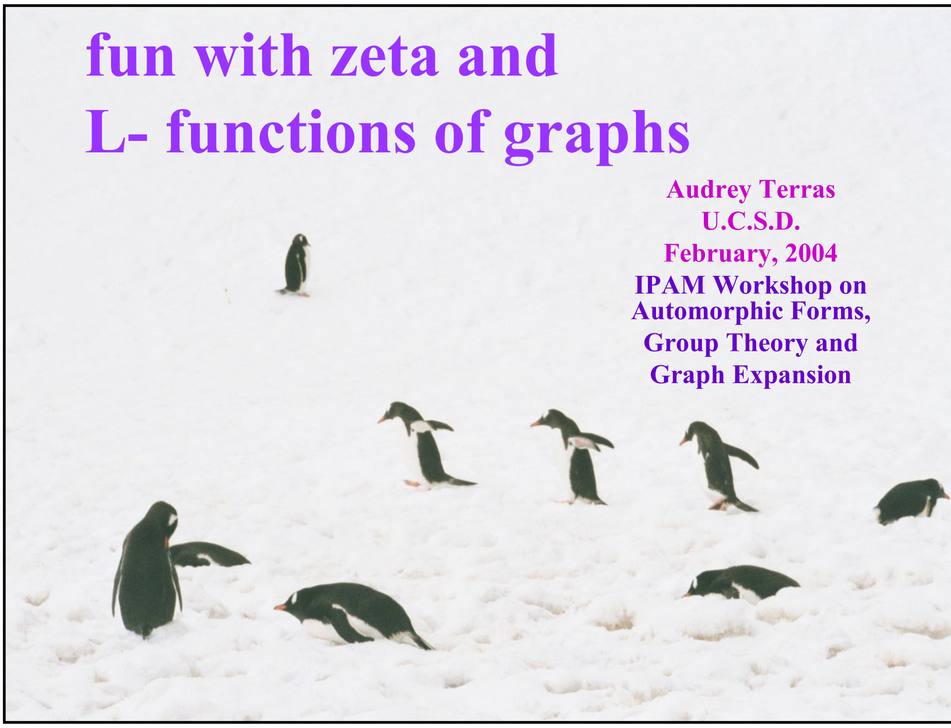

fun with zeta and L- functions of graphs

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U.C.S.D.

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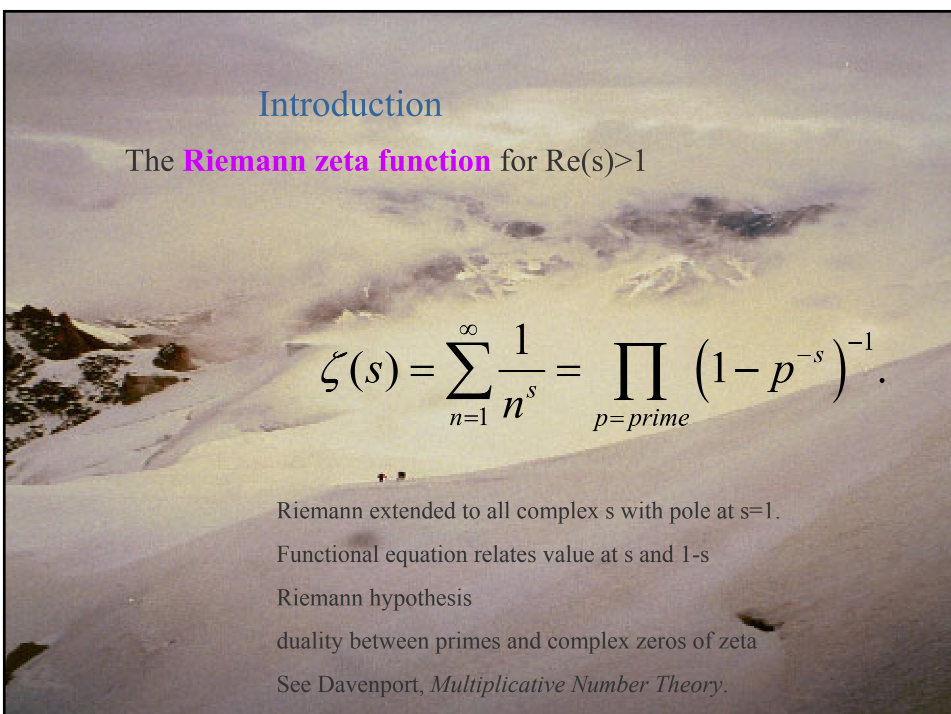
IPAM Workshop on Automorphic Forms, Group Theory and Graph Expansion



Introduction

The **Riemann zeta function** for Re(s)>1

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p=prime}\left(1-p^{-s}\right)^{-1}.$$

Riemann extended to all complex s with pole at s=1.

Functional equation relates value at s and 1-s

Riemann hypothesis

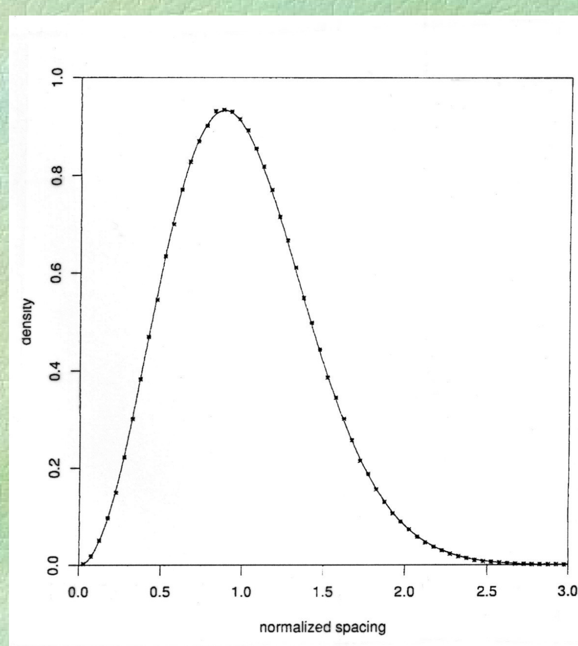
duality between primes and complex zeros of zeta

See Davenport, *Multiplicative Number Theory*.

Graph of |Zeta|

Graph of z=| ζ(x+iy) | showing the pole at x+iy=1 and the first 6 zeros which are on the line x=1/2, of course.
The picture was made by D. Asimov and S. Wagon to accompany their article on the evidence for the Riemann hypothesis as of 1986.

A. Odlyzko's Comparison of Spacings of Zeros of Zeta and Eigenvalues of Random Hermitian Matrix.
See B. Cipra, What's Happening in Math. Sciences, 1998-1999.



