of $C(\widetilde{X})$. What makes this interesting is that if $\widetilde{X}$ is an d-fold covering of X, then $d-1$ of the lifted edges from the $B(X)$ must be used in the tree of $\widetilde{X}$. The remaining $dr-(d-1)=d(r-1)+1$ non-tree edges of $C\left(\widetilde{X}\right)$ give the generators of the fundamental group of $\widetilde{X}$. The specialization algorithm needs to take account of the tree edges.

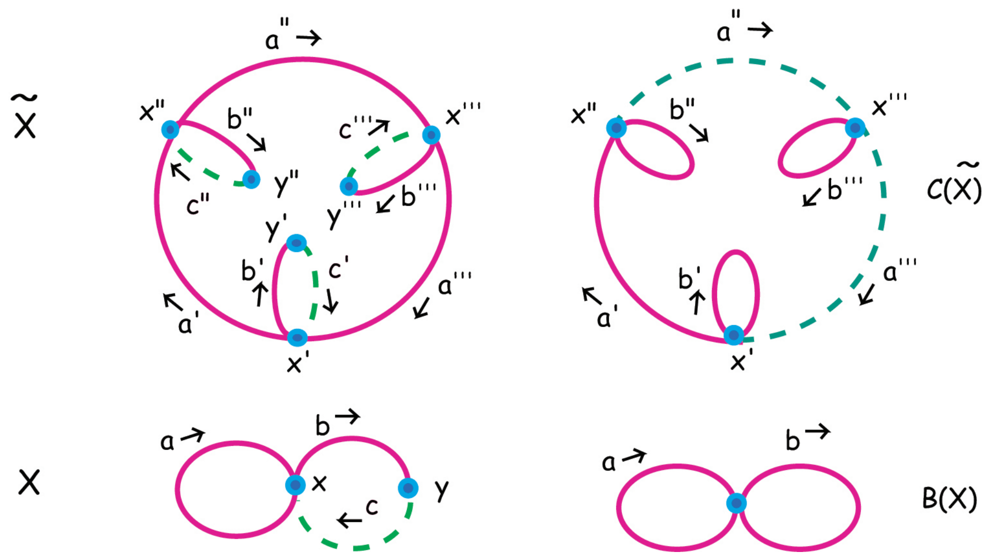


FIGURE 64. **Illustration of the contraction of sheets of a cover corresponding to the spanning trees** contracted below. Here on the left we have a $d=3$-cyclic cover $\widetilde{X}$ of X. A spanning tree of X is shown on the left below with green dashed lines. When we contract the spanning tree of X below, we get the bouquet of loops $B(X)$ on the right. On the right at the top, the graph $C(\widetilde{X})$ is obtained by contracting the sheets of $\widetilde{X}$. In $C(\widetilde{X})$ the new spanning tree is shown with green dashed edges and it will have $d-1=2$ edges.

First specialize variables in the path matrix $\widetilde{Z}\left(\widetilde{X}\right)$ to the edge variables on the contracted graph $C\left(\widetilde{X}\right)$. This turns the path norm into the edge norm on the contracted graph $C\left(\widetilde{X}\right)$. Then specialize the edge variables of the contracted $C\left(\widetilde{X}\right)$ to the edge variables of the contracted base graph $B(X)$ in our usual manner from from the induction theorem for the edge Artin L-function. This turns the edge norm on $\widetilde{X}$ into the edge norm on $B(X)$ which is the same as the path norm on X. This is the desired specialization. Call it $\widetilde{Z}_{spec}$.

Example 37. *Contracted Covers. The contracted versions of X and Y_3 from Figure 50 are shown in Figure 65.*

We will use the following notation for the inverses of edges c and d. We write $C=c^{-1}$ and $D=d^{-1}$.

The tree $\widetilde{T}$ of Y_3 is completed with one of the lifts of the cC pair between the top two sheets of Y_3 and one of the lifts of the dD pair between the bottom two sheets. The remaining four undirected edges of the contracted Y_3 give rise to the fundamental group of Y_3 and the resulting 8×8 path matrix $\widetilde{Z}$. We give these edges directions projecting to either c or d, rather than C or D, and labels I, II, III, IV, as shown. The inverse edges, projecting to C and D, are given labels $V, VI, VII, VIII$, as shown. The rows and columns of $\widetilde{Z}$ are then labeled by the Roman numerals $I-VIII$.

Following our specialization algorithm described above, the resulting specialized matrix $\widetilde{Z}_{spec}$ is then

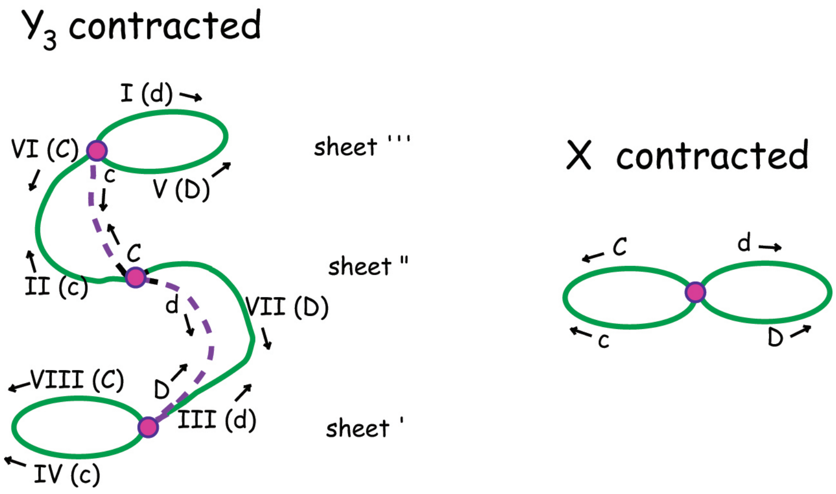


FIGURE 65. **Contracted versions of** X and Y_3 **from Figure 50.** Solid edges are the non-tree edges generating the fundamental group.

$$
\begin{pmatrix}
z_{dd} & z_{dc}z_{cc} & z_{dc}z_{cd}z_{dd} & z_{dc}z_{cd}z_{dc} & 0 & z_{dC} & z_{dc}z_{cD} & z_{dc}z_{cd}z_{dC} \\
z_{cd} & z_{cc}z_{cc} & z_{cc}z_{cd}z_{dd} & z_{cc}z_{cd}z_{dc} & z_{cD} & 0 & z_{cc}z_{cD} & z_{cc}z_{cd}z_{dC} \\
z_{dC}z_{Cd} & z_{dc} & z_{dd}z_{dd} & z_{dd}z_{dc} & z_{dC}z_{CD} & z_{dC}z_{CC} & 0 & z_{dd}z_{dC} \\
z_{cD}z_{DC}z_{Cd} & z_{cD}z_{Dc} & z_{cd} & z_{cc} & z_{cD}z_{DC}z_{CD} & z_{cD}z_{DC}z_{CC} & z_{cD}z_{DD} & 0 \\
0 & z_{Dc}z_{cc} & z_{Dc}z_{cd}z_{dd} & z_{Dc}z_{cd}z_{dc} & z_{DD} & z_{DC} & z_{Dc}z_{cD} & z_{Dc}z_{cd}z_{dC} \\
z_{CC}z_{Cd} & 0 & z_{Cd}z_{dd} & z_{Cd}z_{dc} & z_{CC}z_{CD} & z_{CC}z_{CC} & z_{CD} & z_{Cd}z_{dC} \\
z_{DD}z_{DC}z_{Cd} & z_{DD}z_{Dc} & 0 & z_{Dc} & z_{DD}z_{DC}z_{CD} & z_{DD}z_{DC}z_{CC} & z_{DD}z_{DD} & z_{DC} \\
z_{CD}z_{DC}z_{Cd} & z_{CD}z_{Dc} & z_{Cd} & 0 & z_{CD}z_{DC}z_{CD} & z_{CD}z_{DC}z_{CC} & z_{CD}z_{DD} & z_{CC}
\end{pmatrix}
$$

For example, the IV, I entry $\widetilde{Z}_{IV,I}$ follows directed edge IV (projecting to c), through two edges of $\widetilde{T}$ (projecting to D and C consecutively) to edge I (projecting to d) resulting in the specialized value $z_{cD}z_{DC}z_{Cd}$. This agrees with the fact that any path on Y_3 going through consecutive cut edges IV and I must project to a path on X going consecutively through c, D, C, d.

Exercise 84. *Work out* $\widetilde{Z}_{spec}$ *for the example in Figure 64.*

Theorem 31. ***Induction Property for Path L-functions.***

Suppose Y/X *is normal with Galois group* G*. If* H *is a subgroup of* G *corresponding to the intermediate covering* $\widetilde{X}$, ρ *is a representation of* H, *and* $\rho^{\#}$ *is the representation of* G *induced by* ρ, *then assuming the variables of the path matrix* $\widetilde{Z}$ *for* $Y/\widetilde{X}$ *are specialized according to the specialization rule above,*

$$L_P(\widetilde{Z}_{spec}, \rho, Y/\widetilde{X}) = L_P(Z, \rho^{\#}, Y/X).$$

Proof. Contract each copy of the tree T to a point, both in X and in $\widetilde{X}$. Then both sides of the equality in this theorem become edge L-functions attached to a graph with one vertex and r loops and the corresponding covering of it. Since the induction theorem has been proved in subsection 19.3 for edge L-functions, we are done. $\square$

Remark. From Theorem 30, the equality of Theorem 31 becomes

$$det(I - \widetilde{Z}_{\text{spec},\rho}) = det(I - Z_{\rho^{\#}}).$$

Unlike the analogous equality for the edge L-functions obtained from combining Theorem 28 and Theorem 29, here these determinants have different sizes!

Corollary 7. ***Factorization of the Path Zeta Function.*** *Suppose* Y/X *is normal with Galois group* G*. Then the path zeta function, once the variables are specialized as in the Theorem above, factors into products of Artin L-functions:*

$$\zeta_P\left(\widetilde{Z}_{spec}, Y\right) = \prod_{\rho \in \widetilde{G}} L_P(Z, \rho, Y/X)^{d_\rho}.$$

Proof. The proof is the same as that for the analogous property of the edge Artin L-function. □

Example 38. ***Factorization of the path zeta function of a non-normal cubic cover Y_3 over X from the S_3 Cover in Figure 50.***

Recall Example 32. Here we re-consider the example in light of the factorization theorem above for path zetas. Set $\omega = e^{2\pi i/3}$ and, using our labeling of edges from Figure 50, we write

$$u_1 = z_{cc}, u_2 = z_{cd}, u_3 = z_{cD}, u_4 = z_{dc}, u_5 = z_{dd}, u_6 = z_{dC},$$
$$u_7 = z_{Cd}, u_8 = z_{CC}, u_9 = z_{CD}, u_{10} = z_{Dc}, u_{11} = z_{DC}, u_{12} = z_{DD}.$$

We find that in an analogous manner to Example 32, the product of

$$det\begin{pmatrix} u_1-1 & u_2 & 0 & u_3 \\ u_4 & u_5-1 & u_6 & 0 \\ 0 & u_7 & u_8-1 & u_9 \\ u_{10} & 0 & u_{11} & u_{12}-1 \end{pmatrix}$$

and

$$det\begin{pmatrix} -1 & \omega^2 u_1 & 0 & \omega^2 u_2 & 0 & 0 & 0 & \omega^2 u_3 \\ \omega u_1 & -1 & \omega u_2 & 0 & 0 & 0 & \omega u_3 & 0 \\ 0 & \omega u_4 & -1 & \omega u_5 & 0 & \omega u_6 & 0 & 0 \\ \omega^2 u_4 & 0 & \omega^2 u_5 & -1 & \omega^2 u_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 u_7 & -1 & \omega^2 u_8 & 0 & \omega^2 u_9 \\ 0 & 0 & \omega u_7 & 0 & \omega u_8 & -1 & \omega u_9 & 0 \\ 0 & \omega u_{10} & 0 & 0 & 0 & \omega u_{11} & -1 & \omega u_{12} \\ \omega^2 u_{10} & 0 & 0 & 0 & \omega^2 u_{11} & 0 & \omega^2 u_{12} & -1 \end{pmatrix}$$

must equal the determinant of the matrix $\widetilde{Z}_{spec} - I =$

$$\begin{pmatrix} u_5-1 & u_4u_1 & u_4u_2u_5 & u_4u_2u_4 & 0 & u_6 & u_4u_3 & u_4u_2u_6 \\ u_2 & u_1u_1-1 & u_1u_2u_5 & u_1u_2u_4 & u_3 & 0 & u_1u_3 & u_1u_2u_6 \\ u_6u_7 & u_4 & u_5u_5-1 & u_5u_4 & u_6u_9 & u_6u_8 & 0 & u_5u_6 \\ u_3u_{11}u_7 & u_3u_{10} & u_2 & u_1-1 & u_3u_{11}u_9 & u_3u_{11}u_8 & u_3u_{12} & 0 \\ 0 & u_{10}u_1 & u_{10}u_2u_5 & u_{10}u_2u_4 & u_{12}-1 & u_{11} & u_{10}u_3 & u_{10}u_2u_6 \\ u_8u_7 & 0 & u_7u_5 & u_7u_4 & u_8u_9 & u_8u_8-1 & u_9 & u_7u_6 \\ u_{12}u_{11}u_7 & u_{12}u_{10} & 0 & u_{10} & u_{12}u_{11}u_9 & u_{12}u_{11}u_8 & u_{12}u_{12}-1 & u_{11} \\ u_9u_{11}u_7 & u_9u_{10} & u_7 & 0 & u_9u_{11}u_9 & u_9u_{11}u_8 & u_9u_{12} & u_8-1 \end{pmatrix}.$$

21. Non-Isomorphic Regular Graphs Without Loops or Multiedges Having the Same Ihara Zeta Function.

Algebraic number fields K_1, K_2 can have the same Dedekind zeta functions without being isomorphic. See Perlis [99]. The smallest examples have degree 7 over $\mathbb{Q}$ and come from Artin L-functions of induced representations from subgroups of $G = GL(3, \mathbb{F}_2)$, the simple group of order 168. An analogous example of 2 graphs (each having 7 vertices) which are isospectral but not isomorphic was given by P. Buser. These graphs are found in Figure 66 below. See Buser [23] or Terras [132], Chapter 22. Buser's graphs ultimately lead to 2 planar isospectral drums which are not obtained from each other by rotation and translation, answering the question raised by M. Kac in [65]: Can you hear the shape of a drum? See Gordon et al [45] who show that there are (non-convex) planar drums that cannot be heard using the same basic construction.

Buser's graphs are not simple. That is, they have multiple edges as well as loops. We can use our theory to obtain examples of simple regular graphs with 28 vertices which are isospectral but not isomorphic. See Figure 67. The graphs in Figure 67 are constructed using the same group G and subgroups H_j as in Buser's examples. Sunada [127] shows how to apply the method from number theory to obtain isospectral compact connected Riemannian manifolds that are not isometric.

