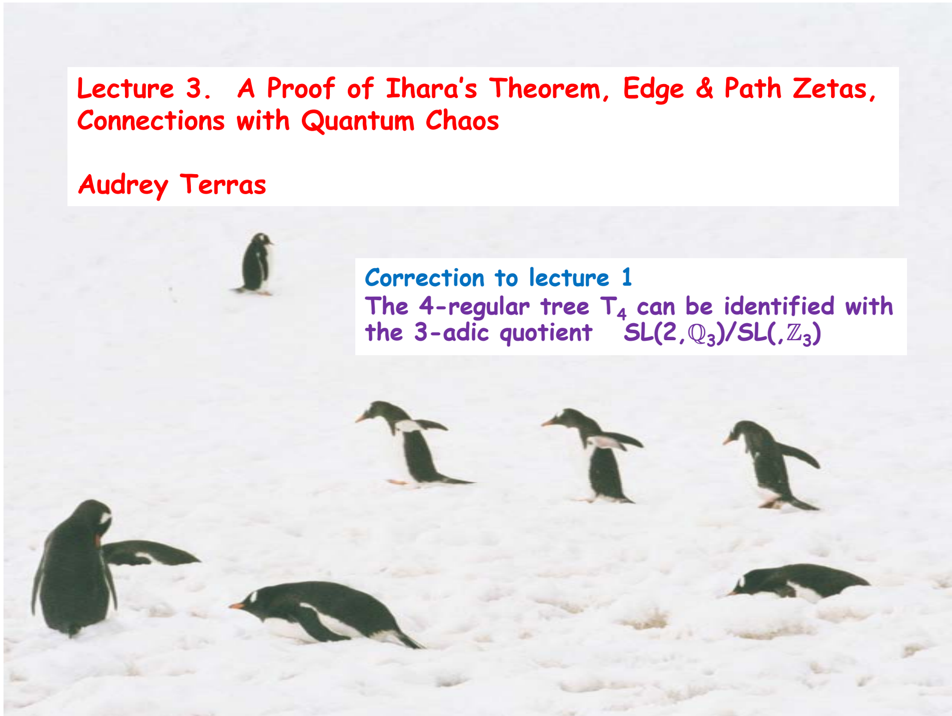


Lecture 3. A Proof of Ihara's Theorem, Edge & Path Zetas, Connections with Quantum Chaos

Audrey Terras

Correction to lecture 1
The 4-regular tree T_4 can be identified with the 3-adic quotient $SL(2,\mathbb{Q}_3)/SL(,\mathbb{Z}_3)$

Ihara Zeta Function

$$\zeta(u,X)=\prod_{\substack{[C]\\ \text{prime}}}\left(1-u^{\nu(C)}\right)^{-1}$$

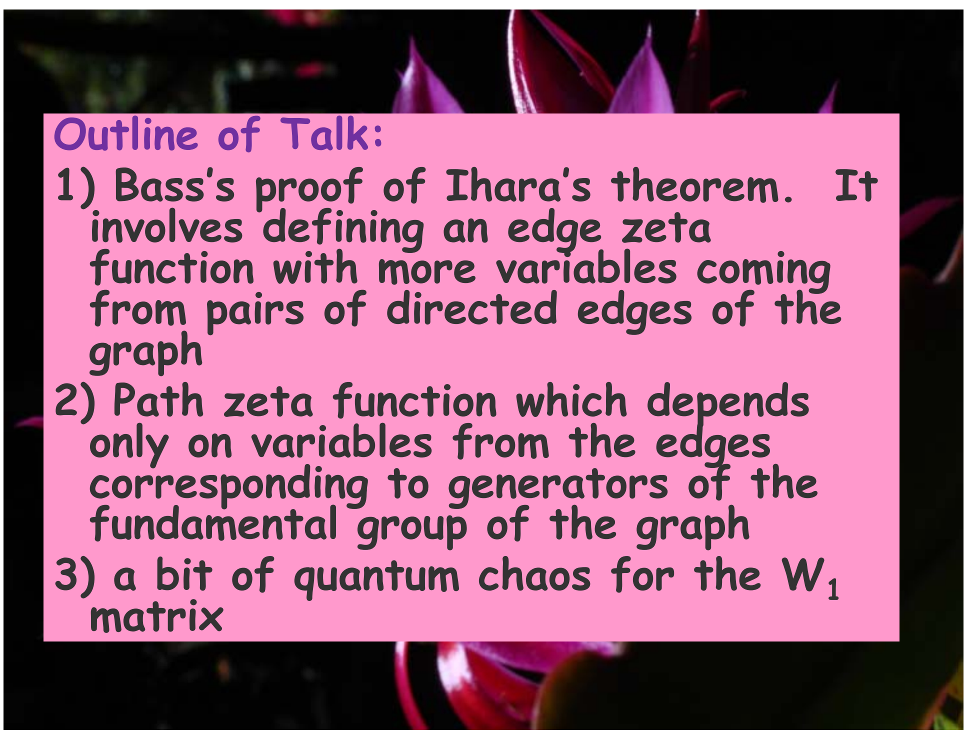
$\nu(C)$ = # edges in C
converges for u complex, |u| small

Ihara's Theorem.

$$\zeta(u,X)^{-1}=(1-u^2)^{r-1}\det(I-Au+Qu^2)$$

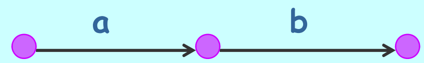
A=adjacency matrix, Q +I = diagonal matrix of degrees, r=rank fundamental group.


Basic Assumptions
graphs are
connected,
with r=rank fundamental group > 1,
no degree 1 vertices (called leaf vertex, hair, danglers, ...)


