

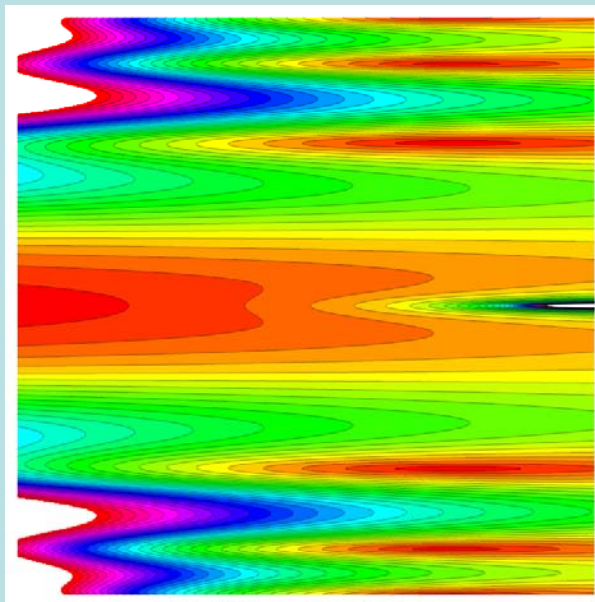
a stroll through the zeta garden

Lecture 1: Riemann, Dedekind, Selberg, and Ihara Zetas

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2.7182818
9 ÷ 1



Audrey Terras
U.C.S.D.
2008



more details can be
found in

my webpage:

[www.math.ucsd.edu
/~aterras/
newbook.pdf](http://www.math.ucsd.edu/~aterras/newbook.pdf)

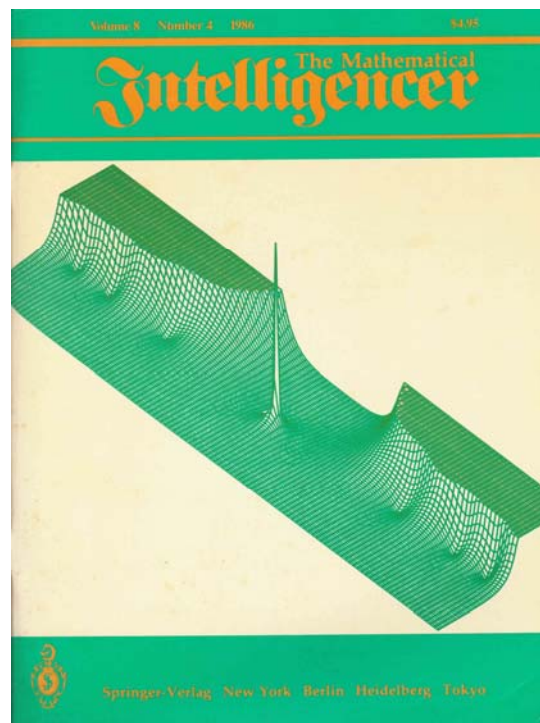
**First the Riemann
Zeta**

The **Riemann zeta function** for $\text{Re}(s) > 1$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=\text{prime}} (1 - p^{-s})^{-1}.$$

- ⌘ **Riemann (1859)** extended to all complex s with pole at $s=1$
- ⌘ **Functional equation** relates value at s and $1-s$
$$\Lambda(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s) = \Lambda(1-s)$$
- ⌘ **Riemann hypothesis** (non real zeros $\zeta(s)=0$ are on the line $\text{Re}(s)=1/2$). This now checked for 10^{13} billion zeros. (work of X. Gourdon and P. Demichel). See Ed Pegg Jr.'s website.

Graph of $z=|\zeta(x+iy)|$ showing the pole at $x+iy=1$ and the first 6 zeros which are on the line $x=1/2$, of course. The picture was made by D. Asimov and S. Wagon to accompany their article on the evidence for the Riemann hypothesis as of 1986.



⌘ duality between primes & complex zeros of zeta using Hadamard product over zeros

⌘ prime number theorem

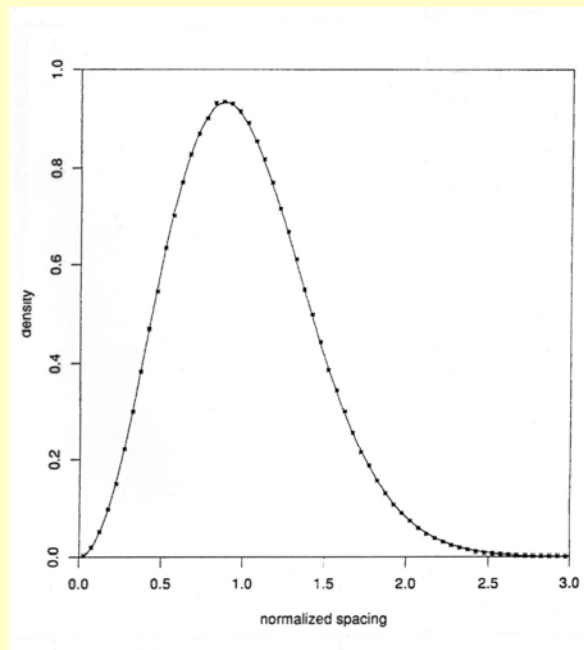
$$\#\{p = \text{prime} \mid p \leq x\} \sim \frac{x}{\log x}, \text{ as } x \rightarrow \infty$$