Define $G = GL(3, \mathbb{F}_2)$, which is the group of all non-singular 3×3 matrices with entries in the finite field with 2 elements. It is a simple group of order 168. Two subgroups H_j of index 7 in G are:

$$H_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \right\} \text{ and } H_2 = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}.$$

Exercise 85. *Show that H_1 and H_2 are not conjugate in G.*

The preceding exercise also follows from the fact that we will construct two non-isomorphic intermediate graphs corresponding to these subgroups H_j of G. One can show that these two groups give rise to equivalent permutation representations of G (i.e., the representation we have called $Ind_{H_i}^G 1$). The same argument as we used in Terras [132] , for Buser's graphs says that the representations $\rho_j = Ind_{H_j}^G 1$ are equivalent because the subgroups H_j are almost conjugate (i.e. $|H_1 \cap \{g\}| = |H_2 \cap \{g\}|$, for every conjugacy class $\{g\}$ in G). This implies that we have equality of the corresponding characters $\chi_{\rho_1} = \chi_{\rho_2}$. Therefore we will get graphs with the same zeta functions (using the induction property of vertex L-functions):

$$\zeta_{\widetilde{X_1}}(u) = L(u, \rho_1) = L(u, \rho_2) = \zeta_{\widetilde{X_2}}(u)$$

See Terras [132] for more information.

Given $g \in G$, all elements of $H_1 g$ have the same first row. The 7 possible non-zero first rows correspond naturally to the numbers 1-7 in binary. Thus order the 7 right cosets $H_1 g_j$ by the numbers represented by the first rows in binary. For example, the first row of g_6 is (110); and $H_1 g_4$ is the identity coset. For any g, it is easy to figure out what coset $H_1 g_j g$ is, as the first row of the product $g_j g$ depends only on the first row of g_j. So for $g \in G$, we find the permutation $\mu(g)$ corresponding to multiplying the 7 cosets $H_1 g_j$ by g on the right; i.e., $H_1 g_j g = H_1 g_{\mu(j)}$.

We need the permutations $\mu(A)$ and $\mu(B)$ for Buser's matrices:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \tag{21.1}$$

Computation shows that

$$\mu(A) = (1436)(2)(57) \text{ and } \mu(B) = (132)(4)(576). \tag{21.2}$$

Exercise 86. *Check these formulas. For example, to find $H_1 g_3 A$, we want the first row of*

$$\begin{pmatrix} 0 & 1 & 1 \\ * & * & * \\ * & * & * \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \in H_1 g_6$$

and so $\mu(A)$ takes 3 to 6.

We have to do the same permutation calculation with the matrices A and B acting on the right cosets of H_2. It might appear that the right cosets of H_2 would be more difficult to deal with. But there is a very useful automorphism of G to help. It is $\varphi(g) = {}^t g^{-1}$, where ${}^t g$ denotes the transpose of $g \in G$. This map φ is an automorphism of G such that $\varphi(H_1) = H_2$. If we apply φ to the right cosets $H_1 g_j$, we get G as a union of the 7 right cosets $H_2 \, {}^t g_j^{-1}$. To figure out how $g \in G$ permutes the cosets, it suffices to consider the action of ${}^tg^{-1}$ on the H_1g_j. Note that

$${}^tA^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } {}^tB^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Therefore the action of ${}^tA^{-1}$ and ${}^tB^{-1}$ on the right cosets H_1g_j is given by the permutations

$$\mu({}^tA^{-1}) = (14)(2376)(5) \text{ and } \mu({}^tB^{-1}) = (123)(4)(567). \tag{21.3}$$

These same permutations give the actions of A and B on the right cosets $H_2\,{}^tg_j^{-1}$.

Exercise 87. *Check these formulas for $\mu({}^tA^{-1})$ and $\mu({}^tB^{-1})$.*

Exercise 88. *Prove that the matrices A and B in formula (21.1) generate the group G.*

Buser [23] used the matrices A and B to construct 2 Schreier graphs corresponding to the 2 subgroups H_1 and H_2. Using the Galois theory we have worked out in preceding sections, this means find coverings $\widetilde{X_1}$ and $\widetilde{X_2}$ of X, where X is the graph consisting of a single vertex and a double loop. Direct each loop resulting in two directed edges a and b, say. Assign the normalized Frobenius elements $\sigma(a) = A$ and $\sigma(b) = B$. The resulting normal cover of X is the Cayley graph of G corresponding to the generators A and B. We want two intermediate graphs $\widetilde{X_1}$ and $\widetilde{X_2}$ corresponding to the subgroups H_1 and H_2 by Theorem 17; which are Schreier graphs. The permutations $\mu(A)$ and $\mu(B)$ that we just found tell us how to lift the edges a and b. This tells us how to draw the graphs $\widetilde{X_1}$ and $\widetilde{X_2}$. See Figure 66

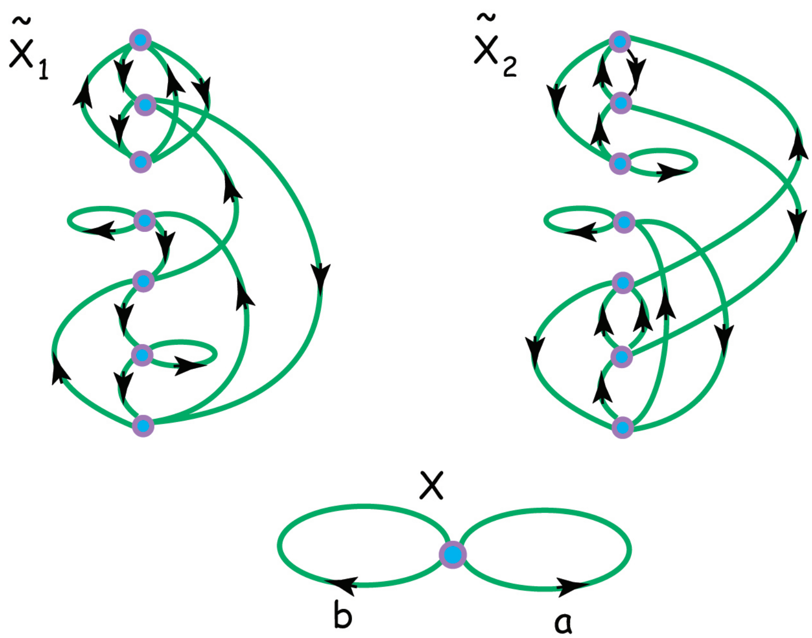


FIGURE 66. **Buser's Isospectral non-Isomorphic Schreier Graphs.** See Buser [23]. The sheets of $\widetilde{X_1}$ and $\widetilde{X_2}$ are numbered 1 to 7 from bottom to top. Lifts of a are on the right in each graph; lifts of b are on the left.

There are many ways to prove that the 2 graphs in Figure 66 are not isomorphic - even as undirected graphs. Look at triple edges; look at double edges; look at distances between loops, etc. Therefore H_1 and H_2 are not conjugate in G. Both graphs are 4-regular; they have the same zeta function and their adjacency matrices have the same spectrum.

Constructing the Graphs in Figure 67 - Simple, 3-Regular Isospectral but not Isomorphic Graphs.

Next we construct graphs like Buser's that have no loops or multiple edges. Use the same G, H_1 and H_2, but take X to be a tetrahedron K_4. Thus X is 3-regular and has a fundamental group of rank 3. Take the cut or deleted edges (directed as in Figure 67) to be a, b, c. Choose the normalized Frobenius automorphisms to be

$$\sigma(a) = A,\ \sigma(b) = \sigma(c) = B.$$

Take 7 copies of the tree of X to be the sheets of $\widetilde{X_1}$ and again for $\widetilde{X_2}$. On $\widetilde{X_1}$, we lift a, b, c using the permutations $\mu(A)$ and $\mu(B)$ from formula (21.2) above. On $\widetilde{X_2}$ we lift a, b, c using the permutations $\mu({}^tA^{-1})$ and $\mu({}^tB^{-1})$ from formula (21.3). This produces graphs $\widetilde{X_1}$ and $\widetilde{X_2}$ shown in Figure 67.

Both graphs are 3-regular; they have the same zeta function and their adjacency matrices have the same spectrum. The proof that

$$\zeta_{\widetilde{X_1}}(u) = L(u, \rho_1) = L(u, \rho_2) = \zeta_{\widetilde{X_2}}(u)$$

is the same argument that we used above and in Terras [132] for Buser's graphs.

More Discussion of the Construction.

The edge c goes from vertex 2 to vertex 3 in X and has the normalized Frobenius automorphism $\sigma(c) = B$. The lifts of c to $\widetilde{X_1}$ are determined by the permutation $\mu(B) = (132)(4)(576)$. This means that c in X lifts to an edge in $\widetilde{X_1}$ from $2'$ to $3^{(3)}$, an edge from $2^{(3)}$ to $3^{(2)}$, an edge from $2''$ to $3'$ and then (beginning a new cycle) to an edge from $2^{(4)}$ to $3^{(4)}$, etc. The edge b lifts in exactly the same manner as c. Similarly, for $\widetilde{X_1}$, the edge a in X corresponds to the permutation $(1436)(2)(57)$. This means that edge a in X lifts to an edge in $\widetilde{X_1}$ from $3'$ to $4^{(4)}$, an edge from $3^{(4)}$ to $4^{(3)}$, an edge from $3^{(3)}$ to $4^{(6)}$ etc.

To see that graphs $\widetilde{X_1}$ and $\widetilde{X_2}$ in Figure 67 are not isomorphic, proceed as follows. There are exactly 4 triangles in each graph (shown by very thick solid lines in Figure 67) and they are connected in pairs in both graphs. This distinguishes in each pair the 2 vertices not on common edges (starred vertices). In $\widetilde{X_1}$ we can go in 3 steps (via dotted lines) from a starred vertex in one pair to a starred vertex in the other pair and, in fact, in 2 different ways. This cannot be done at all in $\widetilde{X_2}$.

We said each $\widetilde{X_i}$ has 4 triangles (up to equivalence and choice of direction). Why? Since X has no loops or multiedges, any triangle on $\widetilde{X_1}$ or $\widetilde{X_2}$ projects to a triangle on X. We saw back in Chapter 1 that (up to equivalence and choice of direction) $X = K_4$ has 8 primes of length 3 and therefore 4 triangles.

Let χ_1 be the trivial character on H_1 or H_2. The induced representations $Ind_{H_i}^G 1$ for $i = 1, 2,$ have the same character $\chi_1^\#$. By Corollary 6, for any directed triangle C on X, there are $\chi_1^\#(\sigma(C))$ directed triangles above C on $\widetilde{X_1}$ and also above C on $\widetilde{X_2}$. Reversing the direction of C reverses the direction of the covering triangles. We choose the most convenient direction for each triangle.

Three of the triangles on X have 2 edges on the tree of X with normalized Frobenius elements = 1 automatically. Thus, with appropriate choice of direction in each case, $\sigma(C) = A, B, B$, for each triangle. The fourth triangle may be taken to be the path $ab^{-1}c$ whose normalized Frobenius is $\sigma(a)\sigma(b)^{-1}\sigma(c) = AB^{-1}B = A$. For $g \in G$, $\chi_1^\#(g)$ is simply the number of 1 cycles in the permutation $\mu(g)$. In particular, $\chi_1^\#(A) = \chi_1^\#(B) = 1$ (the same for both H_1 and H_2). Thus each of the 4 triangles of X has precisely 1 triangle of $\widetilde{X_j}$ above it for $j = 1$ or 2. Thus the triangles shown in Figure 67 are all triangles on $\widetilde{X_1}$ and $\widetilde{X_2}$ as we claimed.

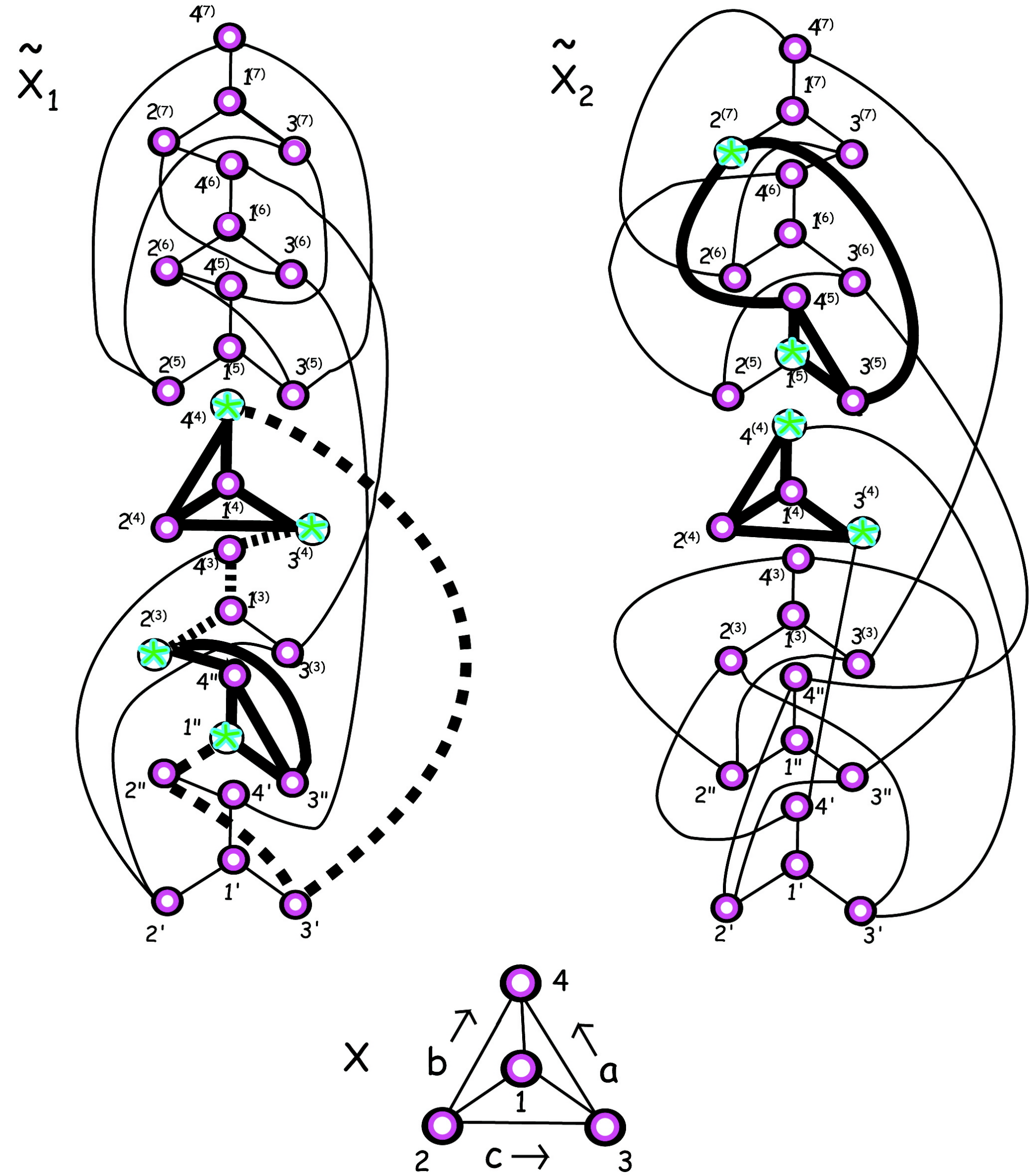


FIGURE 67. **Non-Isomorphic Graphs Without Loops or Multiedges Having the Same Ihara Zeta Functions.** The superscripts number the sheets of $\widetilde{X_1}$ and $\widetilde{X_2}$. The lifts of a are on the right side of each graph, lifts of b are on the left, and lifts of c cross from the left to the right.