Dedekind zeta of an algebraic number field F, where **primes** become prime ideals $\mathfrak{p}$ and infinite product of terms

$(1-N\mathfrak{p}^{-s})^{-1}$,

$N\mathfrak{p}$ = norm of $\mathfrak{p}$ = #(O/$\mathfrak{p}$),

O=ring of integers in F

Selberg zeta associated to a compact Riemannian manifold M=G\H, H = upper half plane with arc length

$ds^2=(dx^2+dy^2)y^{-2}$,

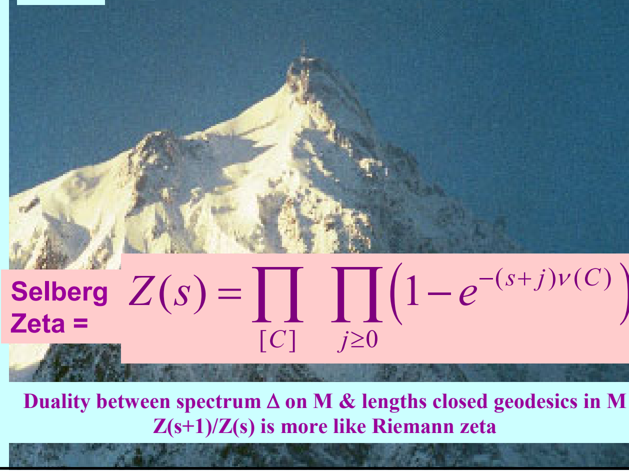
G=discrete group of real fractional linear transformations

primes = primitive closed geodesics C in M of length ν(C),

(primitive means only go around once)

Reference: A.T., Harmonic Analysis on Symmetric Spaces and Applications, I.

We'll say more about number field zetas soon but not Selberg zeta



Many Kinds of Zeta

Selberg Zeta =

$$Z(s)=\prod_{[C]}\prod_{j\geq 0}\left(1-e^{-(s+j)\nu(C)}\right)$$

Duality between spectrum Δ on M & lengths closed geodesics in M

Z(s+1)/Z(s) is more like Riemann zeta

Artin L-Functions

K ⊃ F	**number fields with K/F Galois**
$O_K \supset O_F$	**rings of integers**
$\mathfrak{P} \supset \mathfrak{p}$	**prime ideals ($\mathfrak{p}$ unramified, i.e., $\mathfrak{p} \not\subset \mathfrak{P}^2$)**

Frobenius Automorphism when $\mathfrak{p}$ is unramified.

$$\left(\frac{K/F}{\mathfrak{P}}\right)=\sigma_{\mathfrak{P}}\in Gal(K/F),\quad \sigma_{\mathfrak{P}}(x)\equiv x^{N\mathfrak{p}} \pmod{\mathfrak{P}},\ \text{ for } x\in O_K$$

$\sigma_{\mathfrak{P}}$ generates finite Galois group, Gal$((O_K/\mathfrak{P})/(O_F/\mathfrak{p}))$

determined by $\mathfrak{p}$ up to conjugation if $\mathfrak{P}/\mathfrak{p}$ unramified

f ($\mathfrak{P}/\mathfrak{p}$) = order of $\sigma_{\mathfrak{P}}$ = $[O_K/\mathfrak{P} : O_F/\mathfrak{p}]$

g ($\mathfrak{P}/\mathfrak{p}$)=number of primes of K dividing $\mathfrak{p}$

Artin L-Function for s∈ℂ, π a representation of Gal(K/F).
Give only the formula for unramified primes $\mathfrak{p}$ of F.
Pick $\mathfrak{P}$ a prime in O_K dividing $\mathfrak{p}$.

$$L(s,\pi)\text{"}=\text{"}\prod_{\mathfrak{p}}\det\left(1-\pi\left(\frac{K/F}{\mathfrak{P}}\right)N\mathfrak{p}^{-s}\right)^{-1}$$

Applications

⌘ **Factorization**

$$\zeta_K(s) = \prod_{\substack{\pi \\ \text{irreducible} \\ \text{degree } d_\pi}} L(s,\pi)^{d_\pi}$$

⌘ **Tchebotarev Density Theorem** $\Rightarrow \forall\ \sigma$ in Gal(K/F), $\exists\ \infty$ -ly many prime ideals $\mathfrak{p}$ of O_F such that $\exists\ \mathfrak{P}$ in O_K dividing $\mathfrak{p}$ with Frobenius

$$\left(\frac{K/F}{\mathfrak{P}}\right) = \sigma$$

⌘ **Artin Conjecture:** L(s,π) entire, for non-trivial irreducible rep π (proved in function field case not number field case)

References: Stark's paper in *From Number Theory to Physics*, edited by Waldschmidt et al

Lang or Neukirch, *Algebraic Number Theory*

Rosen, *Number Theory in Function Fields*

Siegel Zeros

These are zeros of the Dedekind zeta function on the real line near 1.

They should not exist if the Riemann hypothesis is true.

Since work of Heilbronn, Stark, etc. it has been realized that the worst case for proving anything is that of quadratic extensions of the rationals.

See Stark, Inv. Math., 23 (1974)

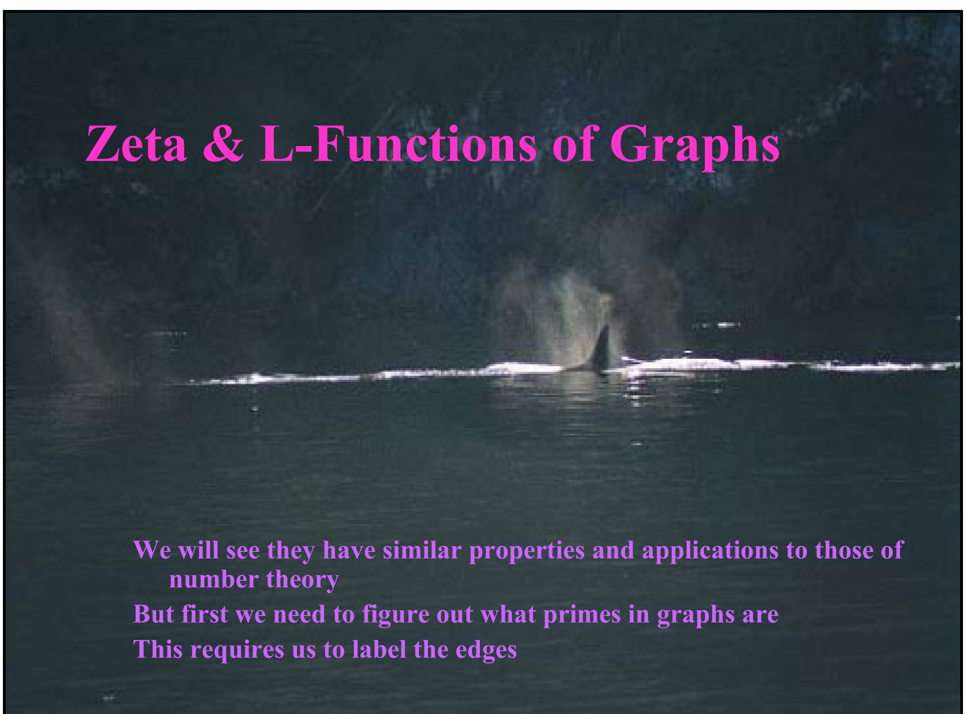
If Siegel zeros did not exist, one would have an easy proof of the Brauer-Siegel theorem on the growth of the class number * regulator as the discriminant goes to infinity. See Lang, Algebraic Number Theory.