Outline of Talk:
1) Bass's proof of Ihara's theorem. It involves defining an edge zeta function with more variables coming from pairs of directed edges of the graph
2) Path zeta function which depends only on variables from the edges corresponding to generators of the fundamental group of the graph
3) a bit of quantum chaos for the W_1 matrix

Edge Zetas

Orient the edges of the graph. Multiedge matrix W has ab entry w_{ab} in $\mathbb{C}$, $w(a,b)=w_{ab}$ if the edges a and b look like



and a is not the inverse of b

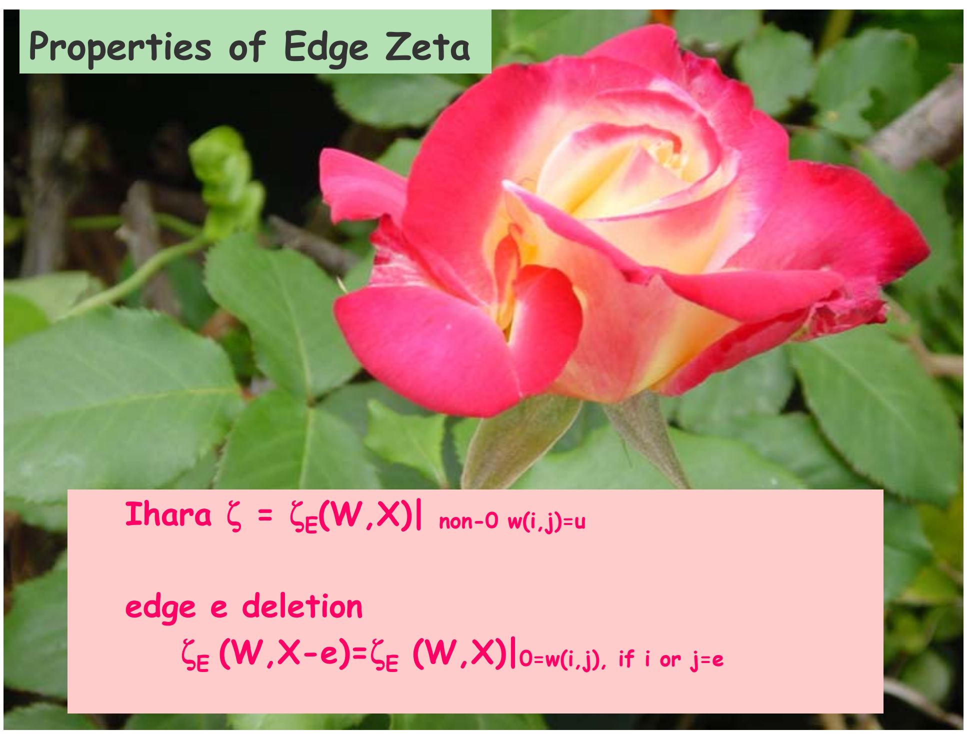
Otherwise set $w_{ab}=0$.

For a prime $C = a_1 a_2 \ldots a_s$, define the edge norm

$$N_E(C) = w(a_s,a_1)w(a_1,a_2)w(a_2,a_3)\cdots w(a_{s-1},a_s)$$

Define the edge zeta for small $|w_{ab}|$ as

$$\zeta_E(W,X) = \prod_{[C]} \left(1 - N_E(C)\right)^{-1}$$



Properties of Edge Zeta

Ihara $\zeta = \zeta_E(W,X)|_{\text{non-0 } w(i,j)=u}$

edge e deletion

$\zeta_E(W,X-e)=\zeta_E(W,X)|_{0=w(i,j),\text{ if } i \text{ or } j=e}$

Determinant Formula For Edge Zeta

$$\zeta_E(W,X) = \det\left(I - W\right)^{-1}$$

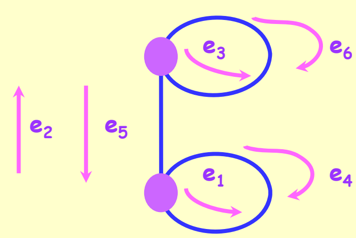
From this Bass gives an ingenious proof of Ihara's theorem.

Reference:

Stark and T., *Adv. in Math.*, Vol. 121 and 154 and 208 (1996 and 2000 and 2007)

Example

D=Dumbbell Graph



$$\zeta_E(W,D)^{-1} = \det\begin{pmatrix} w_{11}-1 & w_{12} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{23} & 0 & 0 & w_{26} \\ 0 & 0 & w_{33}-1 & 0 & w_{35} & 0 \\ 0 & w_{42} & 0 & w_{44}-1 & 0 & 0 \\ w_{51} & 0 & 0 & w_{54} & -1 & 0 \\ 0 & 0 & 0 & 0 & w_{65} & w_{66}-1 \end{pmatrix}$$

e_2 and e_5 are the vertical edges.

Specialize all variables with 2 and 5 to be 0 and get zeta function of subgraph with vertical edge removed. Fission

Diagonalizes the matrix.

Proof of the Determinant Formula

$$\log \zeta_E^{-1} = \sum_{[C]} \sum_{j\geq 1} \frac{N_E(C)^j}{j}$$

$$L = \sum_{i,j} w_{ij} \frac{\partial}{\partial w_{ij}}$$

$$L \log \zeta_E^{-1} = \sum_{m\geq 1} \frac{1}{m} \sum_{\substack{C \\ \nu(C)=m}} \sum_{k\geq 1} \frac{1}{k} L\left(N_E(C)^k\right),$$

C primitive, no backtrack, no tails

$$L N_E(C)^k = km N_E(C)^k$$

$$L \log \zeta_E^{-1} = \sum_{C} N_E(C) = \sum_{m\geq 1} Tr(W^m)$$

Here C need not be primitive, still no backtrack, no tails though.

End of Proof

An exercise in matrix calculus gives

$$L \log \det\left(I - W\right)^{-1} = \sum_{m\geq 1} Tr(W^m)$$

This proves L(log(determinant formula)).