Other Results on Isospectral Graphs.

See Alexander Lubotzky, Beth Samuels and Uzi Vishne [80] for infinite towers of isospectral graphs coming from finite simple groups. There are many examples (not necessarily connected) found by an undergraduate research group at Louisiana State directed by Robert Perlis. See the paper by Rachel Reeds [103] and Yaim Cooper [24] for some of the results they found. We include the Harold and Audrey graphs found by this REU in Figure 68. They were shown to me by Aubi Mellein, who had taken part in the REU.

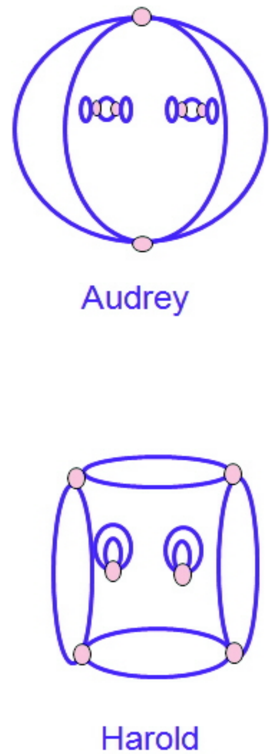


FIGURE 68. **Two isospectral non-isomorphic graphs named Harold and Audrey found by the REU at LSU and drawn for me by Aubi Mellein.** There is a different version of the picture in Rachel Reeds paper [103].

Other questions can be asked. See Gnutzmann et al [10] for the question of whether a count of nodal domains for the eigenvectors resolves the isospectrality of our examples. The question is "Can one count the shape of a drum?" Classically a nodal domain is a maximally connected region where the eigenfunction ψ of the $-\Delta$ for a bounded region $D \subset \mathbb{R}^n$ (ψ satisfying Dirichlet boundary conditions on ∂D, meaning ψ vanishes on ∂D, or Neumann boundary conditions, meaning the normal derivative of ψ vanishes on ∂D) has a constant sign. If $n = 1$, Sturm's oscillation theorem states that the n^{th} eigenfunction has exactly n nodal domains. Here the eigenfunctions are ordered by increasing eigenvalues. In higher dimensions Courant showed that the number of nodal domains of the n^{th} eigenfunction is less than or equal to n.

In [10], the conjecture is stated that nodal counts resolve isospectrality of isospectral quantum graphs. Quantum graphs are weighted graphs which have a Schödinger operator which is the 1 D Laplacian on an edge. There are boundary conditions (say Neumann) at the vertices. A **wavefunction** is a function on each edge (continuous at the vertices and satisfying boundary conditions). Let S_i be the set of edges from vertex i. The wavefunction ψ_b with wave number k can be written if vertex i and vertex j are connected by edge b of length L_b and with coordinate x_b along the edge:

$$\psi_b(x_b) = \frac{1}{\sin(kL_b)} \left\{ \phi_i \sin\left(k\left(L_b - x_b\right)\right) + \phi_j \sin\left(kx_b\right) \right\},$$

$$\sum_{b \in S_i} \frac{d}{dz_b} \psi_b(x_b) \bigg|_{x_b=0} = 0.$$

Here the wave function ψ_b takes the values ϕ_i and ϕ_j at vertex i, j, respectively. Substitute the 1st equation into the 2nd and obtain equations for the ϕ_j given by

$$\sum_{j=1}^{|E|} A_{i,j}\left(L_1, .., L_{|E|}; k\right)\phi_j = 0, \quad \text{for all } 1 \le i \le |V|.$$

The spectrum $\{k_n\}$ is a discrete, positive, unbounded sequence - the zero set of the determinant of the matrix of coefficients $A_{i,j}\left(L_1, .., L_{|E|}; k\right)$. Then one must regularize the determinant function.

There are 2 ways to define the nodal domains for quantum graphs. The discrete way says: a nodal domain is a maximal set of connected interior (meaning degree ≥ 3) vertices where the vertex eigenfunctions ϕ_i have the same sign. This definition is modified if any of the ϕ_i vanishes.

It has been shown that isospectral pairs of quantum graphs must have rationally dependent edge lengths.

22. The Chebotarev Density Theorem

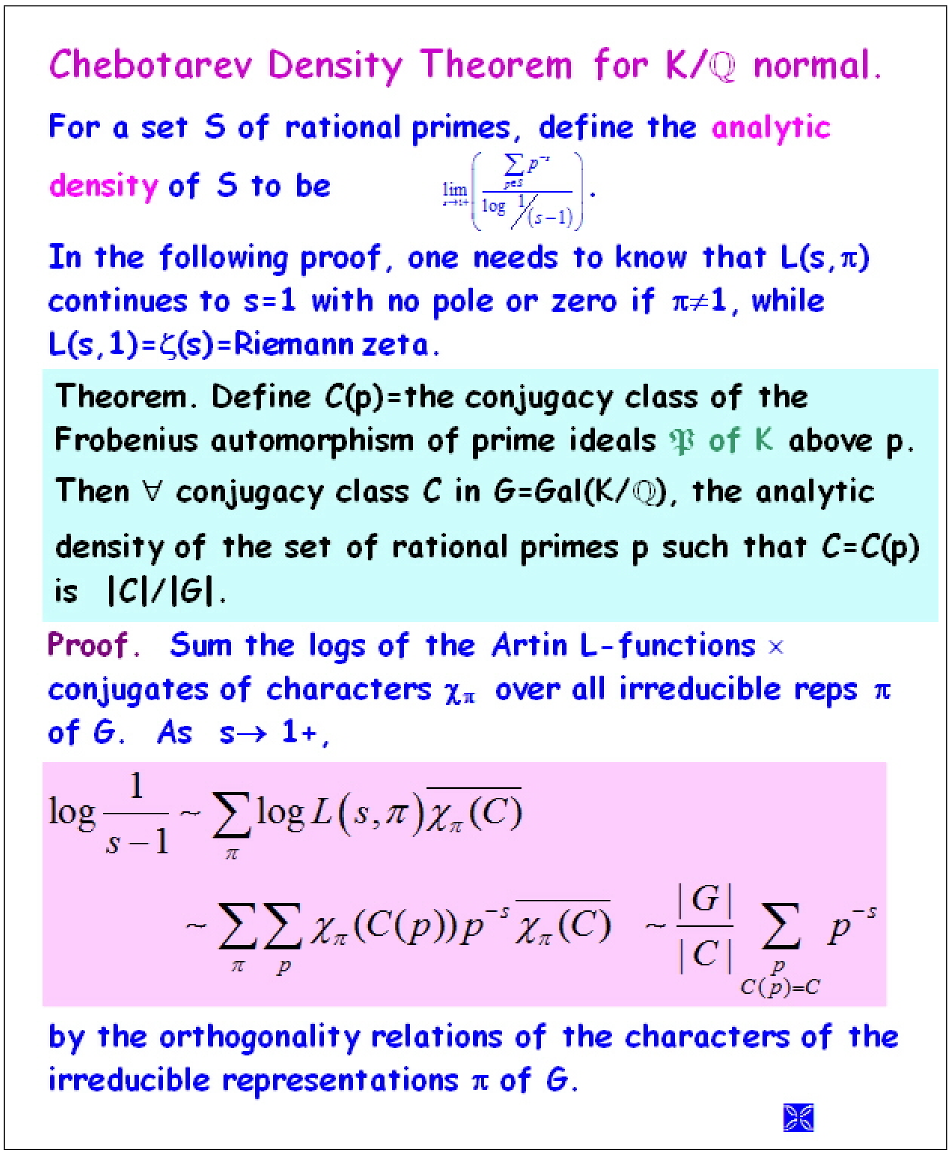


FIGURE 69. **Chebotarev Density Theorem in Number Field Case.**

The Chebotarev density theorem for algebraic number fields was proved in 1922. There are discussions in Stevenhagen and Lenstra [123] and Stark [116]. The Stark version is sketched in Figure 69. The Chebotarev density theorem generalizes the

Dirichlet theorem saying that there are an infinite number of primes in an arithmetic progression of the form $\{a + nb \,|n \in \mathbb{Z}\}$, when a, b are relatively prime. It also generalizes a theorem of Frobenius (1880) concerning a monic irreducible polynomial $f(x) \in \mathbb{Z}[x]$ of degree n, with non-0 discriminant $\Delta(f)$ and the list of degrees $e_1, ..., e_t$ of the irreducible factors mod p (called the **decomposition type** mod p) for primes p not dividing $\Delta(f)$. The decomposition type of f mod p is a partition of n; i.e., $n = e_1 + \cdots + e_t$. Let K be the extension field of $\mathbb{Q}$ obtained by adjoining all the roots of $f(x)$ and let G be the Galois group of $K/\mathbb{Q}$. The **Frobenius density theorem** says that the density (from Definition 54 below) of such p for given decomposition type $e_1, ..., e_t$ is

$$\frac{\#\{\sigma \in G \,|\text{cycle pattern of } \sigma \text{ is } e_1, ..., e_t\}}{|G|}. \tag{22.1}$$

Here by "**cycle pattern**" we mean that you view the Galois group as a subgroup of the symmetric group of permutations of the roots of the polynomial $f(x)$. Then write σ as a product of disjoint cycles (including cycles of length 1). The cycle pattern is the list of cycle lengths. It is also a partition of n.

In order to prove the graph theory analog of the Chebotarev density theorem we will need some information on the poles of zeta and L-functions of graph coverings. We begin with a Lemma for which you need to recall our notation: the largest circle of convergence R_X from Definition 3 and the edge adjacency matrix $W_1 = W_1(X)$ from Definition 8. Recall that R_X denotes the radius of convergence of the Ihara zeta function of X as well as the closest pole of zeta to 0 and R_X is the reciprocal of the Perron-Frobenius eigenvalue of W_1.

Lemma 8. *Suppose Y is an n-sheeted covering of X. The maximal absolute value of an eigenvalue of the edge adjacency matrix $W_1(X)$ is the same as that for $W_1(Y)$. This common value is $R_Y^{-1} = R_X^{-1}$.*

Proof. First note that from Stark and Terras [119], we know $\zeta_X^{-1}(u)$ divides $\zeta_Y^{-1}(u)$. See Proposition 8. It follows that $R_Y \leq R_X$.

Then a standard estimate from the theory of zeta functions of number fields works for graph theory zeta functions as well. For all real $u \geq 0$ such that the infinite product for $\zeta_X(u)$ converges, we have

$$\zeta_Y(u) \leq \zeta_X(u)^n, \tag{22.2}$$

with n equal to the number of sheets of the covering. Thus $R_X \leq R_Y$.

We take the idea of the proof of formula (22.2) from Lang [73] (p. 160). One begins with the product formula for $\zeta_Y(u)$ and the behavior of primes in coverings.

So for real u such that $R_Y > u \geq 0$, we have a product over primes $[D]$ of Y giving

$$\zeta_Y(u) = \prod_{[D]} \left(1 - u^{\nu(D)}\right)^{-1}.$$

Rewrite this as a product over primes $[C]$ of X and then primes of Y $[D_1], ..., [D_g]$, all above $[C]$:

$$\zeta_Y(u) = \prod_{[C]} \prod_{i=1}^{g} \left(1 - u^{\nu(D_i)}\right)^{-1}.$$

Recall that the primes above C are closed paths obtained by lifting C a total of f_i times. This means

$$\zeta_Y(u) = \prod_{[C]} \prod_{i=1}^{g} \left(1 - u^{f_i \nu(C)}\right)^{-1}.$$

We know (from Exercise 67 in the section on behavior of primes in coverings) that $n = \sum_{i=1}^{g} f_i$. It follows that

$$\zeta_Y(u) \leq \prod_{[C]} \left(1 - u^{\nu(C)}\right)^{-g} \leq \zeta_X(u)^n.$$

$\square$

Next let us define the analytic density.

Definition 54. *If S is a set of primes in X, define the **analytic density** $\delta(S)$ to be*

$$\delta(S) = \lim_{u \to R-} \frac{\sum\limits_{[C] \in S} u^{\nu(C)}}{\sum\limits_{[C]} u^{\nu(C)}} = \lim_{u \to R-} \frac{\sum\limits_{[C] \in S} u^{\nu(C)}}{\log \zeta_X(u)} = \lim_{u \to R-} \frac{\sum\limits_{[C] \in S} u^{\nu(C)}}{-\log(R_X - u)}.$$

Here the sums are over primes $[C]$ in X.

Question: Why does $\sum_{[C]} u^{\nu(C)}$ blowup at $u = R_X$ like $\log \zeta_X(u)$?

To answer this, recall that for $0 \leq u < R_X$

$$\zeta_X(u) = \prod_{[C]} \left(1 - u^{\nu(C)}\right)^{-1} = \det\left(I - uW_1(X)\right)^{-1}.$$

Take the logarithm and obtain:

$$\begin{aligned}\log \zeta_X(u) &= -\sum_{[C]} \log\left(1 - u^{\nu(C)}\right) = \sum_{[C]} \sum_{m \geq 1} \frac{1}{m} u^{m\nu(C)} \\ &= \sum_{[C]} u^{\nu(C)} + H(u),\end{aligned}$$

where

$$H(u) = \sum_{[C]} \sum_{m \geq 2} \frac{1}{m} u^{m\nu(C)}.$$

The amazing thing is that $H(u)$ is bounded up to $u = R_X$ and beyond. Thus $\sum_{[C]} u^{\nu(C)}$ must account for the blowup of $\log \zeta_X(u)$ at $u = R_X$.

To see that $H(u)$ is bounded up to $u = R_X$, we do a few estimates:

$$\sum_{m \geq 2} \frac{1}{m} u^{m\nu(C)} \leq \sum_{m \geq 2} u^{m\nu(C)} = \frac{u^{2\nu(C)}}{1 - u^{\nu(C)}} \leq \frac{u^{2\nu(C)}}{1 - u}.$$

Here we are using the fact that $0 \leq u \leq R_X \leq 1$. It follows that

$$H(u) \leq \frac{1}{1-u} \sum_{[C]} u^{2\nu(C)},$$

which converges up to $u^2 = R_X$ or $u = \sqrt{R_X}$, as $\log \zeta_X(u^2)$ converges up to $u = \sqrt{R_X}$.