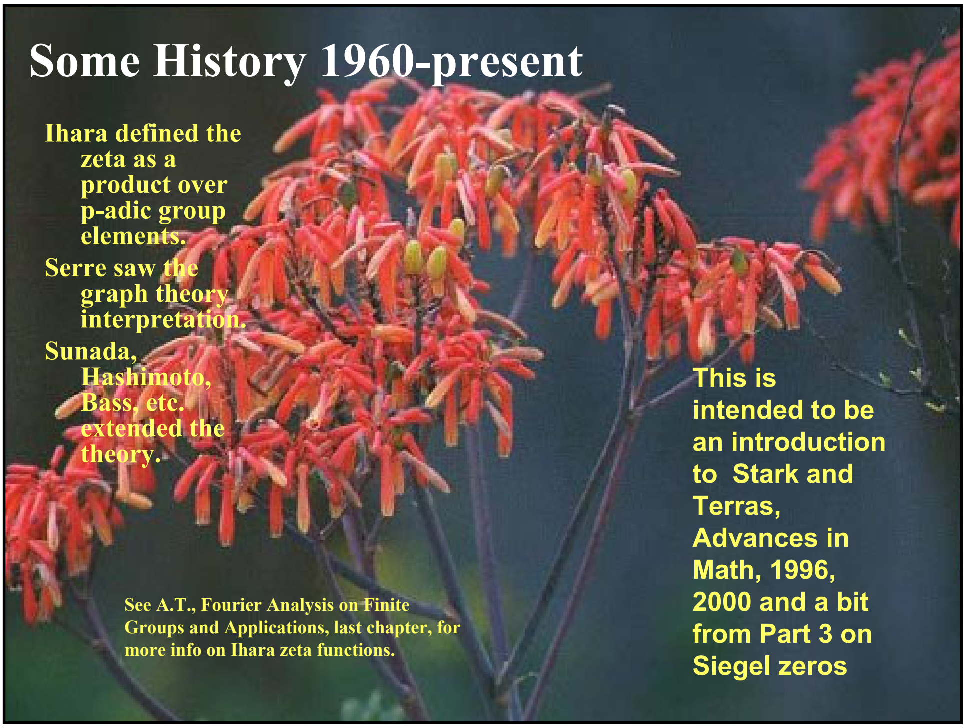
Example. Galois Extension of Non-normal Cubic

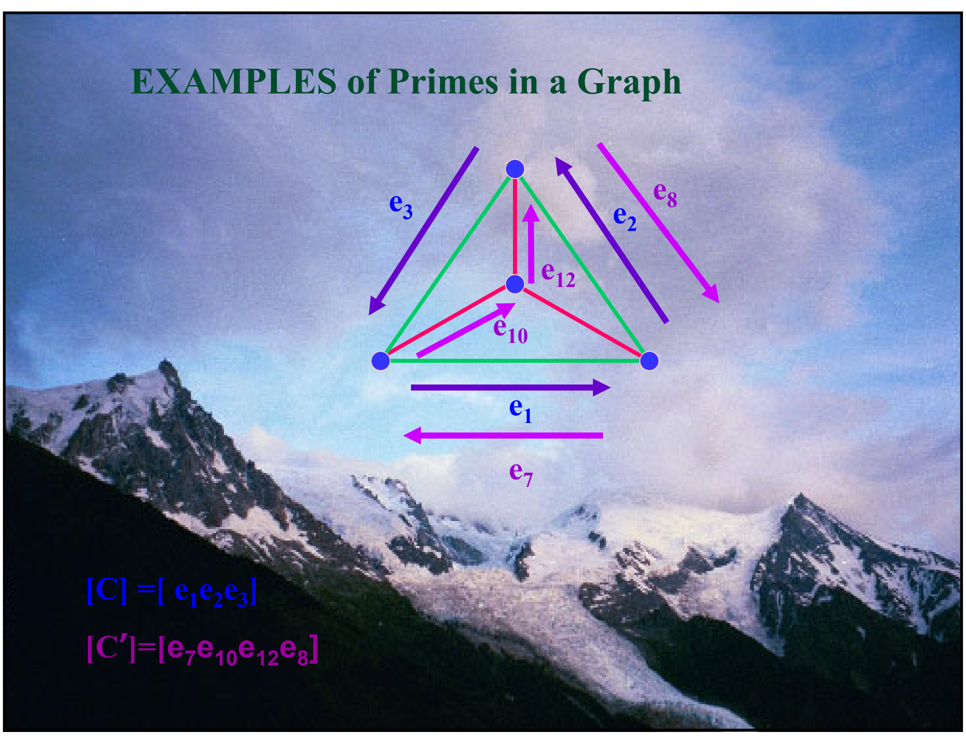
field	ring	prime ideal	finite field
$K=F(e^{2\pi i/3})$	O_K	$\mathfrak{P}$	$O_K/\mathfrak{P}$
2 |	|	|	|
$F=\mathbb{Q}(\sqrt[3]{2})$	O_F	$\mathfrak{p}$	$O_F/\mathfrak{p}$
3 |	|	|	|
$\mathbb{Q}$	$\mathbb{Z}$	$p\mathbb{Z}$	$\mathbb{Z}/p\mathbb{Z}$

More details are in Stark's article in *From Number Theory to Physics*, edited by Waldschmidt et al

Splitting of Rational Primes in O_F - Type 1. Primes that Split Completely:
$pO_F = \mathfrak{p}_1 \mathfrak{p}_2 \mathfrak{p}_3$, with distinct $\mathfrak{p}_i$ of degree 1 ($p=31$ is 1st example),
Frobenius of prime $\mathfrak{P}$ above $\mathfrak{p}_i$ has order 1
density 1/6 by Chebotarev
There are also 2 other types of unramified primes.