So we get the formula

$$\zeta_E(W,X) = \det\left(I - W\right)^{-1}$$

up to a multiplicative constant. The proof ends by noting that both sides are 1 when all the w_{ij} are 0.

❀

Bass Proof of Ihara formula

$$\zeta_E(W,X) = \det\left(I - W\right)^{-1}$$

$$\zeta_V(u,X)^{-1} = (1-u^2)^{r-1}\det(I - Au + Qu^2)$$

Part 1 of Bass Proof

Define $J = \begin{pmatrix} 0 & I_{|E|} \\ I_{|E|} & 0 \end{pmatrix}$

Define starting matrix S and terminal matrix T

$$s_{ve} = \begin{cases} 1, \text{ if v is starting vertex of edge e} \\ 0, \text{ otherwise} \end{cases}$$

$$t_{ve} = \begin{cases} 1, \text{ if v is the terminal vertex of edge e} \\ 0, \text{ otherwise} \end{cases}$$

Then, recalling our edge numbering system, we see that

$$SJ = T, \qquad TJ = S$$

since start (end) of e_j is end (start) of $e_{j+|E|}$

$$A = S\ T^t, \quad Q + I_{|V|} = SS^t = TT^t$$

Note: matrix A counts number of undirected edges connecting 2 distinct vertices and twice # of loops at each vertex. Q+I = diagonal matrix of degrees of vertices

Part 2 of Bass Proof

W_1 matrix obtained from W by setting all non-zero w_{ij} equal to 1

$W_1 + J = T^t S$, where J compensates for not allowing edge e_j to feed into $e_{j\pm|E|}$

Below all matrices are (|V|+2|E|) x (|V|+2|E|), with |V| x |V| 1st block.

The preceding formulas imply that:

$$\begin{pmatrix} I_{|V|} & 0 \\ T^t & I_{2|E|} \end{pmatrix}\begin{pmatrix} I_{|V|}(1-u^2) & Su \\ 0 & I_{2|E|} - W_1 u \end{pmatrix}$$
$$= \begin{pmatrix} I_{|V|} - Au + Qu^2 & Su \\ 0 & I_{2|E|} + Ju \end{pmatrix}\begin{pmatrix} I_{|V|} & 0 \\ T^t - S^t u & I_{2|E|} \end{pmatrix}$$

Then take determinants of both sides to see

$$(1-u^2)^{|V|}\det(I_{2|E|} - W_1 u) = \det(I_{|V|} - Au + Qu^2)\det(I_{2|E|} + Ju)$$

End of Bass Proof

$$(1-u^2)^{|V|}\det(I_{2|E|} - W_1 u) = \det(I_{|V|} - Au + Qu^2)\det(I_{2|E|} + Ju)$$

$$I + Ju = \begin{pmatrix} I & Iu \\ Iu & I \end{pmatrix} \text{ implies } \begin{pmatrix} \mathrm{I} & 0 \\ \text{-Iu} & \mathrm{I} \end{pmatrix}(I + Ju) = \begin{pmatrix} I & Iu \\ 0 & I(1-u^2) \end{pmatrix}$$

So $\det(I+Ju)=(1-u^2)^{|E|}$

Since r-1=|E|-|V|, for a connected graph, the Ihara formula for the vertex zeta function follows from the edge zeta determinant formula.

Next we define a zeta function invented by Stark which has several advantages over the edge zeta.

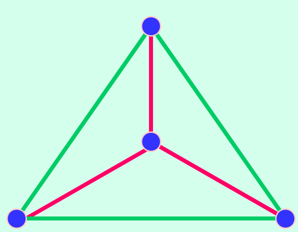
It can be used to compute the edge zeta using smaller determinants.

It gives the edge zeta for a graph in which an edge has been fused.

spanning trees

A tree is a connected graph without cycles.

A spanning tree for a graph X is a subgraph which is a tree and which contains all the vertices of X.



the red graph is a spanning tree for K_4

Path Zeta Function

Fundamental Group of X can be identified with group generated by edges left out of a spanning tree $e_1,\ldots e_r, e_1^{-1},\ldots,e_r^{-1}$

2r ×2r **multipath matrix** Z has ij entry

z_{ij} in $\mathbb{C}$ if $e_j \neq e_i^{-1}$, z_{ij} =0, otherwise.

Imitate the definition of the edge zeta function.

Define for a prime path

$$C = a_1 \cdots a_s,\ \ where\ \ a_j \in \{e_1^{\pm 1},\ldots,e_r^{\pm 1}\}$$

the **path norm** $N_P(C) = z(a_s,a_1)\prod_{i=1}^{s-1} z(a_i,a_{i+1})$

Define the path zeta function

$$\zeta_P(Z,X) = \prod_{[C]}\left(1-N_P(C)\right)^{-1}$$

Specialize Path Zeta to Edge Zeta

edges left out of a spanning tree T of X are $e_1,\ldots e_r$

inverse edges are $e_{r+1} = e_1^{-1},\ldots,e_{2r} = e_r^{-1}$

edges of the spanning tree T are $t_1,\ldots,t_{|X|-1}$

with inverse edges $t_{|X|},\ldots t_{2|X|-2}$

If $e_i \neq e_j^{-1}$, write the part of the path between e_i and e_j as the (unique) product $t_{k_1}\cdots t_{k_n}$

A prime cycle C is first written as a reduced product of generators of the fundamental group e_j and then a product of actual edges e_j and t_k.

Now **specialize the multipath matrix Z to Z(W)** with entries

Then $z_{ij} = w(e_i,t_{k_1})w(t_{k_n},e_j)\prod_{v=1}^{n-1} w(t_{k_v},t_{k_{v+1}})$

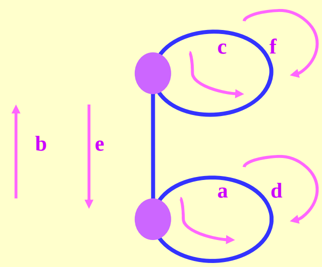
$$\zeta_P(Z(W),X) = \zeta_E(W,X)$$

Example - the Dumbbell

Recall that the edge zeta involved a 6x6 determinant.