What if $R_X = 1$? Then the graph must be a cycle. Why? Then p=q=1.

You might prefer to use a less complicated notion of density than that in Definition 54. Perhaps you would like to say that a set S of primes has **natural density** δ if

$$\frac{\{[P] \mid [P] = \text{prime}, [P] \in S, \text{ and } \nu(P) \leq n\}}{\{[P] \mid [P] = \text{prime and } \nu(P) \leq n\}} \to \delta, \quad \text{as } n \to \infty.$$

At least in the number theory case, the proof of the density theorem is harder with this version of density. For rational primes, if a set of primes has natural density, then it has analytic density and the 2 densities are equal. However, the converse is false. We leave it as a **research problem** to figure out what happens in the graph theory case. Perhaps one can replace $\nu(P) \leq n$ with $\nu(P) = n$, as well.

Theorem 32. ***Graph Theory Chebotarev Density Theorem.***

Suppose the graph X is not a cycle graph. If Y/X is normal and $\{g\}$ is a fixed conjugacy class in the Galois group $G = G(Y/X)$

$$\delta\{\,[C] \text{ prime of } X \mid \sigma(C) = \{g\}\} = \frac{|\{g\}|}{|G|}.$$

Here $\sigma(C)$ is the normalized Frobenius for C.

Proof. We imitate the proof sketched by Stark in [116] for the number field case, where one knows much less about the Artin L-functions. The idea goes back to Dirichlet. The main idea is to sum the terms $\overline{\chi_\pi(g)} \log L(u, \pi, Y/X)$ over all irreducible representations π of G. Here $\chi_\pi = Tr(\pi)$. This gives the following asymptotic formula as u approaches R_X from below:

$$\log\left(\frac{1}{R_X - u}\right) \underset{u \to R_X-}{\sim} \log \zeta_X(u) \sim \sum_{\pi \in \widehat{G}} \overline{\chi_\pi(g)} \log L(s, \pi).$$

Here we use Lemma 8 and the fact that

$$\zeta_Y(u) = \prod_{\rho \in \widehat{G}} L(u, \rho, Y/X)^{d_\rho}.$$

It follows from the Euler product definition of the L-functions that

$$\log\left(\frac{1}{R_X-u}\right)\underset{u\to R_X-}{\sim}\sum_{\pi\in\widehat{G}}\sum_{\substack{[C]\\ \text{prime of } X}}\chi_\pi(\sigma(C))u^{\nu(C)}\overline{\chi_\pi(g)}$$

$$+\sum_{\pi\in\widehat{G}}\sum_{\substack{[C]\\ \text{prime of } X}}\sum_{m\geq 2}\frac{1}{m}\chi_\pi(\sigma(C^m))u^{m\nu(C)}\overline{\chi_\pi(g)}.$$

The second term in the sum is holomorphic as $u\to R_X-$. To see this, note that the second term can be written as

$$\sum_{\substack{[C]\\ \text{prime of } X}}\sum_{m\geq 2}\frac{1}{m}u^{m\nu(C)}\sum_{\pi\in\widehat{G}}\chi_\pi(\sigma(C^m))\overline{\chi_\pi(g)}.$$

Then, using the orthogonality relations for characters of G in formula (18.3), we find that this last sum is for $0\leq u<R_X$

$$\frac{|G|}{|\{g\}|}\sum_{\substack{[C]\\ \{\sigma(C^m)\}=\{g\}}}\sum_{m\geq 2}\frac{1}{m}u^{m\nu(C)}\leq |G|\sum_{[C]}\sum_{m\geq 2}\frac{1}{m}u^{m\nu(C)}$$

$$\leq |G|\sum_{[C]}\frac{u^{2\nu(C)}}{1-u}\leq\frac{|G|}{1-u}\sum_{[C]}u^{2\nu(C)}.$$

This is holomorphic up to $u=\sqrt{R_X}$.

Thus, we have shown that $\log\left(\frac{1}{R_X-u}\right)$ is asymptotic to the following as $u\to R_X-$

$$\sum_{\pi\in\widehat{G}}\sum_{\substack{[C]\\ \text{prime of } X}}\chi_\pi(\sigma(C))u^{\nu(C)}\overline{\chi_\pi(g)}$$

$$=\sum_{\substack{[C]\\ \text{prime of } X}}\sum_{\pi\in\widehat{G}}\chi_\pi(\sigma(C))u^{\nu(C)}\overline{\chi_\pi(g)}$$

$$=\frac{|G|}{|\{g\}|}\sum_{\substack{[C]\\ \{\sigma(C)\}=\{g\}}}u^{\nu(C)}.$$

For the last equality, use the orthogonality relations again. The theorem follows. □

The simplest example of the Chebotarev theorem is the cube over the tetrahedron, where primes with $f=1$ have density 1/2, as do the primes with $f=2$.

A more complicated example of our result can be found in Figure 54. The example concerns the splitting of primes in a non-normal cubic covering Y_3/X, where $X=K_4-edge$. Thus one must consider what happens in the normal cover for which Y_3/X is intermediate.

Exercise 89. *Fill in the details concerning the densities of the primes in various classes in Figure 54 by imitating Stark's arguments for the corresponding example in* [116], *pages 358-360 and 364. The Frobenius version of Chebotarev's theorem in formula (22.1) would simplify the computation.*

23. Siegel Poles

23.1. **Summary of Siegel Poles Results.** In number theory, there is a known zero free region of a Dedekind zeta function which can be explicitly given except for the possibility of a single first order real zero within this region. This possible exceptional zero has come to be known as a **"Siegel zero"** and is closely connected with the Brauer-Siegel Theorem on the growth of the class number times the regulator with the discriminant. See Lang[73] for more information on the implications of the non-existence of Siegel zeros. There is no known example of a Siegel zero for Dedekind zeta functions. In number fields, a Siegel zero (should it exist) "deserves" to arise already in a quadratic extension of the base field. This has now been proved in many cases (see Stark [117]).

$\zeta_X(u)^{-1}$ is a polynomial with a finite number of zeros. Thus there is an $\epsilon>0$ such that any pole of $\zeta_X(u)$ in the region $R_X\leq|u|<R_X+\epsilon$ must lie on the circle $|u|=R_X$. This gives us the graph theoretic analog of a **"pole free region"**, $|u|<R_X+\epsilon$; the only exceptions lie on the circle $|u|=R_X$. We will show that $\zeta_X(u)$ is a function of u^δ with

$\delta = \delta_X$ a positive integer from Definition 56 below. This implies there is a $\delta-$fold symmetry in the poles of $\zeta_X(u)$; i.e., $u = \varepsilon_\delta R$ is also a pole of $\zeta_X(u)$, for all δ^{th} roots of unity ε_δ. Any further poles of $\zeta_X(u)$ on $|u| = R$ will be called **"Siegel poles"** of $\zeta_X(u)$. Thus if $\delta = 1$, any pole $u \neq R$ of $\zeta_X(u)$ with $|u| = R$ will be called a Siegel pole.

All graphs considered in this section are assumed to satisfy our usual hypotheses stated before Definition 1.

Definition 55. *A vertex of X having degree $\geq$ 3 is called a* ***node*** *of X.*

A graph X satisfying our usual hypotheses will have rank $\geq$ 2 and thus always has at least one node.
Now the reader should recall Definition 7 of Δ. Next we consider a closely related quantity.

Definition 56. *Define δ to be*

$$\delta_X = g.c.d.\left\{ \nu(P) \,\middle|\, \begin{array}{c} P = \text{backtrackless path in } X \text{ such that the} \\ \text{initial and terminal vertices are both nodes} \end{array} \right\}.$$

When a path P in the definition of δ_X is closed, the path will be backtrackless but may have a tail. Later we give an equivalent definition of δ_X not involving paths with tails. The relation between δ_X and our earlier Δ_X from Definition 7 is given by the following result. As a result we will see that any k-regular graph X with $k \geq 3$ has $\delta_X = 1$.

Theorem 33. *Suppose X satisfies our usual hypotheses. Then either $\Delta_X = \delta_X$ or $\Delta_X = 2\delta_X$.*

It is easy to see that if Y is a covering graph of X(of rank $\geq$ 2) we have $\delta_Y = \delta_X$ since they are the $g.c.d.$s of the same set of numbers. Therefore δ_X is a covering invariant. Because of this, Theorem 33 gives a useful Corollary.

Corollary 8. *If Y is a covering of a graph X of rank ≥ 2, then*

$$\Delta_Y = \Delta_X \quad \text{or} \quad 2\Delta_X.$$

For a cycle graph X the ratio Δ_Y/Δ_X can be arbitrarily large. The general case of the theorem about Siegel poles can be reduced to the more easily stated case where $\delta_X = 1$, where any pole of $\zeta_X(u)$ on $|u| = R$ other than $u = R$, is a **Siegel pole**.

Theorem 34. ***Siegel Poles when $\delta_X = 1$.*** *Suppose X satisfies our usual hypotheses and $\delta_X = 1$. Let Y be a covering graph of X and suppose $\zeta_Y(u)$ has a Siegel pole μ. Then we have the following facts.*

1. *The pole μ is a first order pole of $\zeta_Y(u)$ and $\mu = -R$ is real.*
2. *There is a unique intermediate graph X_2 to Y/X with the property that for every intermediate graph $\widetilde{X}$ to Y/X (including X_2), μ is a Siegel pole of $\zeta_{\widetilde{X}}(u)$ if and only if $\widetilde{X}$ is intermediate to Y/X_2.*
3. *X_2 is either X or a quadratic (i.e., 2-sheeted) cover of X.*

23.2. Proof of Theorems 33 and 34.

Before proving Theorem 33, we need a Lemma.

Lemma 9. *The invariant δ of Definition 56 equals δ' defined by*

$$\delta' = g.c.d.\left\{ \nu(P) \,\middle|\, \begin{array}{c} P \text{ is backtrackless and the initial and terminal vertices of } P \text{ are} \\ \text{(possibly equal) nodes and no intermediate vertex is a node} \end{array} \right\}.$$

Proof. Clearly $\delta | \delta'$.
To show $\delta' | \delta$, note that anything in the length set for δ is a sum of elements of the length set for δ'. □

Exercise 90. *Use Lemma 9 to show that any k-regular graph X with $k \geq 3$ has $\delta_X = 1$.*

Exercise 91. *Compute Δ_X and δ_X for K_4, the cube, and $K_4 - e$.*

Proof of Theorem 33.

Proof. Theorem 33 says that if Δ_X is odd then $\Delta_X = \delta_X$ and otherwise either $\Delta_X = \delta_X$ or $\Delta_X = 2\delta_X$. First note that $\delta | \Delta$ since every cycle in a graph X of rank ≥ 2 has a node (otherwise X would not be connected). To finish the proof, we show that $\Delta | 2\delta$.

If X has a loop, then the vertex of the loop is a node (if the rank is ≥ 2) and thus $\Delta = \delta = 1$. **So assume X loopless for the rest of the proof .**

By Lemma 9 we may consider only backtrackless paths A between arbitrary nodes α_1 and α_2 without intermediate nodes. There are two cases.

Case 1. $\alpha_1 \neq \alpha_2$.

Let e_1' be an edge out of α_1 not equal to the initial edge i of A (or i^{-1} since there are no loops). Let e_2' be an edge into α_2 not equal to the terminal edge t of A (or t^{-1}). Let $B = P(e_1', e_2')$ from Lemma 2.

Suppose e_1'' is another edge out of α_1 such that $e_1'' \neq i$, $e_1'' \neq e_1'$ (or their inverses). Likewise suppose e_2'' is another edge into α_2 such that $e_2'' \neq t$, $e_2'' \neq e_2'$ (or their inverses). Let $C = P(e_1'', e_2'')$ from Lemma 2. See Figure 70.

Then AB^{-1}, AC^{-1}, BC^{-1} are backtrackless tailless paths from α_1 to α_1.

We have

$$\Delta | \nu(AB^{-1}) = \nu(A) + \nu(B),$$
$$\Delta | \nu(AC^{-1}) = \nu(A) + \nu(C),$$
$$\Delta | \nu(BC^{-1}) = \nu(B) + \nu(C).$$

It follows that Δ divides $2\nu(A)$ since

$$2\nu(A) = (\nu(A) + \nu(B)) + (\nu(A) + \nu(C)) - (\nu(B) + \nu(C)).$$

Case 2. $\alpha_1 = \alpha_2$.

Then A is a backtrackless path from α_1 to α_1 without intermediate nodes. This implies that A has no tail, since then the other end of the tail would have to be an intermediate node. Therefore Δ divides $\nu(A)$ and hence Δ divides $2\nu(A)$.

Thus, in all cases, Δ divides $2\nu(A)$ and hence $\Delta | 2\delta$. □

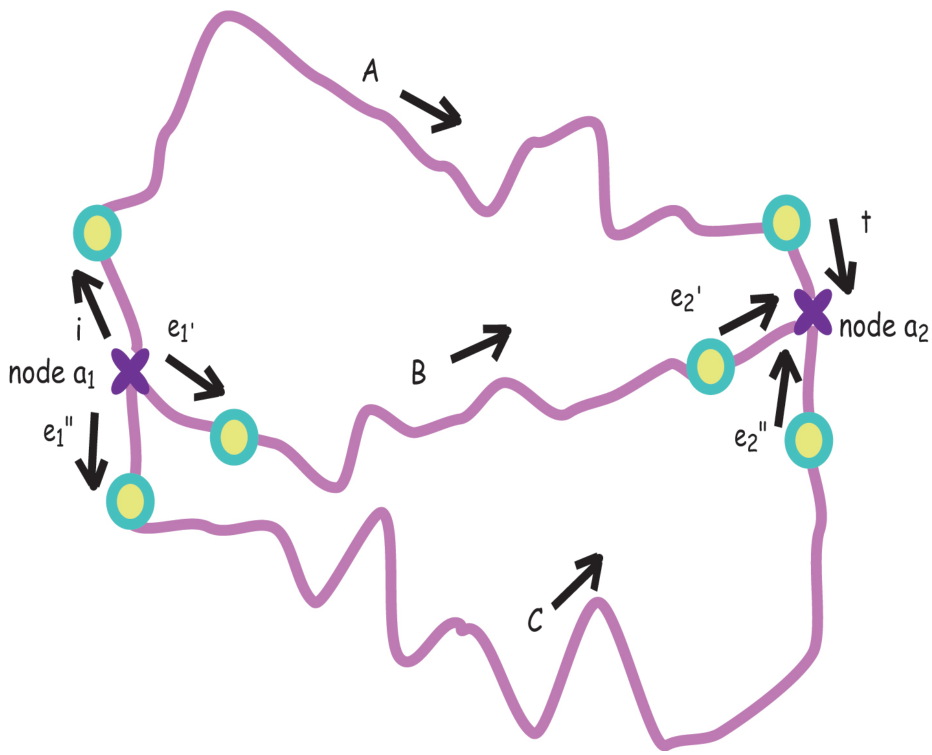


FIGURE 70. **The paths in the proof of Theorem 33 for the case when the nodes are different.**

Lemma 10. *$\zeta_Y(u) = f(u^d)$ if and only if d divides Δ_Y.*

Proof. By definition, $\zeta_Y(u)$ is a function of u^{Δ_Y} and therefore of u^d for all divisors d of Δ_Y.