⌘ statistics of Riemann zero spacings studied by Odlyzko (GUE) proved by Hadamard and de la Vallée Poussin (1896-1900) Their proof requires complex analysis

www.dtc.umn.edu/~odlyzko/doc/zeta.htm

⌘ B. Conrey, The Riemann Hypothesis, Notices, A.M.S., March, 2003

Odlyzko's Comparison of Spacings of 7.8×10^7 Zeros of Zeta at heights $\approx 10^{20}$ & Eigenvalues of Random Hermitian Matrix (GUE).



Many Kinds of Zeta

Dedekind zeta of an algebraic number field F such as $\mathbb{Q}(\sqrt{2})$, where primes become prime ideals \mathfrak{p} and infinite product of terms $(1-N\mathfrak{p}^{-s})^{-1}$, where $N\mathfrak{p}$ = norm of \mathfrak{p} = $\#(O/\mathfrak{p})$, O =ring of integers in F

Functional Equations: $\zeta_K(s)$ related to $\zeta_K(1-s)$
Hecke

Values at 0: $\zeta(0) = -1/2$, $\zeta_K(0) = -hR/w$

h = **class number** (measures how far O_K is from having unique factorization) ($=1$ for $K=\mathbb{Q}(\sqrt{2})$)

R = **regulator** (determinant of logs of units)
 $= \log(1+\sqrt{2})$ when $K=\mathbb{Q}(\sqrt{2})$

w = **number of roots of unity** in K is 2 , when $K=\mathbb{Q}(\sqrt{2})$

Statistics of Prime Ideals and Zeros

- ✳ from information on zeros of $\zeta_K(s)$ obtain
prime ideal theorem

$$\#\{ \mathfrak{p} \text{ prime ideal in } O_K \mid N\mathfrak{p} \leq x \} \sim \frac{x}{\log x}, \text{ as } x \rightarrow \infty$$

- ✳ there are an infinite number of primes such that $\left(\frac{2}{p}\right)=1$.
- ✳ Dirichlet theorem: there are an infinite number of primes p in the progression $a, a+d, a+2d, a+3d, \dots$, when $\text{g.c.d.}(a,d)=1$.
- ✳ **Riemann hypothesis still open:**
GRH or ERH: $\zeta_K(s)=0$ implies $\text{Re}(s)=1/2$,
assuming s is not real.

Selberg zeta associated to a compact Riemannian manifold $M = \Gamma \backslash \mathbb{H}$, \mathbb{H} = upper half plane with

$$ds^2 = (dx^2 + dy^2)y^{-2}$$

Γ = discrete subgroup of group of real fractional linear transformations

primes = primitive closed geodesics C in M of length $\nu(C)$, (primitive means only go around once)

$$Z(s) = \prod_{[C]} \prod_{j \geq 0} (1 - e^{-(s+j)\nu(C)})$$

Duality between spectrum Δ on M & lengths closed geodesics in M
 $Z(s+1)/Z(s)$ is more like Riemann zeta

Realize M as quotient of **upper half plane**

$$\mathbb{H} = \{x+iy \mid x, y \in \mathbb{R}, y > 0\}.$$

Non-Euclidean distance: $ds^2 = y^{-2}(dx^2 + dy^2)$

ds is invariant under

$$z \rightarrow (az+b)/(cz+d),$$

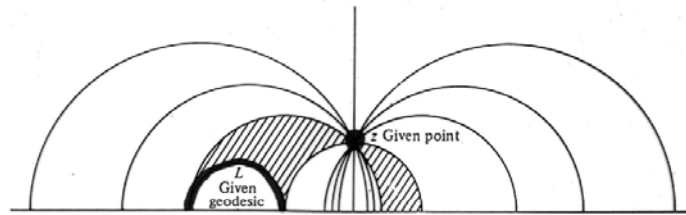
for a, b, c, d real and $ad - bc = 1$. **PSL(2, \mathbb{R}).**

Corresponding **Laplacian** $\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$.

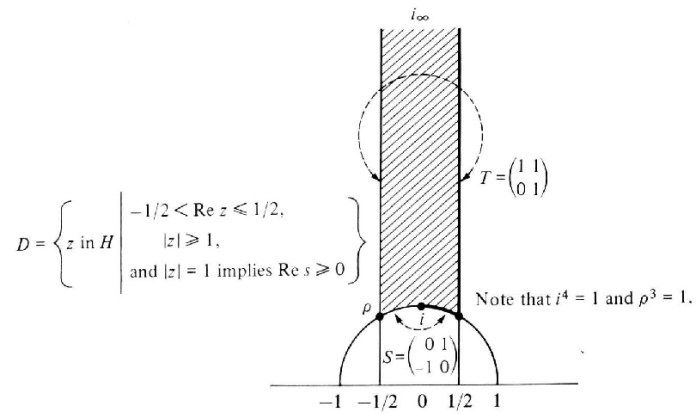
also commutes with action of **PSL(2, \mathbb{R}).**

The curves (geodesics) minimizing arc length are circles and lines in \mathbb{H} orthogonal to real axis. Non-Euclidean geometry.