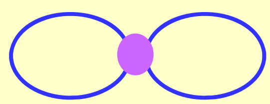
The path zeta is only 4x4.

Maple computes ζ_E much faster than the 6x6.



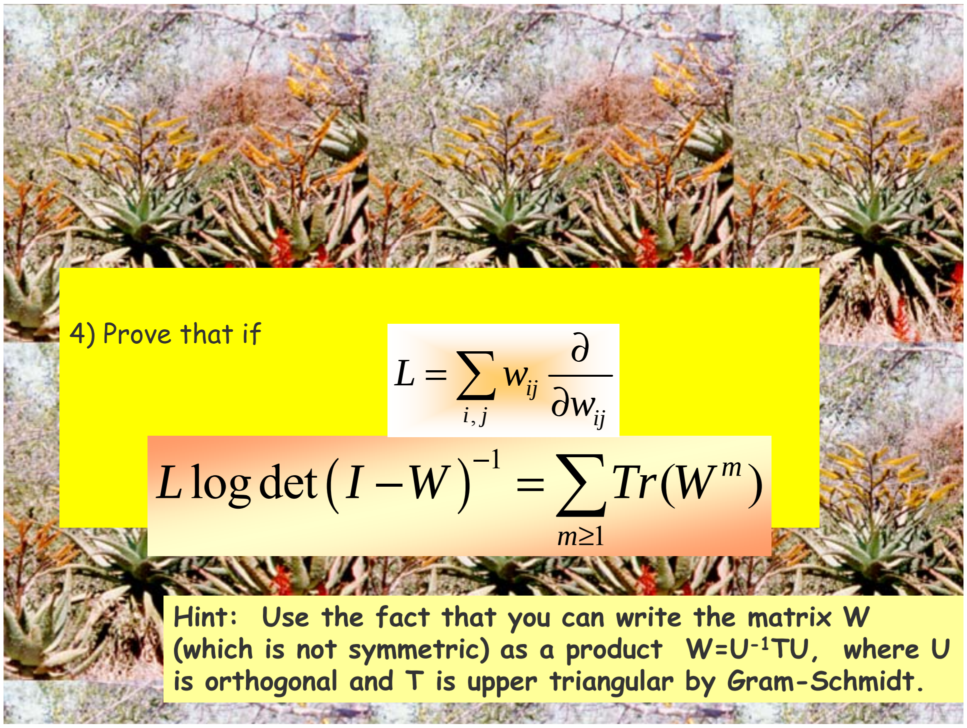
$$\begin{pmatrix} w_{aa}-1 & w_{ab}w_{bc} & 0 & w_{ab}w_{bf} \\ w_{ce}w_{ea} & w_{cc}-1 & w_{ce}w_{ed} & 0 \\ 0 & w_{db}w_{bc} & w_{dd}-1 & w_{db}w_{bf} \\ w_{fe}w_{ea} & 0 & w_{fe}w_{ed} & w_{ff}-1 \end{pmatrix}$$

Fusion: shrink edge b to a point. The edge zeta of the new graph obtained by setting $w_{xb}w_{by}=w_{xy}$ in specialized path zeta & same for e instead of b.



Exercises

1) Fill in the details of the proof that $1/\zeta_E(W,X)=\det(I-W)$.
2) Fill in the details of the proof that the formula in exercise 4 implies Ihara's 3-term determinant formula for the vertex zeta.
3) Write a Mathematica (or whatever) program to do the process that specializes the path zeta to get the edge zeta.