Conversely suppose $\zeta_Y(u)$ is a function of u^d. In the power series $\zeta_Y(u) = \sum_{n\geq 0} a_n u^n$, d divides n for all n with $a_n > 0$. But if P is a prime cycle of Y with length $\nu(P) = n$, then $a_n \geq 1$ and hence for all prime cycles P, d divides $\nu(P)$. Therefore, by Definition 7 of Δ, we see that d divides Δ_Y. □

Proof of Theorem 34.

Proof. First reduce the Theorem to the case that Y/X is normal with Galois group G. To see that this is possible, let $\widetilde{Y}$ be a normal cover of X containing Y. Then $\zeta_{\widetilde{Y}}(u)^{-1}$ is divisible by $\zeta_Y(u)^{-1}$ and both graphs have the same R (by Lemma 8), as well as the same δ. Therefore a Siegel pole of $\zeta_Y(u)$ is a Siegel pole of $\zeta_{\widetilde{Y}}(u)$. Once the Theorem is proved for normal covers of X, the graph X_2 which we obtain will be contained in Y as well as in every graph intermediate to Y/X whose zeta function has the Siegel pole and we will be done. From now on, assume Y/X is normal.

Recall Corollary 2 , $\zeta_X(u)^{-1} = \det(I - W_X u)$ and Definition 8 of the edge adjacency matrix $W_1 = W_X$. Poles of $\zeta_X(u)$ are reciprocal eigenvalues of W_X. For graphs of rank ≥ 2 the edge adjacency matrix W_X satisfies the hypotheses of the Perron-Frobenius theorem, namely that W_X is irreducible.

Lemma 8 and Theorem 14 of Perron and Frobenius imply that if there are d poles of $\zeta_Y(u)$ on $|u| = R_Y = R_X = \frac{1}{\omega}$, then these poles are equally spaced first order poles on the circle and further $\zeta_Y(u)$ is a function of u^d. Lemma 10 implies that Δ_Y has to be divisible by d. But $\delta = \delta_X = \delta_Y = 1$ implies $\Delta_Y = 1$ or 2. Therefore $d = 1$ or 2. If there is a Siegel pole, $d > 1$. Thus if there is a Siegel pole, $d = 2$, $\Delta_Y = 2$ and the equal spacing result says the Siegel pole is $-R_X$ and it is a pole of order one.

Corollary 5 says

$$\zeta_Y(u) = \prod_{\pi \in \widehat{G}} L(u, \pi)^{d_\pi}. \tag{23.1}$$

Therefore $L(u, \pi)$ has a pole at $-R_X$ for some π and $d_\pi = 1$. Moreover π must be real or $L(u, \overline{\pi})$ would also have a pole at $-R_X$.

Thus either π is trivial or it is first degree and $\pi^2 = 1$, $\pi \neq 1$. Then we say π **is quadratic.**

Case 1. π is trivial.

Then $\Delta_X = 2 = \Delta_Y$. Every intermediate graph then has poles at $-R_X$ as well.

Case 2. $\pi = \pi_2$ is quadratic.

No other $L(u, \pi)$ has $-R_X$ as pole since it is a first order pole of $\zeta_Y(u)$. Let

$$H_2 = \{x \in G \,|\pi_2(x) = 1\} = \ker \pi_2.$$

Then $|G/H_2| = 2$ which implies there is a graph X_2 corresponding to H_2 by Galois theory. Moreover X_2 is a quadratic cover of X.

Consider the diagram of covering graphs with Galois groups indicated next to the covering lines in Figure 71. Then

$$\zeta_{\widetilde{X}}(u) = L(u, Ind_H^G 1) = \prod_{\kappa \in \widehat{G}} L(u, \kappa)^{m_\kappa}.$$

$L(u, \kappa)$ appears m_κ times in the factorization and Frobenius reciprocity (see Theorem 24) says

$$m_\kappa = \left\langle \chi_{Ind_H^G 1}, \kappa \right\rangle = \langle 1, \kappa|_H \rangle \leq \deg \kappa.$$

Let $\kappa = \pi_2$, which has $\deg \kappa = 1$. This implies $\zeta_{\widetilde{X}}(u)$ has $-R_X$ as a (simple) pole if and only if $\pi_2|_H =$ identity. Note that $-R_X$ is not a pole of any $L(u, \pi)$, for $\pi \neq \kappa$. We have $\pi_2|_H =$identity if and only if $H \subset H_2 = \ker \pi_2$, which is equivalent to saying $\widetilde{X}$ covers X_2. Here we use part 5) of the fundamental theorem of Galois theory.

Finally X_2 is unique as each version of X_2 would cover the other. □

Note that if $\widetilde{X}$ is intermediate to Y/X in Theorem 34, then $\Delta(\widetilde{X}) = 1$ or 2 and the Perron-Frobenius Theorem says $\zeta_{\widetilde{X}}(u)$ is a function of u^d, where d is the number of poles of $\zeta_{\widetilde{X}}(u)$ on the circle $|u| = R$. Thus the $\widetilde{X}$ with $\Delta(\widetilde{X}) = 2$ are exactly the $\widetilde{X}$ with $\zeta_{\widetilde{X}}(u)$ having $-R$ as a Siegel pole and these are the $\widetilde{X}$ which cover X_2. Since $\Delta(\widetilde{X}) = 2$ is the condition for $\widetilde{X}$ to be bipartite, this says that $\widetilde{X}$ is bipartite. $\widetilde{X}$ is not quadratic unless $\widetilde{X} = X_2$. All remaining intermediate graphs $\widetilde{X}$ to Y/X have $\Delta_{\widetilde{X}} = 1$.

Every graph X of rank ≥ 2 has a covering Y with zeta function having a Siegel pole as we will see. This is probably not the case for algebraic number fields.

Corollary 9. *Under the hypotheses of Theorem 34 with X_2 the unique graph defined in that Theorem, the set of intermediate bipartite covers to Y/X is precisely the set of graphs intermediate to Y/X_2.*

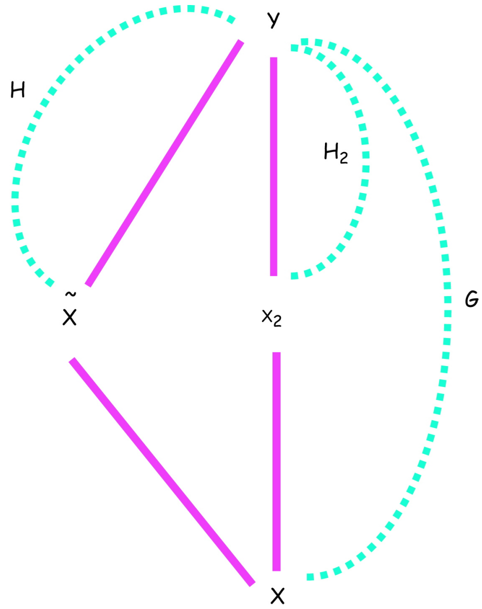


FIGURE 71. **The covering appearing in Theorem 34.** Galois groups are indicated with dashed lines.

23.3. **General Case, Inflation and Deflation.** For the next result, we need some definitions.

Definition 57. *The* ***inflation*** $I^{\delta}(X)$ *is defined by putting* $\delta - 1$ *vertices on every edge of* X.

Definition 58. *The* ***deflation*** $D_{\delta}(X)$ *is obtained from* X *by collapsing* δ *consecutive edges between consecutive nodes to one edge.*

See Figure 72 for examples.

Theorem 35. ***Siegel Poles in General.*** *Suppose* X *and a cover* Y *of* X *satisfy our usual hypotheses and* $\delta = \delta_X = \Delta_X$. *Suppose that* $\Delta_Y = 2\Delta_X = 2\delta$. *Then we have the following facts.*

(1) *There is a unique intermediate quadratic cover* X_2 *to* Y/X *such that* $\Delta_{X_2} = 2\delta$.
(2) *Let* $\widetilde{X}$ *be any graph intermediate to* Y/X. *Then* $\Delta_{\widetilde{X}} = 2\delta$ *if and only if* $\widetilde{X}$ *is intermediate to* Y/X_2.

Proof. When $\delta > 1$, this is proved by deflation. The deflated graph $D_{\delta}(X) = X'$ contains all the information on X and its covers. This graph X' has $\delta_{X'} = 1$ and $\zeta_X(u) = \zeta_{X'}(u^{\delta})$. Every single Y/X has a corresponding deflated covering Y' of X' such that

$$\zeta_Y(u) = \zeta_{Y'}(u^{\delta_X}).$$

There is also a relation between all the Artin L-functions

$$L_{Y/X}(u, \pi) = L_{Y'/X'}(u^{\delta_X}, \pi).$$

where π is a representation of $Gal(Y/X) = Gal(Y'/X')$. Theorem 35 now follows from Theorem 34 which contains the case $\delta = 1$ of Theorem 35. □

If $\delta = 1$ in Theorem 35, the graphs $\widetilde{X}$ with $\Delta_{\widetilde{X}} = 2\delta$ are the bipartite covering graphs intermediate to Y/X and, in particular, X_2 is bipartite. Even when $\Delta_X = \delta_X = \delta$ is odd, the $\widetilde{X}$ with $\Delta_{\widetilde{X}} = 2\delta$ are precisely the bipartite covering graphs intermediate to Y/X. But, if $\Delta_X = \delta_X = \delta$ is even, then every graph intermediate to Y/X, including X itself, is bipartite, and thus being bipartite does not determine which quadratic cover of X is X_2. Note also that when the rank of X is ≥ 2, we have proved the following purely graph theoretic equivalent theorem.

Theorem 36. ***The Story of Bipartite Covers.***
Suppose X *satisfies our usual hypotheses except that the rank of its fundamental group may be 1, and that* Y *is a bipartite covering graph of* X. *Then we have the following facts.*

(1) *When* X *is bipartite, every intermediate covering* $\widetilde{X}$ *to* Y/X *is bipartite.*

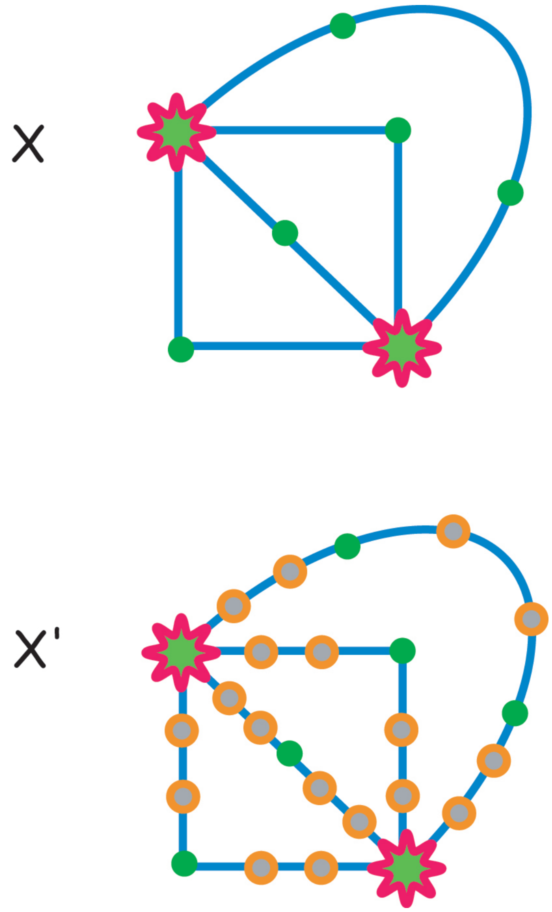


FIGURE 72. X' is the inflation of X increasing the length of paths by a factor of 3.

(2) *When X is not bipartite, there is a unique quadratic covering graph X_2 intermediate to Y/X such that any intermediate graph $\widetilde{X}$ to Y/X is bipartite if and only if $\widetilde{X}$ is intermediate to Y/X_2.*

Example 39. *Let $X = K_4$ and Y=the cube.*
1) $\Delta_X = 1 = \delta_{X.}$ $\delta_Y = 1,\ \Delta_Y = 2.$
2) ζ_X does not have a Siegel zero. ζ_Y does have a Siegel zero.
3) $X_2 = Y = Y_2$.

Exercise 92. *Perform the same calculations as in the last example if $Y = Y_{6,}$ the S_3 cover of $X = K_4 - e$ in Figure 50.*

Part 5. **Last Look at the Garden**

24. An Application to Error-Correcting Codes

Unlike cryptographic codes, the error-correcting ones are used to make a message understandable, even when it is corrupted by some problem in transmission or recording. References for the subject include Vera Pless and Terras [132], Chapter 11. The application of edge zetas that we are considering comes from the papers of Ralf Koetter, Winnie Li, Pascal Vontobel and Judy Walker [70], [71].

Definition 59. *A binary* $[n,k]$ *linear* **code** C *is a* k*-dimensional subspace of the vector space* $\mathbb{F}_2^n$*, where* $\mathbb{F}_2$ *denotes the field with two elements.*

One way to specify such a code C uses the **parity check** or Hamming matrix H, writing

$$C = \left\{ x \in \mathbb{F}_2^n \;\middle|\; Hx = 0 \right\}.$$

If H is an $s \times n$ matrix, it must have rank $n-k$, in order for C to be an $[n,k]$ code.

Let's consider an example from [70], [71].

Example 40. *A* ***parity check matrix*** *for a* $[7,2]$ *code is the matrix* H *below.*

$$H = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Next we define the **Tanner graph** $T(H)$ associated to the parity check matrix H in Example 40. Assume that H is an $s \times n$ matrix. The graph $T(H)$ is a bipartite graph with bit vertices $X_1, ..., X_n$ and check vertices $p_1, ..., p_s$. Connect p_i to X_j with an edge iff the the parity check matrix has ij entry $h_{ij} = 1$. The Tanner graph for the example above is in Figure 73 below along with a quadratic covering.

We call the code corresponding to the parity check matrix H in Example 40 a **cycle code** since each bit vertex X_j in $T(H)$ in Figure 73 has degree 2. The **normal graph** $N(H)$ associated to H is obtained by collapsing or deflating the edges with the bit nodes X_j in $T(H)$ to just an edge. The next figure shows the normal graph $N(H)$ for the parity check matrix in Example 40.