Some History 1960-present
Ihara defined the zeta as a product over p-adic group elements.
Serre saw the graph theory interpretation.
Sunada, Hashimoto, Bass, etc. extended the theory.
See A.T., Fourier Analysis on Finite Groups and Applications, last chapter, for more info on Ihara zeta functions.
This is intended to be an introduction to Stark and Terras, Advances in Math, 1996, 2000 and a bit from Part 3 on Siegel zeros


EXAMPLES of Primes in a Graph
e_3
e_8
e_2
e_{12}
e_{10}
e_1
e_7
$[C] = [e_1 e_2 e_3]$
$[C'] = [e_7 e_{10} e_{12} e_8]$

Ihara Zeta Function

$$\zeta_V(u,X)=\prod_{\substack{[C]\\ prime}}\left(1-u^{\nu(C)}\right)^{-1}$$

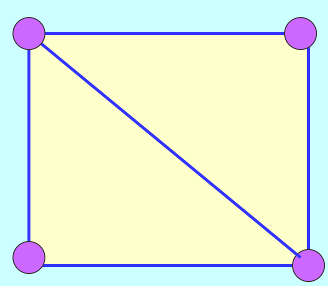
Ihara's Theorem (Bass, Hashimoto, etc.)

A = adjacency matrix of X
Q = diagonal matrix jth diagonal entry = degree jth vertex -1;
r = rank fundamental group = |E|-|V|+1

$$\zeta_V(u,X)^{-1}=(1-u^2)^{r-1}\det(I-Au+Qu^2)$$

Here V is for vertex

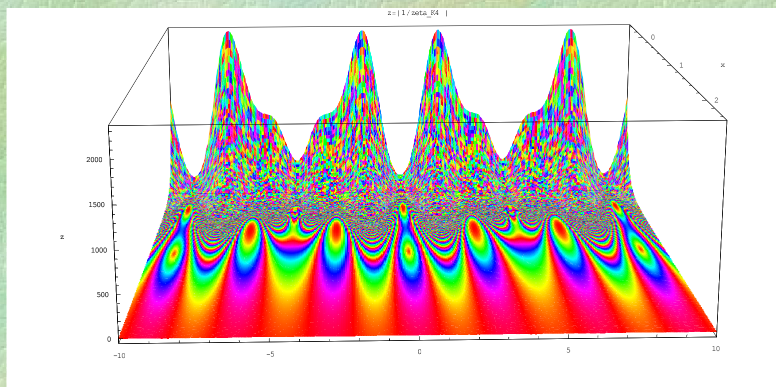
2 Examples K_4 and $X=K_4$-edge



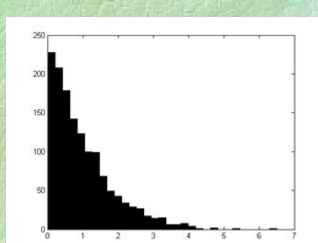
$$\zeta_V(u,X)^{-1}=$$
$$(1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3)$$

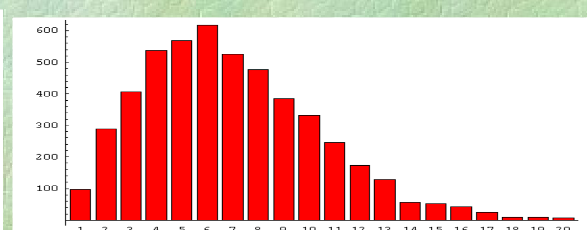
$$\zeta_V(u,K_4)^{-1}=$$
$$(1-u^2)^2(1-u)(1-2u)(1+u+2u^2)^3$$

Graph of $z=1/|Z_{K4}(2^{-(x+iy)})|$ Drawn by Mathematica



Derek Newland's Experiments

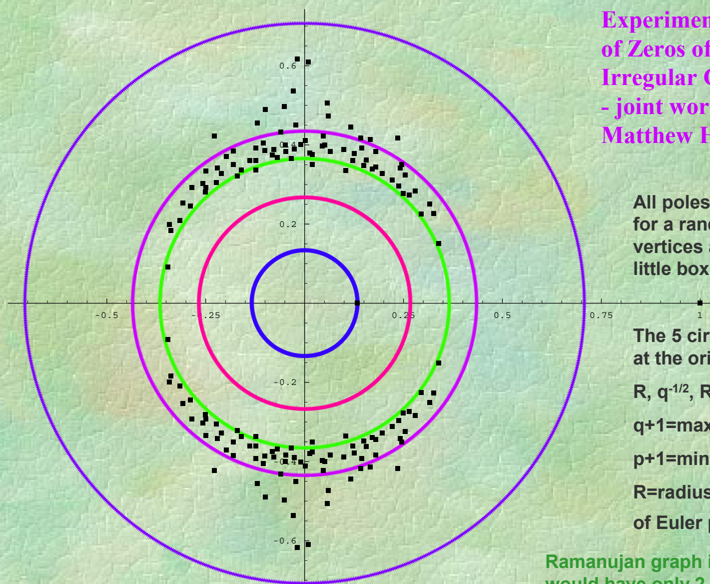




Spacings of Zeros of Ihara Zetas of Regular Graphs

On the Left the Graph is a Finite Euclidean Graph mod 1499 as in Chapter 5 of my book on Fourier Analysis on Finite Groups and Applications.

On the Right is a Random Regular Graph as given by Mathematica with 5000 vertices.



Experiments on Locations of Zeros of Ihara Zeta of Irregular Graphs - joint work with Matthew Horton

All poles except -1 of $\zeta_X(u)$ for a random graph with 80 vertices are denoted by little boxes.

The 5 circles are centered at the origin and have radii

R, $q^{-1/2}$, $R^{1/2}$, $(pq)^{-1/4}$, $p^{-1/2}$

q+1=max degree,

p+1=min degree

R=radius of convergence of Euler product for $\zeta_X(u)$

Ramanujan graph in regular case would have only 2 circles; inner and rest are same

All poles but ±q on green circle; radius √q

Kotani & Sunada, J. Math. Soc. U. Tokyo, 7 (2000) show imaginary poles lie between pink and outside circles; all poles between inner circle and circle of radius 1

Remarks for q+1-Regular Graphs Mostly

- κ(X)=the number of spanning trees of X, the complexity

$$\left(\frac{1}{\zeta_X}\right)^{(r)}(1) = (-1)^{r+1} 2^r (r-1)\kappa(X)$$

  analogue of value of Dedekind zeta at 0
- Riemann Hypothesis, for case of trivial representation (poles), means graph is Ramanujan i.e., non-trivial spectrum of adjacency matrix is contained in the spectrum for the universal covering tree which is the interval (-2√q, 2√q) [see Lubotzky, Phillips & Sarnak, *Combinatorica*, 8 (1988)]. Here $u=q^{-s}$.
- Ihara zeta has functional equations relating value at u and 1/(qu), q=degree - 1

Alon conjecture says RH is true for "most" graphs but it can be false

Hashimoto [Adv. Stud. Pure Math., 15 (1989)] proves Ihara ζ for certain graphs is essentially the ζ function of a Shimura curve over a finite field

The Prime Number Theorem Let $\pi_X(m)$ denote the number of primes [C] in X with length m. Assume X is finite connected (q+1)-regular. Since 1/q is the absolute value of the closest pole(s) of $\zeta(u,X)$ to 0, then

$$\pi_X(m) \sim q^m/m \text{ as } m \to \infty.$$

The proof comes from the method of generating functions (See Wilf, *generatingfunctionology*) and (as in Stark & Terras, *Advances in Math*, 121 & 154):

$$u\frac{d\log\zeta(u,X)}{du} = \sum_{m=1}^{\infty} n_X(m)u^m$$

$n_X(m)$=# closed paths length m no backtrack, no tails

You can also produce an exact formula for $\pi_X(m)$ by the analogous method to that of Rosen, *Number Theory in Function Fields*, page 56.