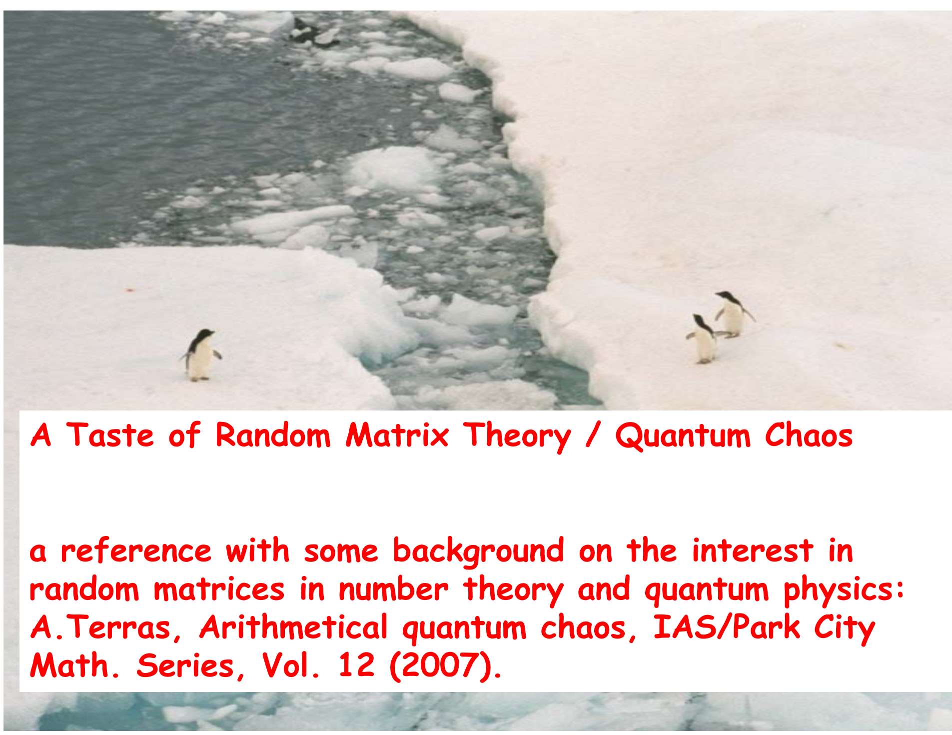


4) Prove that if

$$L=\sum_{i,j} w_{ij}\frac{\partial}{\partial w_{ij}}$$

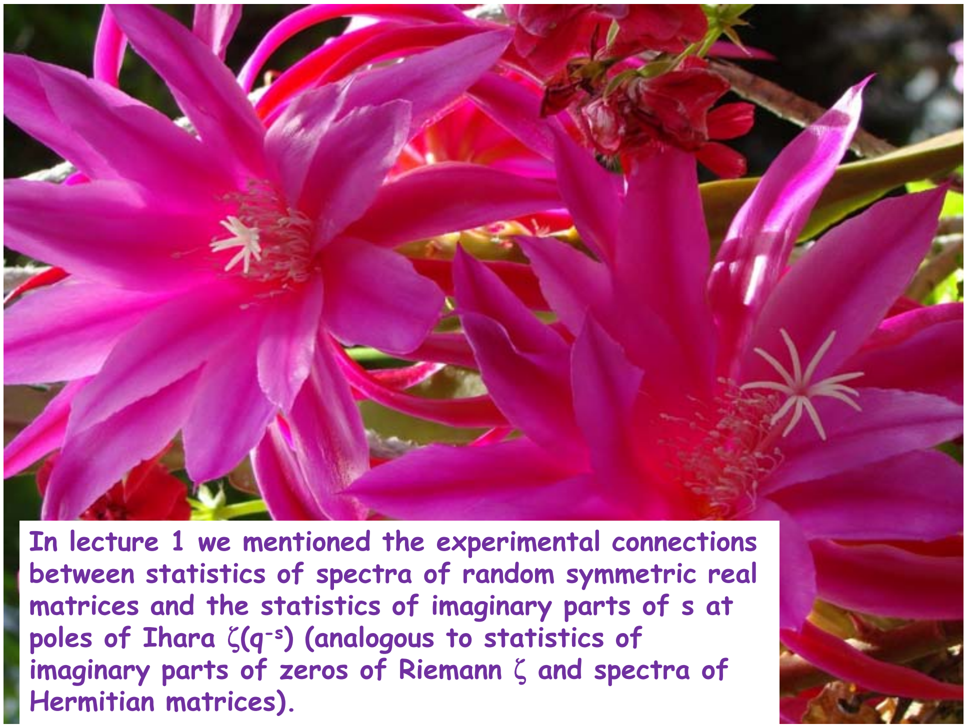
$$L\log\det\left(I-W\right)^{-1}=\sum_{m\geq 1} Tr(W^m)$$

Hint: Use the fact that you can write the matrix W (which is not symmetric) as a product $W=U^{-1}TU$, where U is orthogonal and T is upper triangular by Gram-Schmidt.

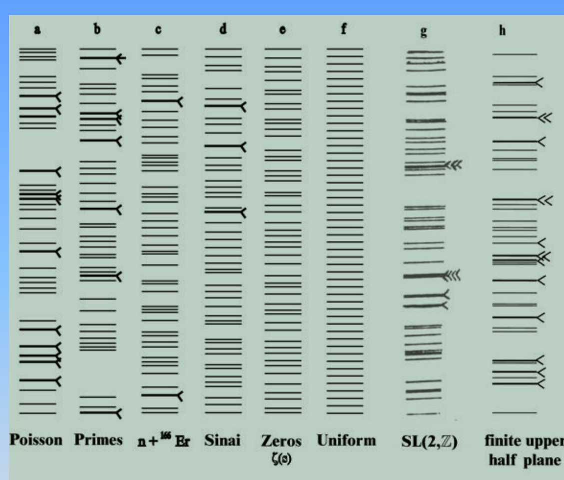


A Taste of Random Matrix Theory / Quantum Chaos

a reference with some background on the interest in random matrices in number theory and quantum physics: A.Terras, Arithmetical quantum chaos, IAS/Park City Math. Series, Vol. 12 (2007).



In lecture 1 we mentioned the experimental connections between statistics of spectra of random symmetric real matrices and the statistics of imaginary parts of s at poles of Ihara $\zeta(q^{-s})$ (analogous to statistics of imaginary parts of zeros of Riemann ζ and spectra of Hermitian matrices).

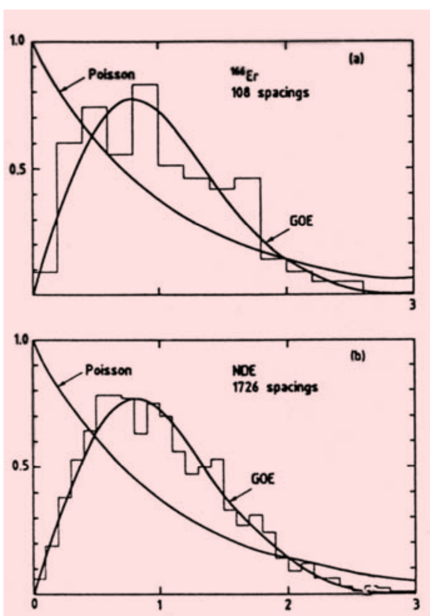


from O. Bohigas and M.-J. Giannoni, Chaotic motion and random matrix theories, *Lecture Notes in Physics*, 209, Springer-Verlag, Berlin, 1984:

arrows mean lines are too close to distiguish

O. Bohigas and M.-J. Giannoni, Chaotic motion and random matrix theories, *Lecture Notes in Physics*, 209, Springer-Verlag, Berlin, 1984: “The question now is to discover the stochastic laws governing sequences having very different origins, as illustrated in” the Figure, each column with 50 levels ...” Note that the spectra have been rescaled to the same vertical axis from 0 to 49.