Of course, the usefulness of error-correcting codes has much to do with the efficiency of the decoding algorithms. Iterative decoding operates locally and thus has trouble with codewords from coverings. This leads to the definition of a pseudo-codeword below. One of the main results of Koetter et al [70], [71] says that edge zetas give a list of pseudo-codewords.

Write the bit vertices of the quadratic cover in the order $X_1', X_1'', ..., X_n', X_n''$. List the check vertices in a similar manner. The quadratic covering in Figure 73 corresponds to a code with parity check matrix given by the matrix with block form

$$\widetilde{H} = \begin{pmatrix} I & I & 0 & 0 & 0 & 0 & 0 \\ 0 & J & I & I & 0 & 0 & 0 \\ I & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & I & 0 & J \\ 0 & 0 & 0 & 0 & I & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I & I \end{pmatrix}.$$

Here $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Every element c of the original code C lifts to an element $\widetilde{c}$ of the covering code $\widetilde{C}$. For example, the code word (1110000) in our example code lifts to (11 11 11 00 00 00 00). The codeword $\widetilde{c} = $ (10 10 10 11 10 10 10) in $\widetilde{C}$ is not a lift of a codeword in C.

Exercise 93. *Using the preceding definitions of* H *and* $\widetilde{H}$*, check that* $H^t(1110000) = 0$. *and* $\widetilde{H}^t(10\ 10\ 10\ 11\ 10\ 10\ 10) = 0$.

Definition 60. *Suppose our original* $[n,k]$ *code is* C *with corresponding Tanner graph* $T(H)$*. The* ***unscaled pseudo-codeword*** *corresponding to codeword* $\widetilde{c}$ *coming from an* M*-sheeted covering of the Tanner graph* $T(H)$ *is defined to be* $\omega(\widetilde{c}) \in \mathbb{Z}^n$*, with jth entry obtained by summing the* M *entries of* $\widetilde{c}$ *above* c_j *in* $\widetilde{c}$*. Note that we do not sum in the finite field but instead sum the 0's and 1's as ordinary integers. The* ***normalized pseudo-codeword*** *is obtained by dividing* $\omega(\widetilde{c})$ *by* M*. It lies in* $\mathbb{Q}^n$*.*

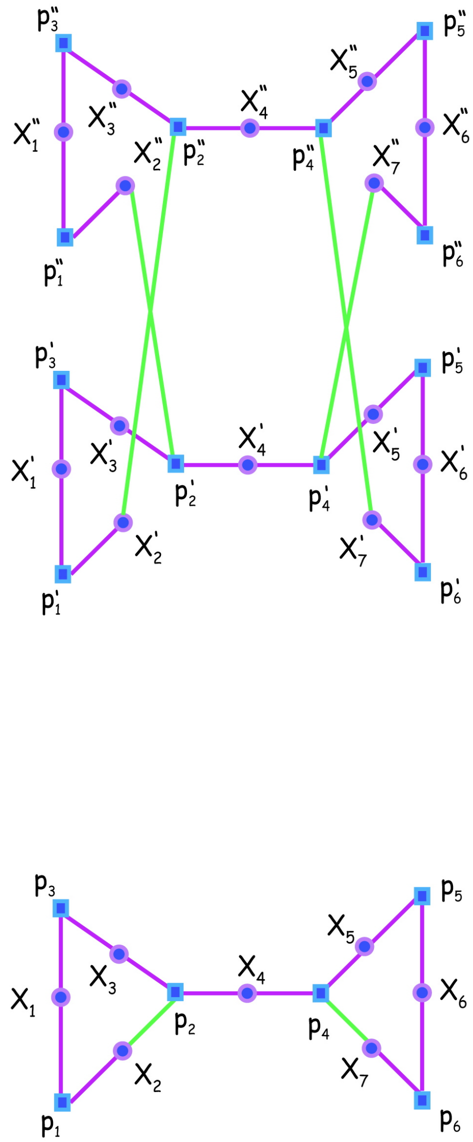


FIGURE 73. **The Tanner graph** of the code corresponding to the parity check matrix H in Example 40 is shown **along with a quadratic cover.** The sheets of the cover are pink. The edges left out of a spanning tree for $T(H)$ are green as are their lifts to the cover.

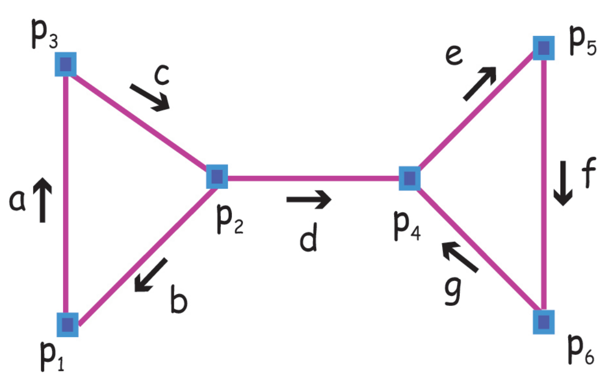


FIGURE 74. **The normal graph** $N(H)$ corresponding to the Tanner graph $T(H)$ on the lower part of the preceding Figure.

Thus, for example, consider the codeword $\widetilde{c} = (10\ 10\ 10\ 11\ 10\ 10\ 10)$ coming from the quadratic cover in Figure 73. Then the corresponding pseudo-codeword is $\omega(\widetilde{c}) = (1\ 1\ 1\ 2\ 1\ 1\ 1)$. The normalized version is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. The normalized pserudo-codeword coming from the lift of a codeword in C will consist entirely of 0's and 1's and conversely.

The key problem with iterative decoding is that the algorithm cannot distinguish between codewords coming from finite covers of $T(H)$. Thus one becomes interested in pseudo-codewords. See the website

http : //www.hpl.hp.com/personal/Pascal_Vontobel/pseudocodewords/papers/papers_pseudocodewords.html

for more information on pseudocodewords.

The following Theorem is proved in [70] for cycle codes and in [71] more generally. We restrict to cycle codes here.

Theorem 37. *(**Ralf Koetter, Winnie Li, Pascal Vontobel and Judy Walker**). Let C be an $[n, k]$ cycle code with parity check matrix H and normal graph $N = N(H)$. Let $\zeta_N(u_1, ..., u_n)$ denote the edge zeta of N with edge matrix W specialized to have e, f entry $W_{ef} = u_i$, if e is a directed edge corresponding to the ith edge of N (i.e, the edge corresponding to the ith bit vertex X_i). Then the monomial $u_1^{a_1} \cdots u_n^{a_n}$ appears with a nonzero coefficient in $\zeta_N(u_1, ..., u_n)$ iff the corresponding exponent vector $(a_1, ..., a_n)$ is an unscaled pseudo-codeword for C.*

Let us consider our example again. Then the zeta function of the normal graph $N(H)$ is found by computing

$$\zeta_N(a, b, c, d, e, f, g)^{-1} =$$

$$\det\begin{pmatrix}
-1 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c & -1 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\
0 & 0 & 0 & 0 & -1 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g & 0 & -1 & 0 & 0 & 0 & g & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & b & 0 & 0 & 0 & 0 & -1 & b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & -1 & 0 & e \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & -1
\end{pmatrix}.$$

$$= -4a^2b^2c^2d^2e^2f^2g^2 + 4a^2b^2c^2d^2efg + a^2b^2c^2e^2f^2g^2 - 2a^2b^2c^2efg$$
$$+ 4abcd^2e^2f^2g^2 - 4abcd^2efg - 2abce^2f^2g^2 + 4abcefg + e^2f^2g^2 - 2efg + a^2b^2c^2 - 2abc + 1$$

The first few terms in the power series for $\zeta_N(a, b, c, d, e, f, g)$ are

$$1 + 2abc + 2efg + 3a^2b^2c^2 + 3e^2f^2g^2 + 4abcefg + 4abcd^2efg + 6abce^2f^2g^2$$
$$+ 6a^2b^2c^2efg + 12abcd^2e^2f^2g^2 + 12a^2b^2c^2d^2efg + 9a^2b^2c^2e^2f^2g^2 + 36a^2b^2c^2d^2e^2f^2g^2.$$

The degrees of the terms are 0,3,3,6,6,6,8,9,9,11,11,12,14. According to the preceding Theorem, these terms lead to the pseudo-codewords:

(0, 0, 0, 0, 0, 0, 0), (1, 1, 1, 0, 0, 0, 0), (0, 0, 0, 0, 1, 1, 1), (2, 2, 2, 0, 0, 0, 0), (0, 0, 0, 0, 2, 2, 2),
(1, 1, 1, 0, 1, 1, 1), (1, 1, 1, 2, 1, 1, 1), (1, 1, 1, 0, 2, 2, 2), (2, 2, 2, 0, 1, 1, 1), (1, 1, 1, 2, 2, 2, 2),
(2, 2, 2, 2, 1, 1, 1), (2, 2, 2, 0, 2, 2, 2), (2, 2, 2, 2, 2, 2, 2).

25. Explicit Formulas

We can also produce analogs of the explicit formulas of analytic number theory. That is, we seek an analog of Weil's explicit formula for the Riemann zeta function. In Weil's original work he used the result to formulate an equivalent statement to the Riemann hypothesis. See Weil [141]. One can also view the explicit formulas as analogs of Selberg's trace formula, the result used to study Selberg's zeta function defined in formula (3.1).

Our analog of the Von Mangoldt function from elementary number theory is N_m. Using formula (4.4), we have

$$u\frac{d}{du}\log\zeta(u,X) = -u\frac{d}{du}\sum_{\lambda\in Spec(W_1)}\log(1-\lambda u) = \sum_{\lambda\in Spec(W_1)}\frac{\lambda u}{1-\lambda u} = -\sum_{\rho\text{ pole of }\zeta}\frac{u}{u-\rho}. \tag{25.1}$$

Then it is not hard to prove the following result following the method of Murty [91], p. 109.

Proposition 15. ***An Explicit Formula.*** *Let $0 < a < R$, where R is the radius of convergence of $\zeta(u, X)$. Assume $h(u)$ is meromorphic in the plane and holomorphic outside the circle of center 0 and radius $a - \varepsilon$, for small $\varepsilon > 0$. Assume also that $h(u) = O(|u|^p)$ as $|u| \to \infty$ for some $p < -1$. Finally assume that its transform $\widehat{h}_a(n)$ decays rapidly enough for the right hand side of the formula to converge absolutely. Then if N_m is as in Definition 9, we have*

$$\sum_{\rho}\rho h(\rho) = \sum_{n\geq 1}N_n\widehat{h}_a(n),$$

where the sum on the left is over the poles of $\zeta(u, X)$ and

$$\widehat{h}_a(n) = \frac{1}{2\pi i}\oint_{|u|=a} u^n h(u)du.$$

Proof. We follow the method of Murty [91], p. 109. Look at

$$\frac{1}{2\pi i}\oint_{|u|=a}\left\{u\frac{d}{du}(\log\zeta(u,X))\right\}h(u)du.$$

Use Cauchy's integral formula to move the contour over to the circle $|u| = b > 1$. Then let $b \to \infty$. Also use formulas (25.1) and (10.1). Note that $N_n \sim \frac{\Delta_X}{R_X^m}$, as $m \to \infty$. $\square$

Such explicit formulas are basic to work on the pair correlation of complex zeros of zeta (see Montgomery [90]). They can also be viewed as an analog of Selberg's trace formula. See [57], [136] for discussion of Selberg's trace formula for a $q + 1$ regular graph. In these papers various kernels (e.g., Green's, characteristic functions of intervals, heat) were plugged in to the trace formula deducing various things such as McKay's theorem on the distribution of eigenvalues of the adjacency matrix and the Ihara determinant formula for the Ihara zeta. It would be an interesting research project to do the same sort of thing for irregular graphs.

Exercise 94. *Plug the function $h(u) = u^{-m-1}$, $m = 1, 2, 3, ...$ into the explicit formula. You should get a result that is well known to us.*

26. Again Chaos

In our earlier section on chaos we considered the spacing of zeta poles of regular graphs as well as the distribution of the eigenvalues of the adjacency matrices of regular graphs. See Figure 24. In particular, we noted in Table 1 of that section that there is a conjectural dichotomy between the behavior of zetas of random regular graphs versus zetas of Cayley graphs of abelian groups for example.

Our plan for this section is to investigate the spacings of the poles of the Ihara zeta function of a random irregular graph and compare the result with spacings for covering graphs both random and with abelian Galois group. By formula (4.4), this is essentially the same as investigating the spacings of the eigenvalues of the edge adjacency matrix W_1 from Definition 8. Here, although W_1 is not symmetric, the nearest neighbor spacing can be studied. If the eigenvalues of the matrix are λ_i, $i = 1, ..., 2m$, we want to look at $\upsilon_i = \min\{\left|\lambda_i - \lambda_j\right| \mid j \neq i\}$. The question becomes: what function best approximates the histogram of the υ_i, assuming they are normalized to have mean 1? [4]

References for the study of spacings of eigenvalues of non-symmetric matrices include Ginibre [43], LeBoeuf [74], and Mehta [85]. The **Wigner surmise for non-symmetric matrices** is

$$4\Gamma\left(\frac{5}{4}\right)^4 x^3\exp\left(-\Gamma\left(\frac{5}{4}\right)^4 x^4\right). \tag{26.1}$$

Before looking at spacings coming from poles of a zeta, let's consider the eigenvalues of a random matrix with block form $\begin{pmatrix} A & B \\ C & A^T \end{pmatrix}$ where B and C are symmetric and 0 on the diagonal. This is the type of matrix that we get for the edge adjacency matrix W_1 of a graph. We used Matlab's randn(N) command to get matrices A, B, C with normally distributed entries. There

[4] In the following figures all spacings have been normalized to have mean 1.

is a result known as the Girko circle law which says that the eigenvalues of a set of random $n \times n$ real matrices with independent entries with a standard normal distribution should be approximately uniformly distributed in a circle of radius $\sqrt{n}$ for large n. References are Bai [6], Girko [44], Tao and Vu [131]. The plot of the eigenvalues of a random matrix with the properties of W_1 is to be found in Figure 75. Note the symmetry with respect to the real axis, since our matrix is real. Another interesting fact is that the circle radius is not exactly that which Girko predicts. The spacing distribution for this random matrix is compared with the non-symmetric Wigner surmise in formula (26.1) in Figure 76.