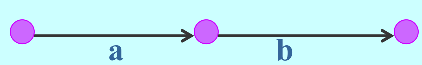
Edge Zetas

Orient the edges of the graph

Multiedge matrix W has ab entry w_{ab} in $\mathbb{C}$,

$w(a,b)=w_{ab}$

if the edges a and b look like those below and $a \neq b^{-1}$



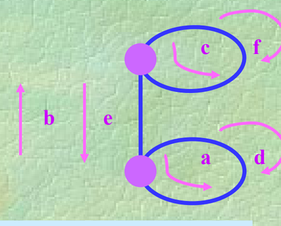
Otherwise set $w_{ab} = 0$

$C = a_1 a_2 \ldots a_s$

$$N_E(C) = w(a_s,a_1)w(a_1,a_2)w(a_2,a_3)\cdots w(a_{s-1},a_s)$$

$$\zeta_E(W,X) = \prod_{\substack{[C]\\ prime}} \left(1 - N_E(C)\right)^{-1}$$

Example. Dumbbell Graph



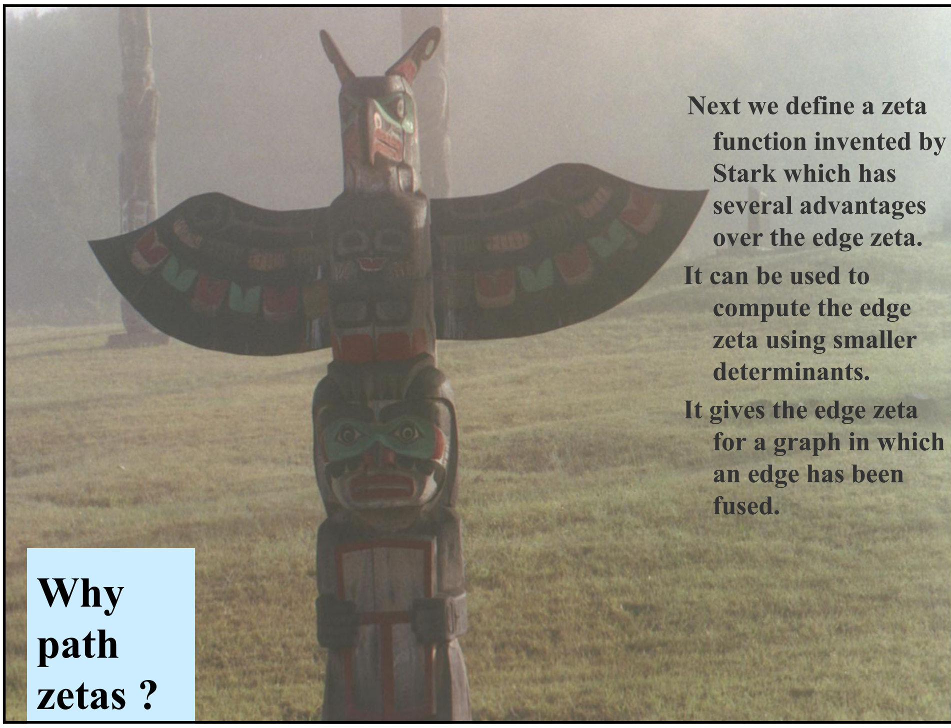
$$\zeta_E(W,D) = \det\begin{pmatrix} w_{11}-1 & w_{12} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{23} & 0 & 0 & w_{26} \\ 0 & 0 & w_{33}-1 & 0 & w_{35} & 0 \\ 0 & w_{42} & 0 & w_{44}-1 & 0 & 0 \\ w_{51} & 0 & 0 & w_{54} & -1 & 0 \\ 0 & 0 & 0 & 0 & w_{65} & w_{66}-1 \end{pmatrix}$$

Here e_2=b and e_5=e are the vertical edges.
Specialize all variables with 2 and 5 to be 0 and get zeta function of subgraph with vertical edge removed. **Fission.**

Properties of Edge Zeta

- **Set all non-0 variables w_{ab}=u in the edge zeta & get Ihara zeta**
- **If you cut an edge of a graph, compute the edge zeta by setting all variables equal to 0 if the cut edge or its inverse appear in subscripts**
- **Edge zeta is the reciprocal of a polynomial given by a much simpler determinant formula than the Ihara zeta**
- **Even better, the proof is simpler (compare Bowen & Lanford proof for dynamical zetas) and Bass deduces Ihara from this**

$$\zeta_E(W,X) = \det(I-W)^{-1}$$



Next we define a zeta function invented by Stark which has several advantages over the edge zeta.

It can be used to compute the edge zeta using smaller determinants.

It gives the edge zeta for a graph in which an edge has been fused.

Why path zetas ?

Multipath Zeta Function

Fundamental Group of X can be identified with group generated by edges left out of a spanning tree

$$e_1,\ldots e_r, e_1^{-1},\ldots, e_r^{-1}$$

2r ×2r **multipath matrix** Z has ij entry

z_{ij} in $\mathbb{C}$ if $e_j \neq e_i^{-1}$ and $z_{ij}=0$, otherwise.

Imitate the definition of the edge zeta function. Define for a prime path $C = a_1 \cdots a_s,\ where\ a_j \in \{e_1^{\pm 1},\ldots,e_r^{\pm 1}\}$ the **path norm**

$$N_P(C) = z(a_s,a_1)\prod_{i=1}^{s-1} z(a_i,a_{i+1})$$

Define the **path zeta**

$$\zeta_P(Z,X) = \prod_{\substack{[C]\\ prime}} \left(1-N_P(C)\right)^{-1}$$

Specialize Path Zeta to Edge Zeta

edges left out of a spanning tree T of X are $e_1, \ldots e_r$

inverse edges are $e_{r+1} = e_1^{-1}, \ldots, e_{2r} = e_r^{-1}$

edges of the spanning tree T are $t_1, \ldots, t_{|X|-1}$

with inverse edges $t_{|X|}, \ldots t_{2|X|-2}$

If $e_i \neq e_j^{-1}$, write the part of the path between e_i and e_j as the (unique) product $t_{k_1} \cdots t_{k_n}$

A prime cycle C is first written as a product of the generators of the fundamental group e_j and then a product of actual edges e_j and t_k.

