

# a stroll through the zeta garden

Lecture 2: Ruelle Zeta and Prime Number  
Theorem for Graphs

10784.36  
5.9 ÷ 1  
2.71828

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## Ihara Zeta Function

$$\zeta_V(u, X) = \prod_{[C] \text{ primes in } X} (1 - u^{V(C)})^{-1}$$

Ihara's Theorem (Bass, Hashimoto, etc.)

$A$  = adjacency matrix of  $X$

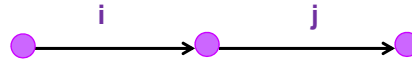
$Q$  = diagonal matrix  $j$ th diagonal entry = degree  $j$ th vertex  $-1$ ;

$r$  = rank fundamental group =  $|E| - |V| + 1$

$$\zeta(u, X)^{-1} = (1 - u^2)^{r-1} \det(I - Au + Qu^2)$$

## The Edge Matrix $W_1$

Define  $W_1$  to be the  $2|E| \times 2|E|$  matrix with  $i j$  entry 1 if edge  $i$  feeds into edge  $j$ , (end vertex of  $i$  is start vertex of  $j$ ) provided  $i \neq$  opposite of  $j$ , otherwise the  $i j$  entry is 0.



**Theorem.**  $\zeta(u, X)^{-1} = \det(I - W_1 u)$ .

**Corollary.** The poles of Ihara zeta are the reciprocals of the eigenvalues of  $W_1$ . Recall that  $R$  = radius of convergence of Dirichlet series for Ihara zeta.

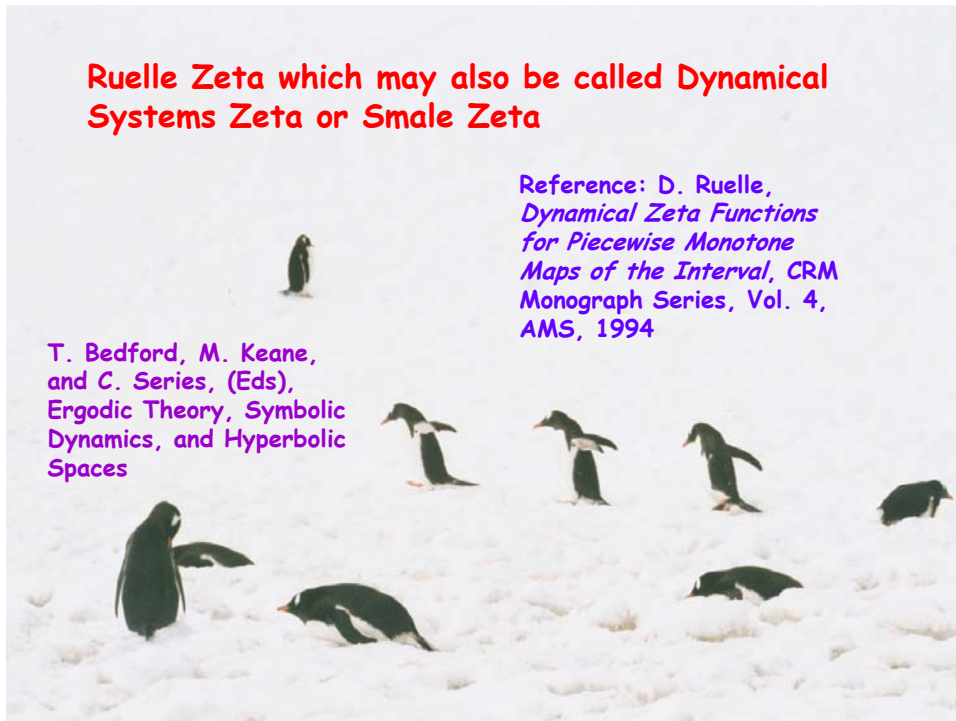
Note:  $R$  is closest pole of zeta to 0.

The pole  $R$  of zeta is:  $R = 1/\text{Perron-Frobenius eigenvalue of } W_1$ . See Horn & Johnson, *Matrix Analysis*

## Ruelle Zeta which may also be called Dynamical Systems Zeta or Smale Zeta

Reference: D. Ruelle, *Dynamical Zeta Functions for Piecewise Monotone Maps of the Interval*, CRM Monograph Series, Vol. 4, AMS, 1994

T. Bedford, M. Keane, and C. Series, (Eds), *Ergodic Theory, Symbolic Dynamics, and Hyperbolic Spaces*



Ruelle's motivation for his definition came partially from Artin and Mazur, *Annals of Math.*, 81 (1965).