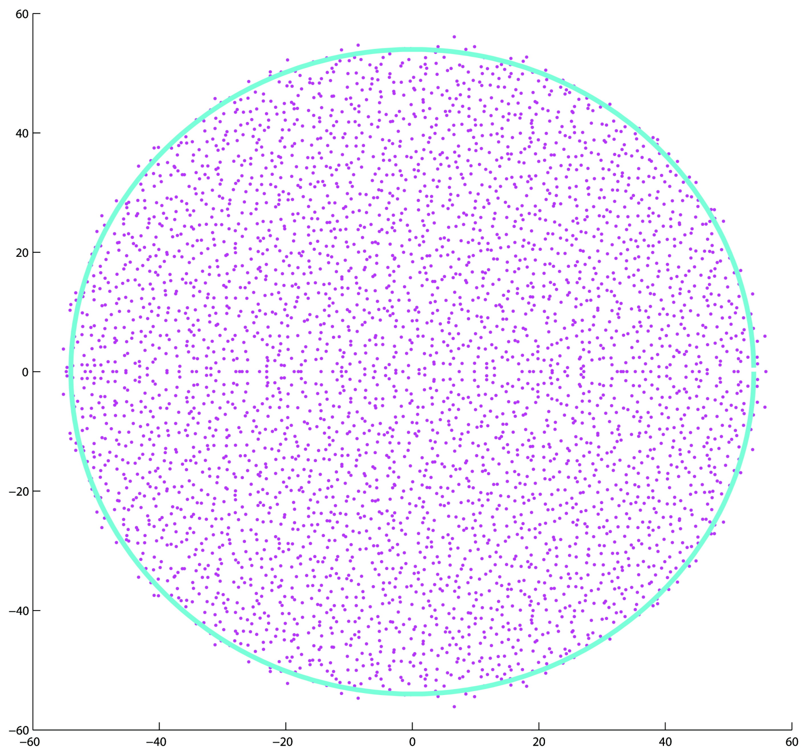


FIGURE 75. **A Matlab experiment showing the spectrum of a random** 2000×2000 **matrix** with the properties of W_1 except that the entries are not 0 and 1. The circle has radius $r = \frac{1}{2}(1+\sqrt{2})\sqrt{2000}$ rather than $\sqrt{2000}$ as in Girko's circle law.

Figure 33 in the earlier section on the edge zeta shows a Matlab experiment giving the spectrum of the edge adjacency matrix W_1 for a "random graph". Figure 77 shows the histogram of the nearest neighbor spacings of the spectrum of the random graph from Figure 33 versus various cases of the **modified Wigner surmise**

$$(\omega+1)\Gamma\left(\frac{\omega+2}{\omega+1}\right)^{\omega+1} x^{\omega} \exp\left(-\Gamma\left(\frac{\omega+2}{\omega+1}\right)^{\omega+1} x^{\omega+1}\right). \tag{26.2}$$

When $\omega = 3$, this is the original Wigner surmise from formula (26.1).

For covering graphs, one can say more about the expected shape of the spectrum of the edge adjacency matrix or equivalently describe the region bounding the poles of the Ihara zeta. Angel, Friedman and Hoory [2] give a method to compute the region encompassing the spectrum of the analogous operator to the edge adjacency matrix W_1 on the universal cover of a graph X. In section 2, we mentioned the Alon conjecture for regular graphs. Angel, Friedman and Hoory give an **analog of the Alon conjecture for irregular graphs**. Roughly their conjecture says that the new edge adjacency spectrum of a large random covering graph is near the edge adjacency spectrum of the universal covering. Here "new" means not occurring in the spectrum

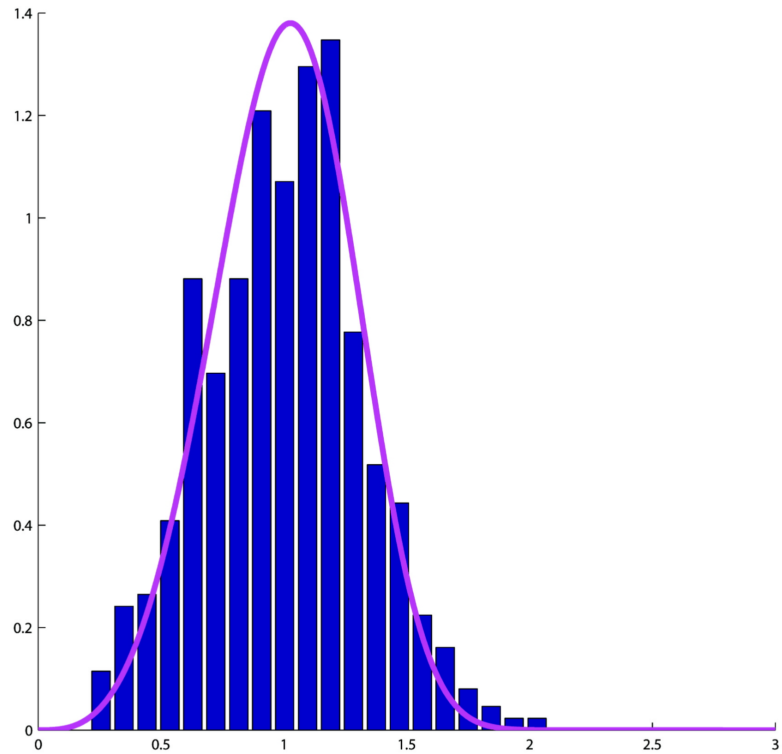


FIGURE 76. **The normalized nearest neighbor spacing for the spectrum of the matrix in Figure 75.** The curve is the Wigner surmise from formula (26.1).

of W_1 for the base graph. This conjecture can be shown to imply the approximate Riemann hypothesis for the new poles of a large random cover.

We show some examples related to this conjecture. Figure 80 shows the spectrum of the edge adjacency matrix of a random cover of the base graph consisting of 2 loops with an extra vertex on 1 loop. The inner circle has radius 1. The middle circle has radius $1/\sqrt{R}$. The outer circle has radius $\sqrt{3}$. The Riemann hypothesis is approximately true for this graph zeta as is the analog of the Alon conjecture made by Angel, Friedman and Hoory.

The truth of the Angel, Friedman and Hoory analog of the Alon conjecture is visible from Figure 78 drawn by Tom Petrillo [100] using the methods of Angel, Friedman, and Hoory for another large random cover of the base graph X consisting of 2 loops with an extra vertex on 1 loop. The light blue region shows the spectrum of the edge adjacency operator on the universal cover of the base graph X.

To produce a figure such as Figure 80 for a random cover Y of the base graph X consisting of 2 loops with an extra vertex on one of them, we can use the formula for the edge adjacency matrix W_1 of Y in terms of the start matrix S and the terminal matrix T from Proposition 4. It is also convenient to write $S = (MN)$, $T = (NM)$ where M and N have $m = |E|$ columns. We used this fact in the proof of the Kotani & Sunada theorem in the same section as Proposition 4. It follows that

$$W_1 = \begin{pmatrix} {}^tNM & {}^tNN - I \\ {}^tMM - I & {}^tMN \end{pmatrix}.$$

Now we arrange the columns of M so that the columns corresponding to lifts of a given edge of the base graph are listed in order of the sheet on which the lift starts. And the lifts of a given vertex of the base graph are also listed together in the order of the sheets where they live. Then

$$M = \begin{pmatrix} I_n & I_n & 0 \\ 0 & 0 & I_n \end{pmatrix}, \quad \text{and} \quad N = \begin{pmatrix} A & 0 & I_n \\ 0 & B & 0 \end{pmatrix},$$

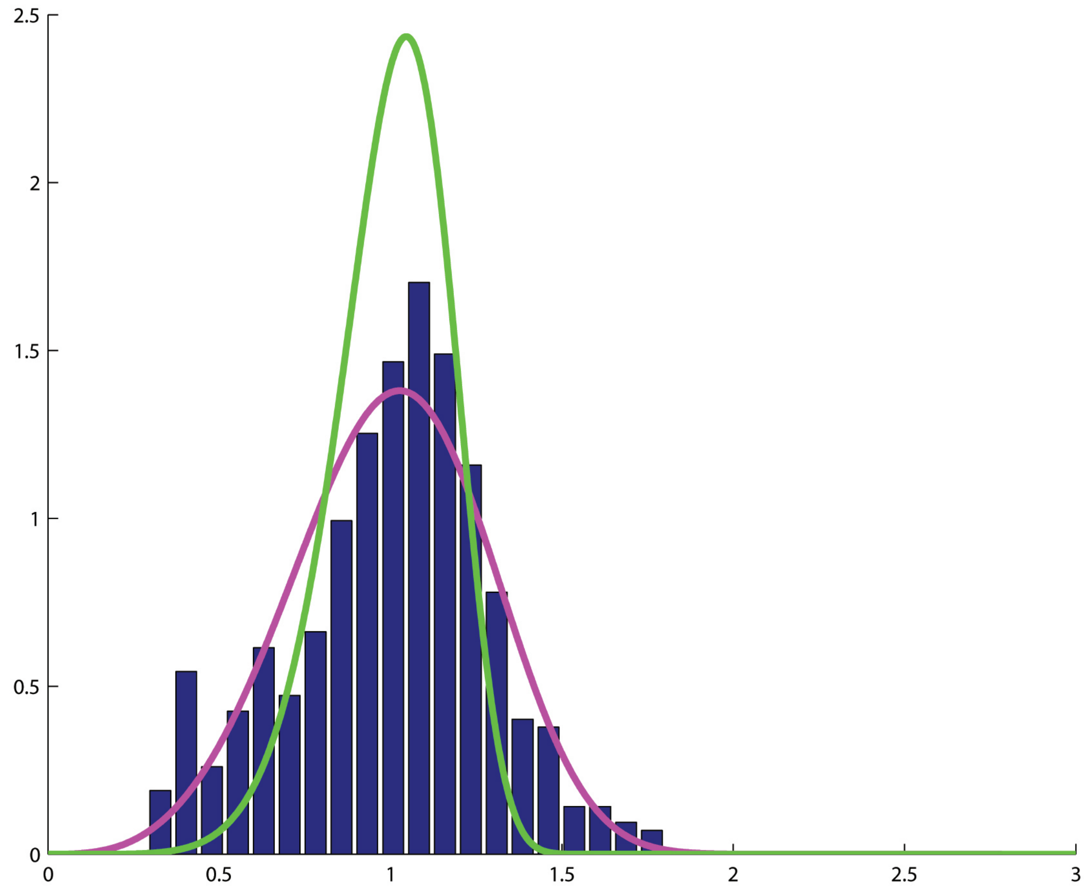


FIGURE 77. **The histogram of the nearest neighbor spacings of the spectrum of the random graph from Figure 33** versus the modified Wigner surmise from formula (26.2) with $\omega = 3$ and 6.

where A and B are permutation matrices. Suppose $n = 3$ and the lift of edge a corresponds to the permutation (12) while the lift of edge b corresponds to the permutation (13). Then we get the graph in Figure 79.

We used Matlab to plot the eigenvalues of W_1 for covers in which A and B are random permutation matrices (found using the command randperm in Matlab). With $n = 801$, we obtain the spectrum of Figure 80. If we compare this with the picture found by Angel, Friedman, and Hoory [2] for random covers of the base graph $K_4 - edge$, we see that there is much similarity, though their Frobenius eigenvalue is 1.5 while ours is approximately 2.1304.

Figure 81 shows the nearest neighbor spacings for the points in Figure 80 compared with the modified Wigner surmise in formula (26.2), for various small values of ω.

Figure 82 shows the spectrum of the edge adjacency matrix for a Galois $\mathbb{Z}_{163} \times \mathbb{Z}_{45}$ covering of the base graph consisting of 2 loops with an extra vertex on 1 loop. The inner circle has radius 1. The middle circle has radius $1/\sqrt{R}$, with R as in Definition 3. The outer circle has radius $\sqrt{3}$. The Riemann hypothesis looks very false.

Figure 83 shows the histogram of the nearest neighbor spacings for the spectrum of the edge adjacency matrix of the graph in the preceding figure compared with spacings of a Poisson random variable (e^{-x}) and the Wigner surmise from formula (26.1).

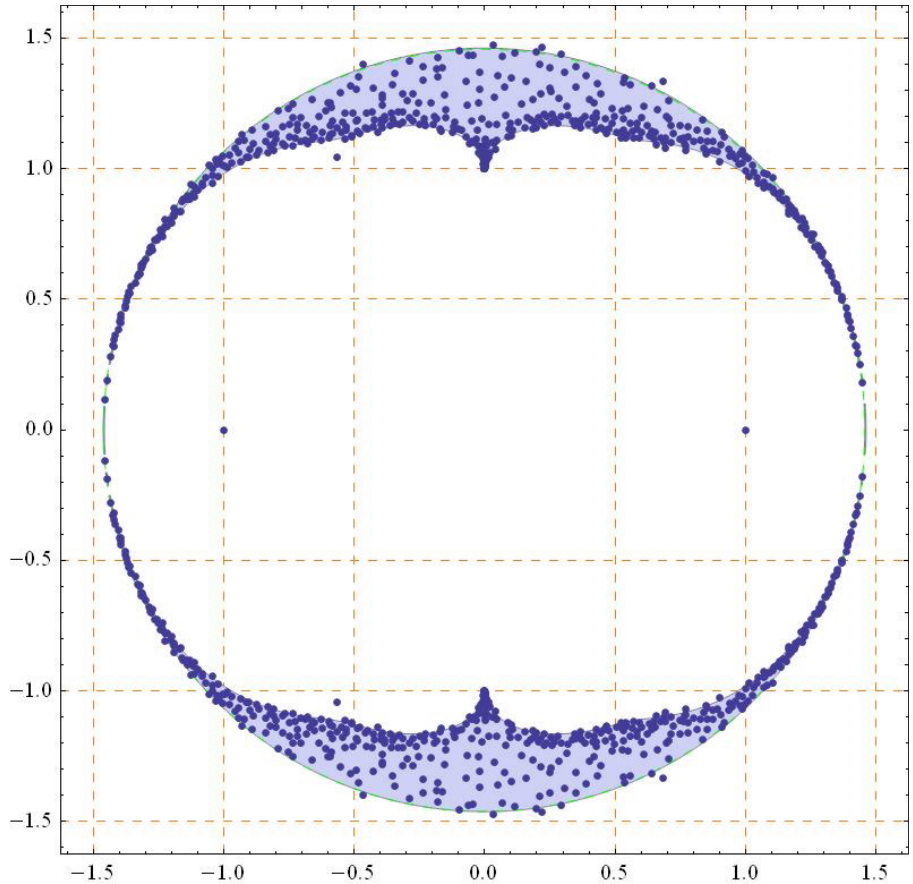


FIGURE 78. **Tom Petrillo's figure** (from [100]) showing in light blue the region bounding the spectrum of the edge adjacency operator on the universal cover of the base graph consisiting of 2 loops with an extra vertex on 1 loop. The dark blue points are the eigenvalues of the edge adjacency matrix of a large random graph. The Angel, Friedman and Hoory analog of the Alon conjecture for irregular graphs appears valid.