Now **specialize the multipath matrix Z to Z(W)** with entries

$$z_{ij} = w(e_i, t_{k_1}) w(t_{k_n}, e_j) \prod_{\nu=1}^{n-1} w(t_{k_\nu}, t_{k_{\nu+1}})$$

Then

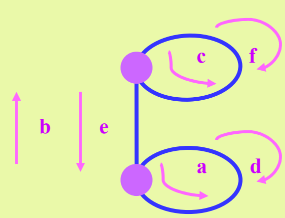
$$\zeta_P(Z(W), X) = \zeta_E(W, X)$$

Example - Dumbbell

Recall that the edge zeta involved a 6x6 determinant.

The path zeta is only 4x4.

Maple computes it much faster than the 6x6.



e.g., specialize z_{ac} to $w_{ab}w_{bc}$

Fusion: shrink edge b to a point. The edge zeta of the new graph obtained by setting $w_{xb}w_{by}=w_{xy}$ in specialized path zeta & same for e instead of b.

$$\zeta_E(W,X)^{-1} = \det\begin{pmatrix} w_{aa}-1 & w_{ab}w_{bc} & 0 & w_{ab}w_{bf} \\ w_{ce}w_{ea} & w_{cc}-1 & w_{ce}w_{ed} & 0 \\ 0 & w_{db}w_{bc} & w_{dd}-1 & w_{db}w_{bf} \\ w_{fe}w_{ea} & 0 & w_{fe}w_{ed} & w_{ff}-1 \end{pmatrix}$$

Why Graph Galois Theory?

Gives generalization of Cayley & Schreier graphs

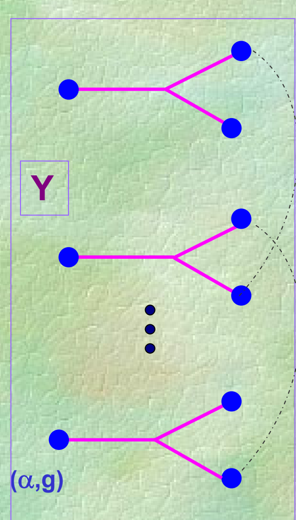
Graph Y an <u>unramified covering</u> of Graph X means
(assuming no loops or multiple edges)

π:Y$\to$X is an onto graph map such that
for every x$\in$X & for every y $\in \pi^{-1}$(x),
π maps the points z $\in$ Y adjacent to y
1-1, onto the points w $\in$ X adjacent to x.

Normal d-sheeted Covering means:
$\exists$ d graph isomorphisms g_1 ,..., g_d mapping Y $\to$ Y
such that $\pi\, g_j\, (y) = \pi\, (y)$ $\forall$ y $\in$ Y
The Galois group G(Y/X) = { g_1 ,..., g_d }.

Note: We do not assume graphs are regular!
We do assume that they are connected,
without "danglers" (degree 1 vertices).

How to Label the Sheets of a Covering

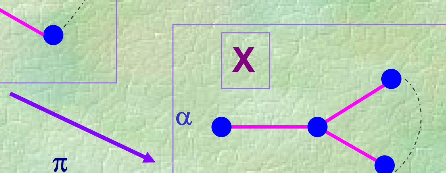


First pick a spanning tree in X (no cycles, connected, includes all vertices of X).

Second make n=|G| copies of the tree T in X. These are the sheets of Y.
Label the sheets with g$\in$G. Then

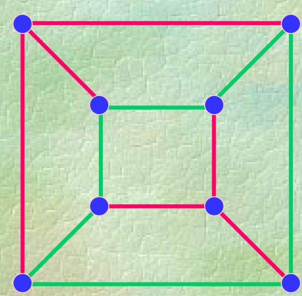
g(sheet h)=sheet(gh)
g(α,h)=(α,gh)

g(path from (α,h) to (β,j))
= path from (α,gh) to (β,gj)

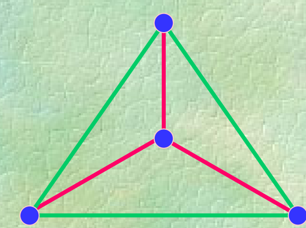


Given G, get examples Y by giving permutation representation of generators of G to lift edges of X left out of T.

Example 1. Quadratic Cover



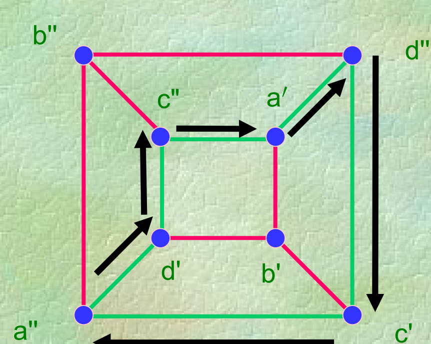
Cube covers
Tetrahedron

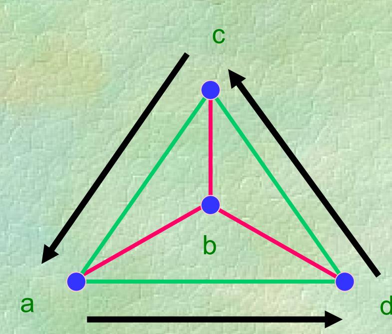


Spanning Tree in X is red.
Corresponding sheets of Y are also red

Example of Splitting of Primes in Quadratic Cover

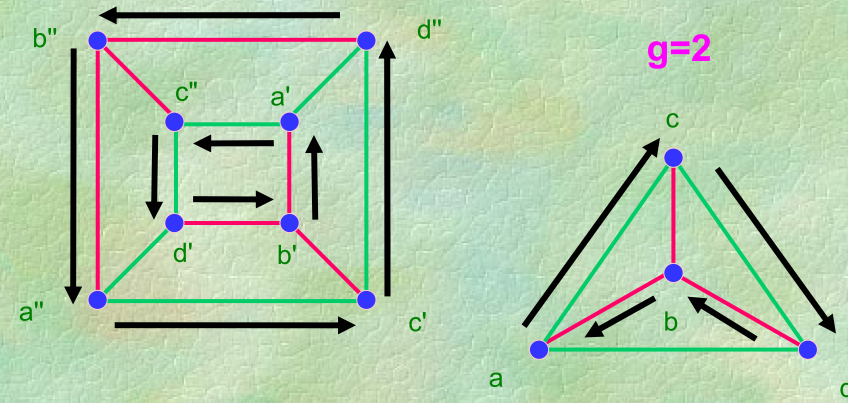
f=2





Picture of Splitting of Prime which is inert;
i.e., f=2, g=1, e=1
1 prime cycle D above, & D is lift of C^2.