(a) Poisson spectrum, i.e., of a random variable with spacings of probability density e^{-x}.
(b) primes between 7791097 and 7791877.
(c) resonance energies of compound nucleus observed in the reaction n+166Er.
(d) from eigenvalues corresponding to transverse vibrations of a membrane whose boundary is the Sinai billiard which is a square with a circular hole cut out centered at the center of the square.
(e) the positive imaginary parts of zeros of the Riemann zeta function (from the 1551th to the 1600th zero).
(f) is equally spaced - the picket fence or uniform distribution.
(g) from P. Sarnak, Arithmetic quantum chaos, *Israel Math. Conf. Proc.*, 8 (1995), (published by Amer. Math. Soc.) : eigenvalues of the Poincaré Laplacian on the fundamental domain of the modular group SL(2,$\mathbb{Z}$), 2 × 2 integer matrices of determinant 1.
(h) spectrum of a finite upper half plane graph for p=53 (a = δ = 2), without multiplicity (see my book *Fourier Analysis on Finite Groups*)



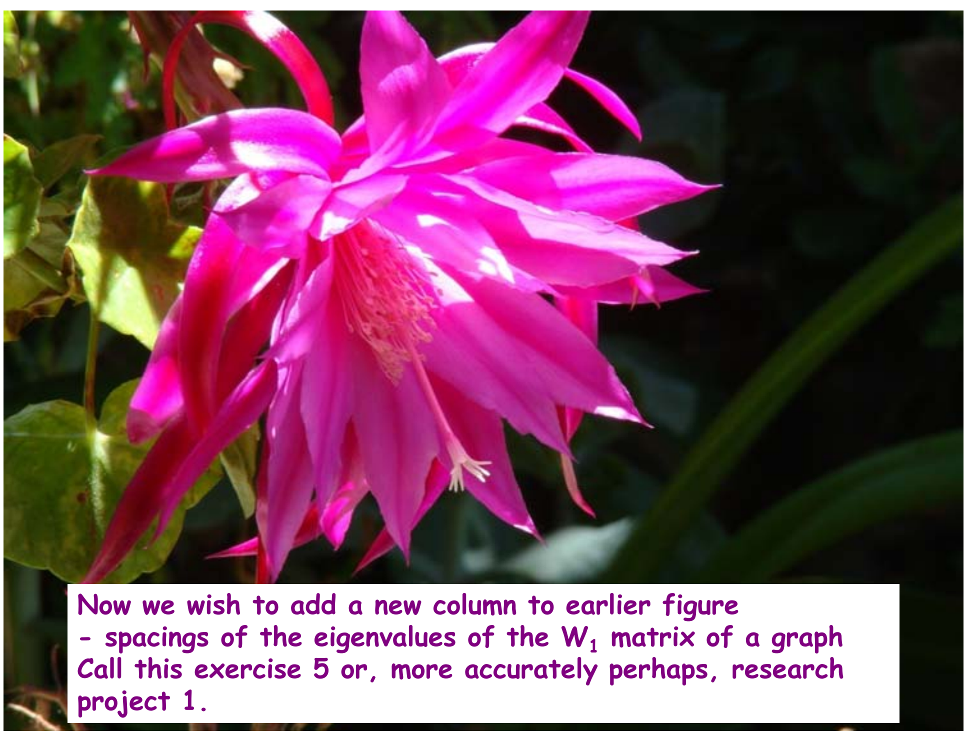
The Figure is from from Bohigas, Haq, and Pandey, Fluctuation properties of nuclear energy levels and widths: comparison of theory with experiment, in K.H. Bockhoff (Ed.), *Nuclear Data for Science and Technology*, Reidel, Dordrecht, 1983)
Level spacing histogram for
(a) 166Er and
(b) a nuclear data ensemble.

Wigner surmise for spacings of spectra of random symmetric real matrices

This means that you arrange the eigenvalues) E_i in decreasing order: $E_1 \geq E_2 \geq \cdot \quad \cdot \quad \cdot \geq E_n$. Assume that the eigenvalues are normalized so that the mean of the level spacings $|E_i - E_{i+1}|$ is 1.

Wigner's Surmise from 1957 says the level (eigenvalue) spacing histogram is ≈ the graph of the function $\frac{1}{2}\pi x \exp(-\pi x^2/4)$,if the mean spacing is 1. In 1960, Gaudin and Mehta found the correct distribution function which is close to Wigner's. The correct graph is labeled GOE in the Figure preceding. Note the level repulsion indicated by the vanishing of the function at the origin. Also in the preceding Figure, we see the Poisson density which is e^{-x}.

A reference is Mehta, *Random Matrices*.



Here although W_1 is not symmetric, the nearest neighbor spacing (i.e., histogram of minimum distances between eigenvalues) is also of interest.

$\nexists$ many references on the study of spacings of spectra of non-Hermitian or non-symmetric matrices. I did find one: P. LeBoef, Random matrices, random polynomials, and Coulomb systems.
He studies the ensemble of matrices introduced by J. Ginibre, J. Math. Phys. 6, 440 (1965).

An approximation to the distribution of spacings of eigenvalues of a complex matrix (analogous to the Wigner surmise for Hermitian matrices) is:

$$4\Gamma\left(\frac{5}{4}\right)^4 s^3 e^{-\Gamma\left(\frac{5}{4}\right)^4 s^4}$$

Since our matrix is real, this will probably not be the correct Wigner surmise.

I haven't done this experiment yet. In what follows, I just plot the reciprocals of the eigenvalues of W_1 - the poles of Ihara zeta for various graphs. The distribution looks rather different than that of a random real matrix with the properties of W_1.

Statistics of the poles of Ihara zeta or reciprocals of eigenvalues of the Edge Matrix W_1

Define W_1 to be the 0,1 matrix you get from W by setting all non-0 entries of W to be 1.