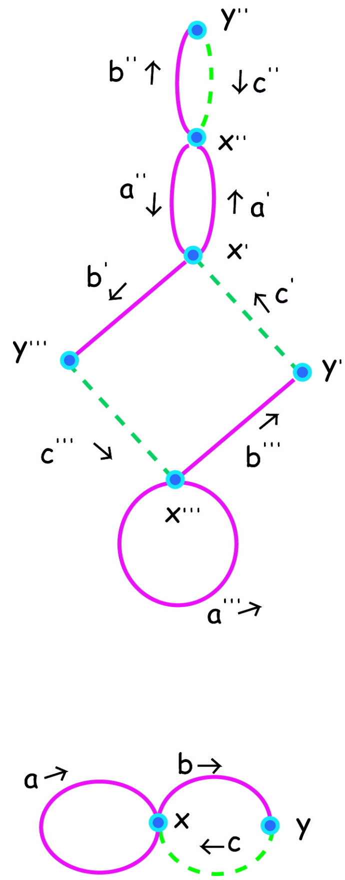


FIGURE 79. **The random 3-cover of 2 loops with and extra vertex** with the lift of *a* corresponding to the permutation (12) and the lift of *b* corresponding to the permutation (13). The green dashed line in the base graph is the spanning tree and the green dashed lines in the cover are the sheets of the cover.

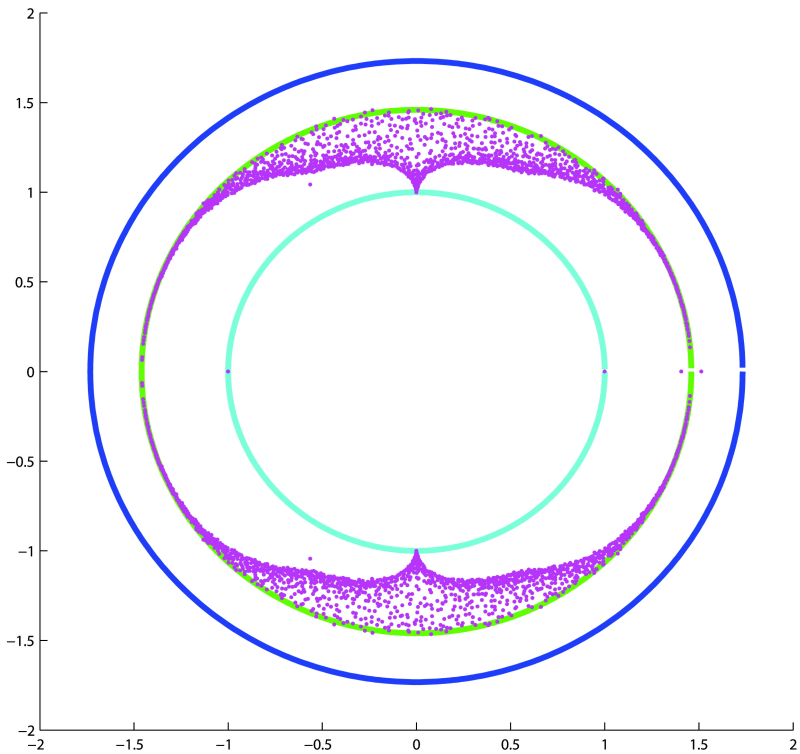


FIGURE 80. **A Matlab experiment in which the purple points are the eigenvalues of the edge adjacency matrix of a random cover (with 801 sheets) of the base graph consisting of 2 loops with an extra vertex on 1 loop.** That is we plot the reciprocals of the poles of the zeta function of the covering. The inner circle has radius 1. The middle circle has radius $1/\sqrt{R}$. The outer circle has radius $\sqrt{3}$. The Riemann hypothesis is approximately true.

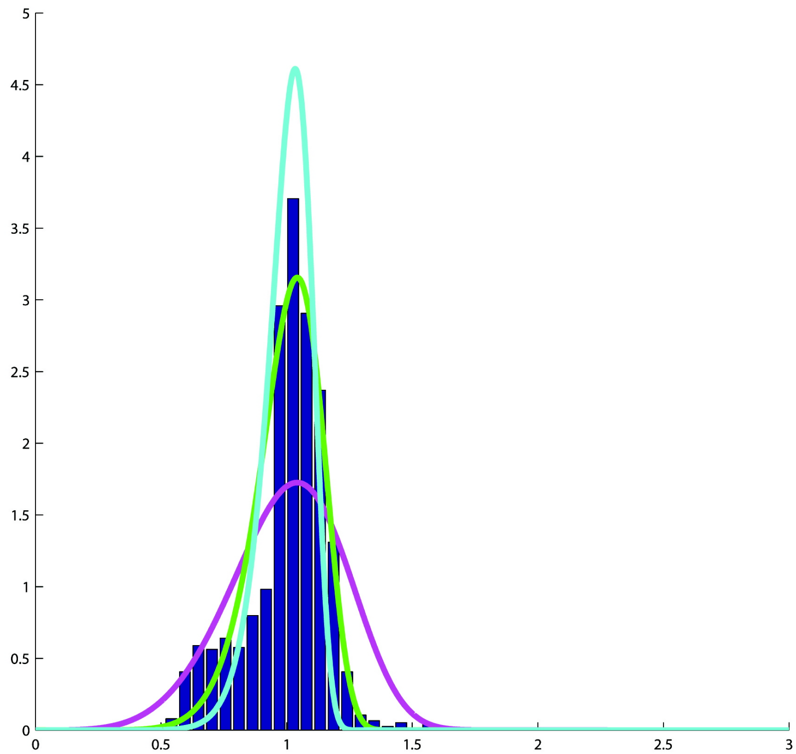


FIGURE 81. **The nearest neighbor spacings for the spectrum of the edge adjacency matrix of the previous graph** compared with 3 versions of the modified Wigner surmise from formula (26.2). Here $\omega = 3, 6, 9$.

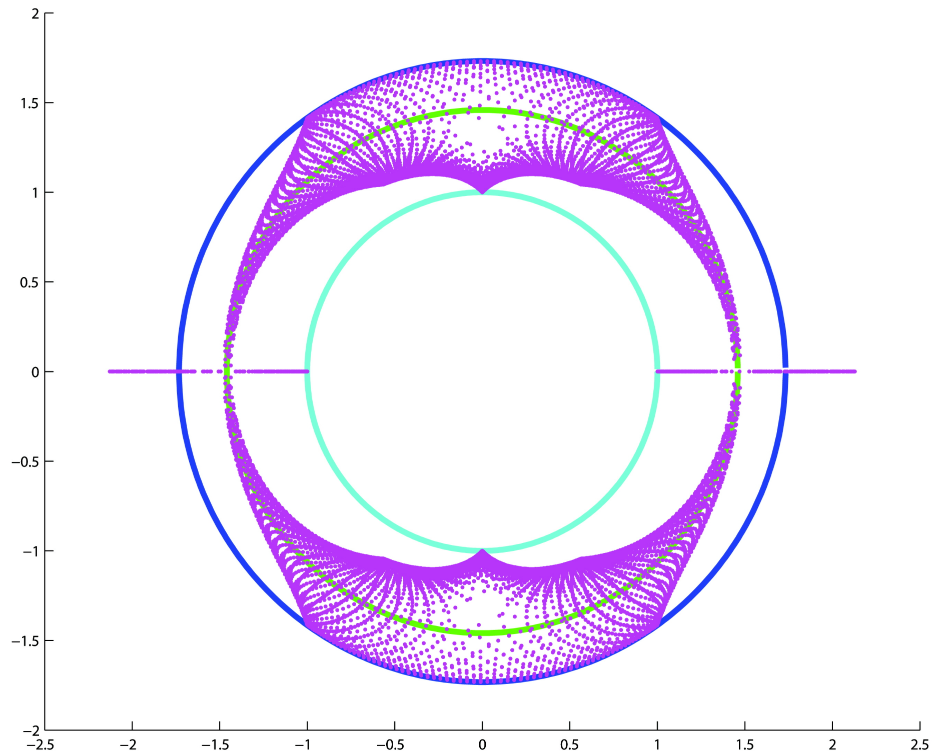


FIGURE 82. **A Matlab experiment in which the purple dots are the eigenvalues of the edge adjacency matrix W_1 for a Galois $\mathbb{Z}_{163} \times \mathbb{Z}_{45}$ covering of the graph consisting of 2 loops with an extra vertex on 1 loop.** The inner circle has radius 1. The middle circle has radius $1/\sqrt{R}$. The outer circle has radius $\sqrt{3}$. The Riemann hypothesis is very false.

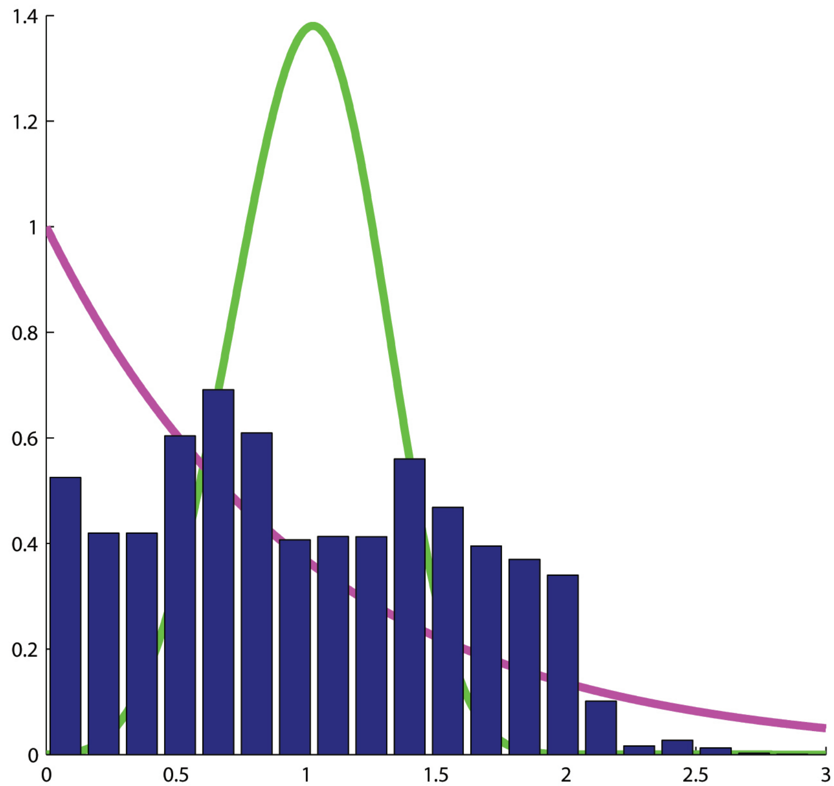


FIGURE 83. **The histogram of the nearest neighbor spacings for the spectrum of the edge adjacency matrix W_1 of the graph in the preceding figure** compared with spacings of a Poisson random variable (e^{-x}) and the Wigner surmise from formula (26.1).

27. Final Research Problems

My Problem List.

1) Do experiments on differences between properties of zetas of weighted or quantum and unweighted graphs. See Horton et al [60], [61]. In particular, consider the connections with random matrix theory. See also Smilansky [115].

2) a) Investigate poles of zeta and the RH for random graphs. How does the distribution of poles of zeta depend on the probability of an edge or the degree sequence?

b) Can one find a Galois graph covering Y of a base graph X such that the poles of the Ihara zeta of Y behave like those of a random cover of X? One could experiment with various finite non-abelian groups. I tried abelian groups in the preceding sections. Pole distributions of zetas for abelian covers were seen to be very different from those of random covers. One idea is to imitate the work of A. Lubotzky, R. Phillips and P. Sarnak [79] using the group $SL(2,F)$, F=finite field.

3) Can you prove an analog of the theorem of Katz and Sarnak [68] for function field zeta functions at least in the case of regular graphs? This says for almost all curves over a finite field, as the genus and order of the field go to infinity, the imaginary parts of the zeros of zetas approach GUE level spacing (meaning the spacings look like those Odlyzko found for the high zeros of Riemann zeta as in Figure 23. An elementary reference giving background on this subject is the book by S. J. Miller and R. Takloo-Bighash [86]. See their bibliography on the web too! A graph theory version should have examples and easier proofs. One expects pole spacings of regular graph zetas to be related to GOE spacings (i.e. those of eigenvalues of real symmetric matrices) here. See the experiments of D. Newland in Figure 24 and the preceding section. Equivalently one expects the level spacings for eigenvalues of the adjacency matrices in a sequence of regular graphs satisfying the hypotheses of McKay's Theorem 2 to approach GOE as $n \to \infty$.

4) Figure out what is a ramified covering and how the zeta function of such a covering will factor. More on this question is found in the section on coverings and Malmskog and Manes [81] or M. Baker and S. Norine [8].

5) Connect the zeta polynomials of graphs to other polynomials associated to graphs and knots (e.g., Tutte, Alexander, and Jones polynomials). Papers exist. But the connection is mysterious to me. See Lin et al [75].

6) Find more graph theoretic analogs of number theoretic results. Galois theory of graph coverings allows us to view Ihara zetas of graph covers as analogs of Dedekind zetas of extensions of number fields. We found analogs of the prime number theorem, the Frobenius automorphism, the Chebotarev density theorem, the explicit formulas of Weil from analytic number theory, Siegel zeros. The analog of the ideal class group is the Jacobian of a graph and has order equal to the number of spanning trees. See R. Bacher, P. de la Harpe, and Tatiana Nagnibeda [5] as well as Baker and Norine [7]. Is there a graph analog of regulator, Stark Conjectures (see formula (10.6), Figure 56 and [118]), class field theory for abelian graph coverings? Or more simply a quadratic reciprocity law, fundamental units? Ihara zeta functions are closer to zeta functions of function fields than to the zetas of number fields. See Rosen [104].

7) Look at Avi Wigderson's website (http://www.math.ias.edu/~avi/) and find out what zig-zag products of graphs are. Does the definition depend on the labeling? Compute their zeta functions. Are there any divisibility properties? Infinite families of regular expanders of arbitrary constant degree are obtained via the modified zig-zag product by Cristina M. Ballantine and Matthew D. Horton [9].

8) Investigate the explicit formula for the Ihara zeta function. See [59] for applications of the regular graph explicit formula known as the Selberg trace formula. Find analogs of the applications of explicit formulas in number theory. See Lang [73] and Murty [91].

9) Investigate the conjecture in Hoory et al [55] saying that every d-regular graph X has a 2 covering Y such that if A_Y is the adjacency matrix of Y, then

$$Spectrum(A_Y) - Spectrum(A_X) \subset [-2\sqrt{d-1}, 2\sqrt{d-1}].$$

See Proposition 11.

10) What basic invariants of the graph X can be determined by the Ihara zeta function? For example, $2|E|$ is the degree of the reciprocal of zeta. See Yaim Cooper [24], Debra Czarneski [33], Matthew Horton [57], [58], and Christopher Storm [125].

11) Investigate the Angel, Friedman and Hoory analog of the Alon conjecture for irregular graphs in [2].

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