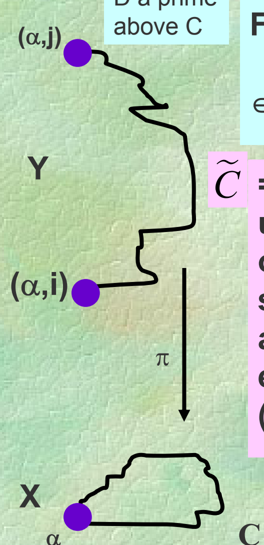
Example of Splitting of Primes in Quadratic Cover



Picture of Splitting of Prime which splits completely; i.e., **f=1, g=2, e=1**
2 primes cycles above

Frobenius Automorphism

D a prime above C



Frob(D) = $\left(\frac{Y/X}{D}\right)$ **= ji^{-1}**

$\in$ **G=Gal(Y/X) where ji^{-1} maps sheet i to sheet j**

$\widetilde{C}$ = the unique lift of C in Y starting at (α,i) ending at (α,j)

$\widetilde{C}$ is not necessarily closed

$$length\left(\widetilde{C}\right) = length\left(C\right)$$

(D a prime above C is closed and is obtained by f liftings like $\widetilde{C}$)

Exercise: Compute Frob(D) on preceding pages, G={1,g}.

Properties of Frobenius

1) Replace (α,i) with (α,hi). Then Frob(D) = ji^{-1} is replaced with $hji^{-1}h^{-1}$. Or replace D with different prime above C and see that Conjugacy class of Frob(D) $\in$ Gal(Y/X) unchanged.
2) Varying α does not change Frob(D).
3) $\mathrm{Frob}(D)^j = \mathrm{Frob}(D^j)$.

Artin L-Function

ρ = representation of G=Gal(Y/X), $u \in \mathbb{C}$, |u| small

$$L(u,\rho,Y/X) = \prod_{[C]} \det\left(1-\rho\left(\frac{Y/X}{D}\right)u^{\nu(C)}\right)^{-1}$$

[C]=primes of X

$\nu(C)$=length C, D a prime in Y over C

Properties of Artin L-Functions

Copy from Lang, *Algebraic Number Theory*

1) L(u,1,Y/X) = $\zeta(u,X)$ = **Ihara zeta function** of X (our analogue of the **Dedekind zeta** function, also **Selberg zeta)**

2)

$$\zeta(u,Y) = \prod_{\rho\in\widehat{G}} L(u,\rho,Y/X)^{d_\rho}$$

product over all irreducible reps of G, d_ρ=degree ρ

Proofs of 1) and 2) require basic facts about reps of finite groups. See A. T., *Fourier Analysis on Finite Groups and Applications.*

Ihara Theorem for L-Functions

$$L(u,\chi_\rho,Y/X)^{-1}$$
$$=(1-u^2)^{(r-1)d_\rho}\det(I'-A'_\rho u+Q'u^2)$$

r=rank fundamental group of X = |E|-|V|+1
ρ= representation of G = Gal(Y/X), d = d_ρ = degree ρ

Definitions. nd×nd matrices A′, Q′, I′, n=|X|

nxn matrix A(g), g ∈ Gal(Y/X), has entry for α,β∈X given by
$(A(g))_{\alpha,\beta}$ = # { edges in Y from (α,e) to (β,g) }, e=identity ∈ G.

$$A'_\rho=\sum_{g\in G}A(g)\otimes\rho(g)$$

Q = diagonal matrix,
jth diagonal entry = q_j = (degree of jth vertex in X)-1,
Q′ = Q⊗I_d , I′ = I_{nd} = identity matrix.

EXAMPLE

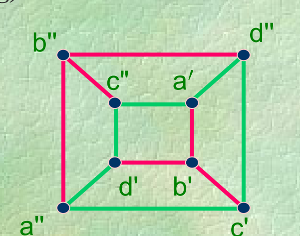
Y=cube, X=tetrahedron: G = {e,g}
representations of G are 1 and ρ: ρ(e) = 1, ρ(g) = -1
$A(e)_{u,v}$ = #{ length 1 paths u′ to v′ in Y}
$A(g)_{u,v}$ = #{ length 1 paths u′ to v″ in Y}

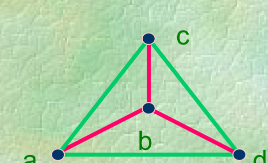
(u,e)=u'
(u,g)=u"

A'_1 = A = adjacency matrix of X = A(e)+A(g)

$$A(e)=\begin{pmatrix}0&1&0&0\\1&0&1&1\\0&1&0&0\\0&1&0&0\end{pmatrix}\qquad A(g)=\begin{pmatrix}0&0&1&1\\0&0&0&0\\1&0&0&1\\1&0&1&0\end{pmatrix}$$

$$A'_\rho=A(e)-A(g)=\begin{pmatrix}0&1&-1&-1\\1&0&1&1\\-1&1&0&-1\\-1&1&-1&0\end{pmatrix}$$



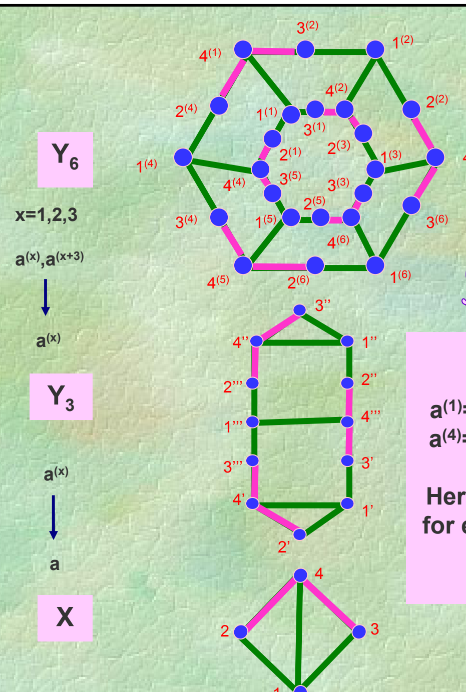


Zeta and L-Functions of Cube & Tetrahedron

- $L(u,\rho,Y/X)^{-1} = (1-u^2)(1+u)(1+2u)(1-u+2u^2)^3$
- $\zeta(u,Y)^{-1} = L(u,\rho,Y/X)^{-1}\,\zeta(u,X)^{-1}$
- $\zeta(u,X)^{-1} = (1-u^2)^2(1-u)(1-2u)(1+u+2u^2)^3$
- poles of $\zeta(u,X)$ are 1,1,1,-1,-1, ½, r,r,r
  where $r=(-1\pm\sqrt{-7})/4$ and $|r|=1/\sqrt{2}$
- ½=Pole of $\zeta(u,X)$ closest to 0 governs prime number thm
- Coefficients of generating function below = number of closed paths without backtracking or tails of length n

$$u\frac{d}{du}\log\zeta(u,X)$$
$$=24u^3+24u^4+96u^6+168u^7+168u^8$$
$$+528u^9+1200u^{10}+1848u^{11}+O(u^{12})$$

So there are 8 primes of length 3 in X, for example.