They based their zeta on the zeta function of a projective non-singular algebraic variety  $V$  of dimension  $n$  over a finite field  $k$  with  $q$  elements.

If  $N_m$  is the number of points of  $V$  with coordinates in the degree  $m$  extension field of  $k$ , the zeta function of  $V$  is:

$$Z_V(z) = \exp\left(\sum_{m \geq 1} \frac{N_m z^m}{m}\right).$$

$N_m = |\text{Fix}(F^m)|$ , where  $F$  is the Frobenius map taking a point with coordinates  $x_i$  to the point with coordinates  $(x_i)^q$ .

Weil conjectures, proved by Deligne, say

$$Z_V(z) = \prod_{j=0}^{2n} P_j(z)^{(-1)^{j+1}},$$

where the  $P_j$  are polynomials with zeros of absolute value  $q^{-j/2}$ . Moreover the  $P_j$  have a cohomological meaning as  $\det(1 - zF^* | H^j(V))$ .

Artin and Mazur replace the Frobenius of  $V$  with a diffeomorphism  $f$  of a smooth compact manifold  $M$  - defining their zeta function

$$\zeta(z) = \exp\left(\sum_{m=1}^{\infty} \frac{z^m}{m} |\text{Fix}(f^m)|\right).$$

### Ruelle zeta function

Suppose  $M$  is a compact manifold and  $f: M \rightarrow M$ .

Assume the following set finite:  $\text{Fix}(f^m) = \{x \in M \mid f^m(x) = x\}$ .

1st type Ruelle zeta is defined for matrix-valued function  $\varphi: M \rightarrow \mathbb{C}^{d \times d}$

$$\zeta(s) = \exp \left\{ \sum_{m \geq 1} \frac{z^m}{m} \sum_{x \in \text{Fix}(f^m)} \text{Tr} \left( \prod_{k=0}^{m-1} \varphi(f^k(x)) \right) \right\}$$

A special case:  $\varphi = 1$

$$\zeta(z) = \exp \left( \sum_{m=1}^{\infty} \frac{z^m}{m} |\text{Fix}(f^m)| \right).$$

$I$  = finite non-empty set (our alphabet). For a graph  $X$ ,  $I$  is the set of directed edges.

The transition matrix  $t$  is a matrix of 0's and 1's with indices in  $I$ . In the case of a graph  $X$ ,  $t$  is the 0,1 edge matrix  $W_1$  defined earlier, which has  $i, j$  entry 1 if edge  $i$  feeds into edge  $j$  (meaning that terminal vertex of  $i$  is the initial vertex of  $j$ ) provided edge  $i$  is not the inverse of edge  $j$ .

Note:  $I^{\mathbb{Z}}$  is compact and so is the closed subset

$$\Lambda = \left\{ (\xi_k)_{k \in \mathbb{Z}} \mid t_{\xi_k \xi_{k+1}} = 1, \forall k \right\}.$$

In the graph case  $\xi \in \Lambda$  corresponds to a path without backtracking.

A continuous function  $\tau: \Lambda \rightarrow \Lambda$  such that  $\tau(\xi)_k = \xi_{k+1}$  is called a **subshift of finite type**.

**Prop. 1. (Bowen & Lanford).** The Ihara zeta is the reciprocal of a polynomial:

$$\begin{aligned} \zeta(z) &= \exp \left( \sum_{m \geq 1} \frac{z^m}{m} \text{Tr}(t^m) \right) \\ &= \det(I - zt)^{-1}. \end{aligned}$$

**Proof.**

By the first exercise below, we have the first equality in the theorem. Then  $|\text{Fix}(\tau^m)| = \text{Tr}(t^m)$ . This implies using the 2nd exercise:

$$\zeta(z) = \exp(\text{Tr}(-\log(1-zt))) = \det(I-zt)^{-1}.$$

**Exercise A.** Show that in the graph case,  $|\text{Fix}(\tau^m)|$  = the number of length  $m$  closed paths without backtracking or tails in the graph  $X$  with  $t = W_1$ , from our previous discussion of graphs.