Theorem. $\zeta(u,X)^{-1}=\det(I-W_1u)$.

Corollary. The poles of Ihara zeta are the reciprocals of the eigenvalues of W_1.

The pole R of zeta is:

R=1/Perron-Frobenius eigenvalue of W_1.

Properties of W_1

1) $W_1 = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}$ B and C symmetric real, A real

2) Row sums of entries are q_j+1=degree vertex which is start of edge j.

Poles Ihara Zeta are in region $q^{-1} \le R \le |u| \le 1$, q+1=maximum degree of vertices of X.

Theorem of Kotani and Sunada

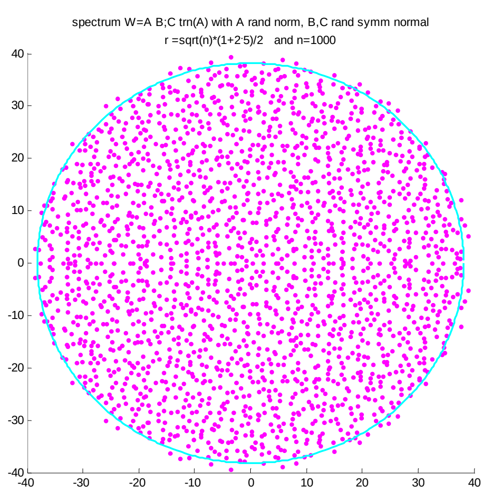
If p+1=min vertex degree, and q+1=maximum vertex degree, non-real poles u of zeta satisfy

$$\frac{1}{\sqrt{q}} \le |u| \le \frac{1}{\sqrt{p}}$$

Kotani & Sunada, *J. Math. Soc. U. Tokyo*, 7 (2000)

or see my manuscript on my website: www.math.ucsd.edu\~aterras\newbook.pdf

Spectrum of Random Matrix with Properties of W_1-matrix



$W_1 = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}$

B and C symmetric

Girko circle law with a symmetry with respect to real axis since our matrix is real. (Girko, *Theory Prob. Appl.* 22 (1977))

We used Matlab command randn(1000) to get A,B,C matrices with random normally distributed entries mean 0 std dev 1.

What is the meaning of the RH for irregular graphs?

For irregular graph, natural change of variables is $u=R^s$, where

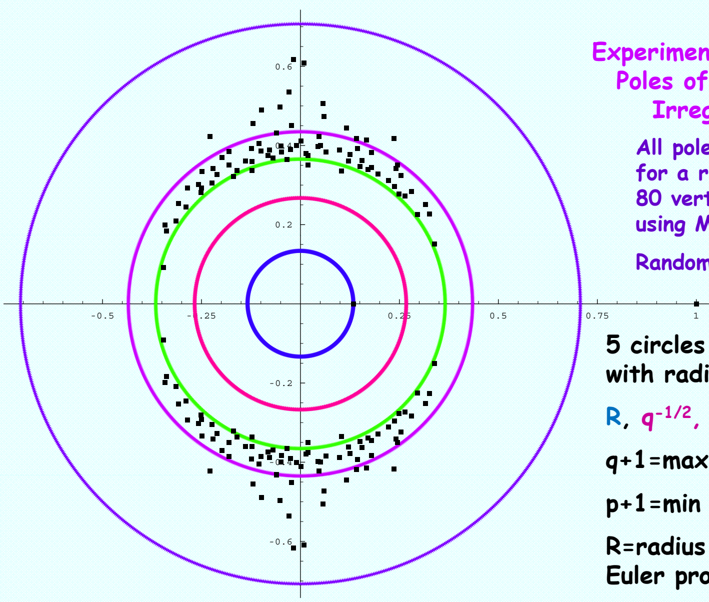
R = radius of convergence of Dirichlet series for Ihara zeta.

Note: R is closest pole of zeta to 0. No functional equation.

Then the critical strip is $0 \le \text{Re}\, s \le 1$ and translating back to u-variable. In the q+1-regular case, R=1/q.

Graph theory RH:

$$\zeta(u) \text{ is pole free in } R < |u| < \sqrt{R}$$



Experiment on Locations of Poles of Ihara Zeta of Irregular Graphs

All poles but -1 of $\zeta_X(u)$ for a random graph with 80 vertices denoted by ■ using Mathematica:

RandomGraph[80,1/10]

5 circles centered at 0 with radii

R, $q^{-1/2}$, $R^{1/2}$, $(pq)^{-1/4}$, $p^{-1/2}$

q+1=max degree,

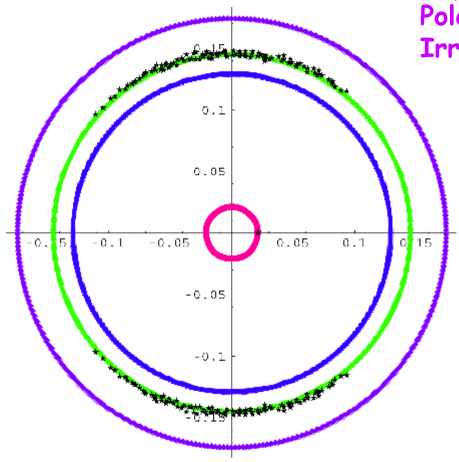
p+1=min degree,

R=radius of convergence of Euler product for $\zeta_X(u)$

RH is false but poles are not far inside circle of radius $R^{1/2}$

RandomGraph[80,1/10] means probability of edge between 2 vertices =1/10.