Example

Galois Cover of Non-Normal Cubic

$G=S_3$, $H=\{(1),(23)\}$ fixes Y_3.

$a^{(1)}=(a,(1))$, $a^{(2)}=(a,(13))$, $a^{(3)}=(a,(132))$, $a^{(4)}=(a,(23))$, $a^{(5)}=(a,(123))$, $a^{(6)}=(a,(12))$.

Here we use standard cycle notation for elements of the symmetric group.

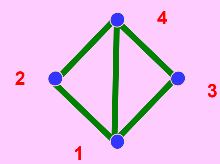
Primes Splitting Completely

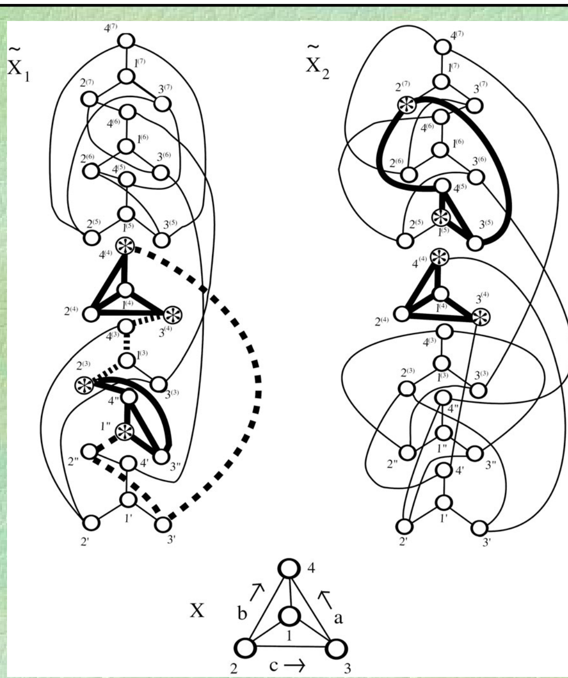
path in X (list vertices) 14312412431

f=1, g=3 **3 lifts to Y_3**

1′4′3‴1‴2‴4″1″2″4‴3′1′
1″4″3″1″2″4‴1‴2‴4″3″1″
1‴4‴3′1′2′4′1′2′4′3‴1‴

Frobenius trivial ⇒ density 1/6



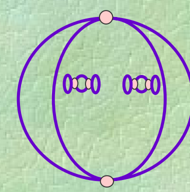


Application of Galois Theory of Graph Coverings. You can't hear the shape of a graph.

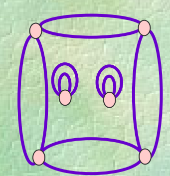
2 connected regular graphs (without loops & multiple edges) which are isospectral but not isomorphic

⌘ See A.T. & Stark in *Adv. in Math.*, Vol. 154 (2000) for the details. The method goes back to algebraic number theorists who found number fields K_i which are non isomorphic but have the same Dedekind zeta. See Perlis, *J. Number Theory*, 9 (1977). Galois group is $GL(3,\mathbb{F}_2)$, order 168. It appears in Buser, also Gordon, Webb & Wolpert (isospectral non-isomorphic planar drums).

⌘ Robert Perlis and Aubi Mellein have used the same methods to find many examples of isospectral non isomorphic graphs with multiple edges and components. 2 such are on the right.



Audrey



Harold

Brauer –Siegel Theory for Ihara Zeta

Let δ be the g.c.d. of lengths of backtrackless paths in X whose 1st and last vertices have degree > 2.

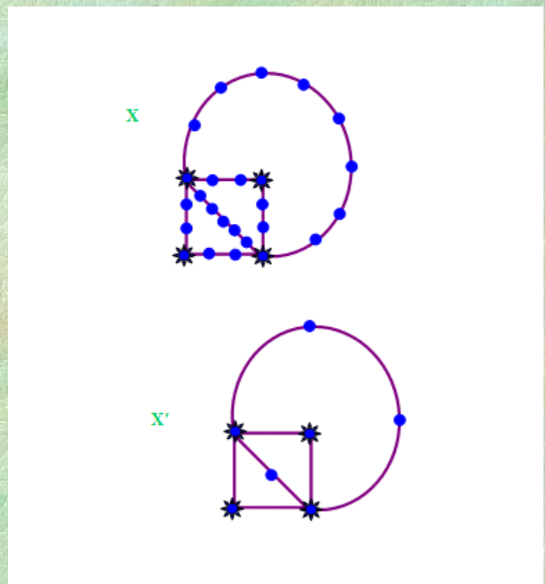
If δ >1, we deflate X to X′ = D(X) obtained by fusing δ consecutive edges between consecutive vertices of degree >2.

$$\zeta_X(u) = \zeta_{X'}(u^\delta).$$

$\zeta_X(u)$ has a pole at R = radius of convergence of the Dirichlet series obtained by expanding the Euler product

So $\zeta_X(u)$ has a δ -fold symmetry producing δ equally spaced poles on a circle of radius R. Any further poles will be called Siegel poles.

Deflating a Graph



Siegel Pole Theorem

Assume rank fundamental group of X > 1, δ=1. If Y is a connected covering graph of X such that $\zeta_y(u)$ has a Siegel pole ρ. Then we have the following

- 1) ρ is 1st order and ρ=-R is real.
- 2) There is a unique intermediate graph X_2 to Y/X such that $\forall$ intermediate graph Z to Y/X, ρ is a Siegel pole of $\zeta_z(u)$ iff Z is intermediate to Y/X_2.
- 3) X_2 is either X or a quadratic cover of X.

Homework Problems

1) Connect constructions of covering graphs using Galois theory with zig-zag product
2) Find the meaning of the Riemann hypothesis for irregular graphs. Are there functional equations?
3) Are there analogs of Artin L-functions for higher dimensional things – buildings ?
4) Connect the zeta polynomials of graphs to other polynomials associated to graphs and knots (Tutte, Alexander, and Jones polynomials)
5) Is there a graph analog of regulator, Stark Conjectures, class field theory for abelian graph coverings? Or more simply a quadratic reciprocity law, fundamental units?