**Exercise B.** Show that  $\exp(\text{Tr}A) = \det(\exp A)$  for any matrix  $A$ . Hint: Use the fact that there is a non-singular matrix  $B$  such that  $BAB^{-1} = T$  is upper triangular.

Define  $R$  as the radius of convergence of the Ihara zeta. It is also the closest pole to 0. For a  $(q+1)$ -regular graph,  $R = 1/q$ .

Define  $\Delta$  as the g.c.d. of the prime lengths.

**Theorem.** (Graph prime number theorem)

If the graph is connected and  $m$  divides  $\Delta$ ,

$$\pi(m) = \#\{\text{prime paths of length } m\} \sim \Delta R^{-m}/m, \\ \text{as } m \rightarrow \infty$$

**Proof.**

If  $N_m = \# \{ \text{closed paths } C, \text{ length } m, \text{ no backtrack, no tails} \}$

we have 
$$u \frac{d \log \zeta(u, X)}{du} = \sum_{m=1}^{\infty} N_m u^m$$

Then  $\zeta(u, X)^{-1} = \det(\mathbf{I} - W_1 u)$  implies

$$\begin{aligned} u \frac{d \log \zeta(u)}{du} &= \sum_{n \geq 1} N_n u^n = - \sum_{\lambda \in \text{Spec}(W_1)} u \frac{d}{du} \log(1 - \lambda u) \\ &= \sum_{m \geq 1} \left( \sum_{\lambda \in \text{Spec}(W_1)} \lambda^m \right) u^m \end{aligned}$$

$$N_m = \sum_{\lambda \in \text{Spec}(W_1)} \lambda^m$$

The dominant terms in this sum are those coming from the eigenvalues  $\lambda$  of  $W_1$  with  $|\lambda| = 1/R$ .

$$N_m = \sum_{\lambda \in \text{Spec}(W_1)} \lambda^m$$

**Theorem.** (Kotani and Sunada using Perron-Frobenius Thm.)

The poles of  $\zeta$  on  $|u| = R$  have the form  $R e^{2\pi i a / \Delta}$ , where  $a = 1, 2, \dots, \Delta$ .

By this theorem,

$$N_m \sim \sum_{\substack{\lambda \in \text{spec}(W_1) \\ |\lambda| \text{ maximal}}} \lambda^m = |R|^{-m} \sum_{a=1}^{\Delta} e^{2\pi i \frac{am}{\Delta}}$$

The sum is 0 unless  $m$  divides  $\Delta$ , when it is  $\Delta R^{-m}$ .

Next we need a formula to relate  $N$  and  $\pi$

$$\zeta(u, X) = \prod_{n \geq 1} (1 - u^n)^{-\pi(n)}$$

This implies

$$u \frac{d \log \zeta(u, X)}{du} = \sum_{m \geq 1} \sum_{d|m} d \pi(d) u^m$$

$$N_m = \sum_{d|m} d \pi(d)$$

Möbius inversion implies

$$\pi(m) = \frac{1}{m} \sum_{d|m} \mu\left(\frac{m}{d}\right) N_d$$

Note that  $\pi(1)=0$  so that when  $m$  is prime  $p$ , we have  $N_p = \pi(p)$ .

Thus the prime number theorem follows. Exercise. Fill in the details in this proof. 

Tetrahedron example

$$x \frac{d}{dx} \log \zeta_x(x) = \sum_{m \geq 1} N_m x^m$$

$$= 24x^3 + 24x^4 + 96x^6 + 168x^7 + 168x^8 + 528x^9 + 1200x^{10} + 1848x^{11} + 3960x^{12} + 8736x^{13} + 16128x^{14} + 31944x^{15} + 66888x^{16}$$

- $\Rightarrow$  8 prime paths of length 3 on the tetrahedron. Check it! We count 4 plus their inverses to get 8.
- $\Rightarrow$  6 paths of length 4. We agree.
- $\Rightarrow$  0 paths of length 5
- $\Rightarrow$  16 paths of length 6. That is harder to check.

**Question:** 528 is not divisible by 9. Shouldn't it be?

**Answer:** You only know  $m$  divides  $N_m$  when  $m$  is prime.

$$\pi(m) = \frac{1}{m} \sum_{d|m} \mu\left(\frac{m}{d}\right) N_d$$