Experiment on Locations of Poles of Ihara Zeta of Irregular Graphs



All poles except -1 of $\zeta_X(u)$ for a random graph with 100 vertices are denoted ■, using Mathematica

RandomGraph[100,1/2]

Circles centered at 0 with radii

R, $q^{-1/2}$, $R^{1/2}$, $p^{-1/2}$

q+1=max degree,

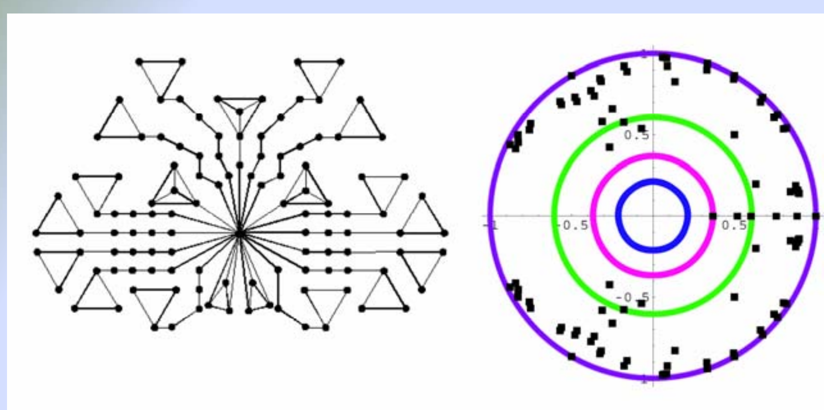
p+1=min degree

R=radius of convergence

of product for $\zeta_X(u)$

RH is false maybe not as false as in previous example with probability 1/10 of an edge rather than ½.

Poles clustering on RH circle (green)

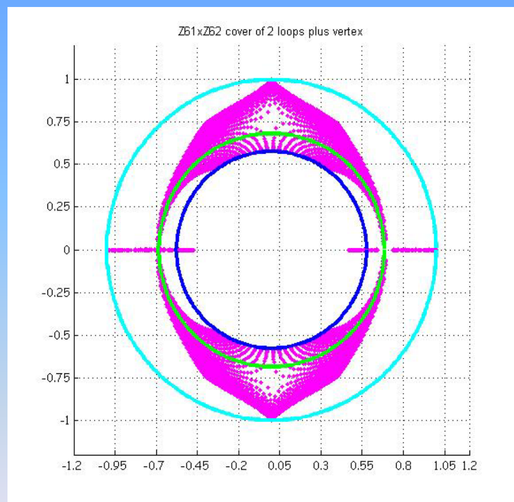


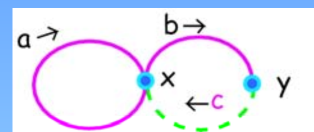
Matthew Horton's Graph has $1/R \cong e$ to 7 digits.

Poles of Ihara zeta are boxes on right. Circles have radii $R, q^{-\frac{1}{2}}, R^{\frac{1}{2}}, p^{-\frac{1}{2}}$, if q+1=max deg, p+1=min deg. Here

The RH is false. Poles more spread out over plane.

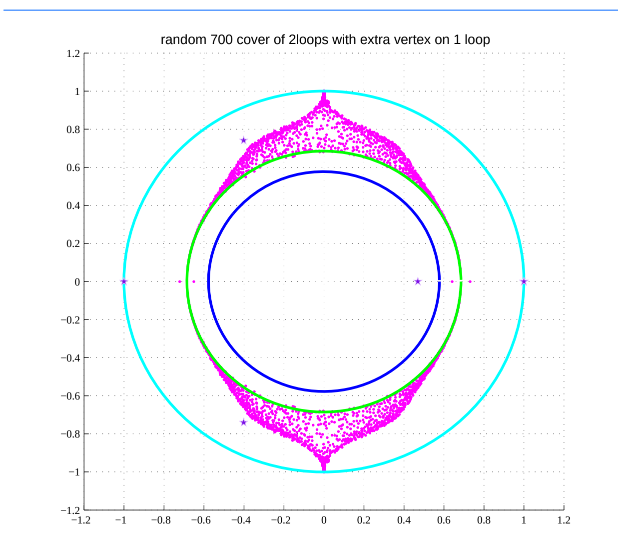
Poles of Ihara Zeta for a $\mathbb{Z}_{61}\times\mathbb{Z}_{62}$-Cover of 2 Loops + Extra Vertex are pink dots

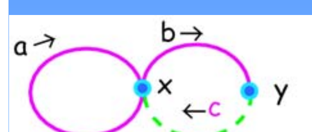


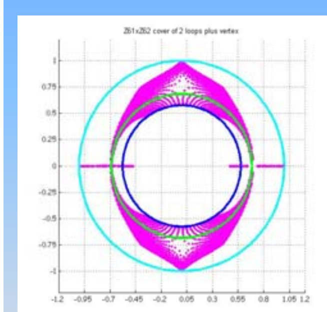


Circles Centers (0,0); Radii: $3^{-1/2}$, $R^{1/2}$,1; R ≅.47

RH very False







Z is random 700 cover of 2 loops plus vertex graph in picture.

The pink dots are at poles of ζ_Z. Circles have radii $q^{-1/2}$, $R^{1/2}$, $p^{-1/2}$, with q=3, p=1, R ≅ .4694. RH approximately True.

References: 3 papers with Harold Stark in *Advances in Math*.

- Paper with Matthew Horton & Harold Stark in Snowbird Proceedings, Contemporary Mathematics, Volume 415 (2006)
  Quantum Graphs and Their Applications, *Contemporary Mathematics*, v. 415, AMS, Providence, RI 2006.
- See my draft of a book:
  www.math.ucsd.edu/~aterras/newbook.pdf
- Draft of new paper joint with Horton & Stark: also on my website
  www.math.ucsd.edu/~aterras/cambridge.pdf
- There was a graph zetas special session of this AMS meeting – many interesting papers some on my website.
- For work on directed graphs, see Matthew Horton, Ihara zeta functions of digraphs, *Linear Algebra and its Applications,* 425 (2007) 130–142.
- work of Angel, Friedman and Hoory giving analog of Alon conjecture for irregular graphs, implying our Riemann Hypothesis (see Joel Friedman's website: www.math.ubc.ca/~jf)