Picture of the Failure of Euclid's 5th Postulate



View compact or finite volume manifold as $\Gamma \backslash H$, where Γ is a discrete subgroup of $PSL(2, \mathbb{R})$. For example,
 $\Gamma = PSL(2, \mathbb{Z})$, **the modular group**.
 Fundamental Domain is a non-Euclidean triangle.



A **geodesic** in $\Gamma \backslash \mathbb{H}$ comes from one in \mathbb{H} . One can show that the endpoints of such in \mathbb{R} (the real line = the boundary of \mathbb{H}) are fixed by hyperbolic elements of Γ ;

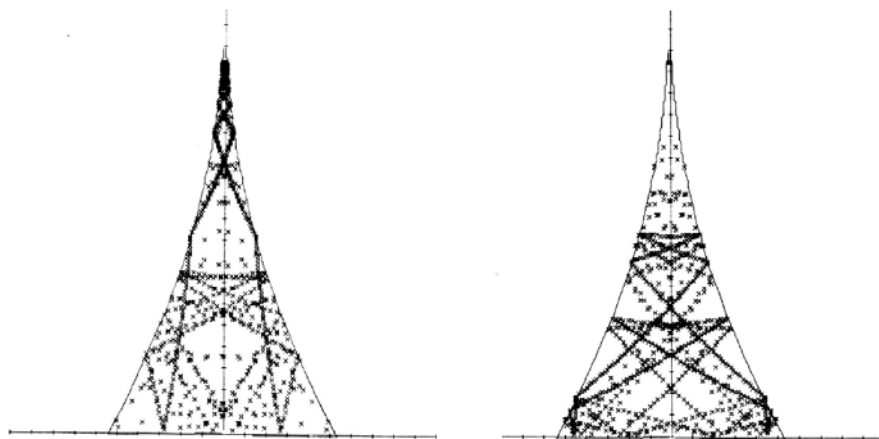
i.e., those $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with trace $= a+d > 2$.

Primitive closed geodesics are traversed only once. They correspond to hyperbolics that generate their centralizer in Γ .

See my book *Harmonic Analysis on Symmetric Spaces*, Vol. I, for more information.

Next a picture of images of points on 2 geodesics circles after mapping them into a fundamental domain of $\text{PSL}(2, \mathbb{Z})$

Images of points on 2 geodesics circles after mapping them into a fundamental domain of $\text{PSL}(2, \mathbb{Z})$



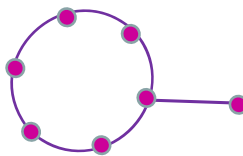
Ihara Zeta Functions of Graphs

We will see they have similar properties and applications to those of number theory.
 But first we need to figure out what primes in graphs are.
 This requires us to label the edges.

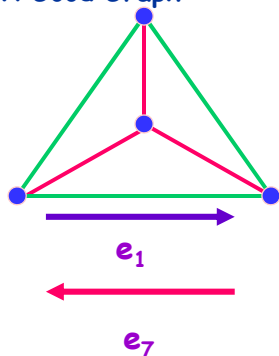
Labeling Edges of Graphs

X = finite connected (not-necessarily regular graph).
 Usually we assume: graph is not a cycle or a cycle with degree 1 vertices

A Bad Graph



A Good Graph



Orient the edges. Label them as follows.
 Here the inverse edge has opposite orientation.

$$e_1, e_2, \dots, e_{|E|},$$

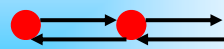
$$e_{|E|+1} = e_1^{-1}, \dots, e_{2|E|} = e_{|E|}^{-1}$$

Primes in Graphs

(correspond to geodesics in compact manifolds)
 are equivalence classes $[C]$ of closed backtrackless
 tailless primitive paths C

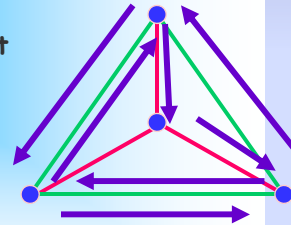
DEFINITIONS

backtrack



equivalence class: change starting point

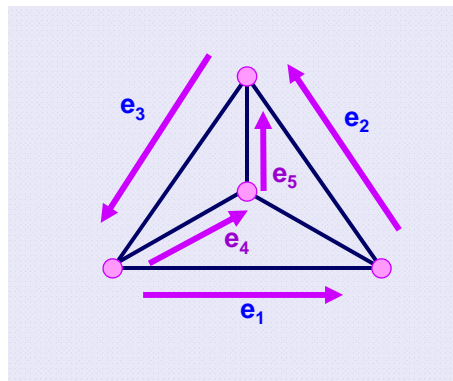
tail (backtrack if you change
 starting vertex)



a path with a backtrack & a tail

non-primitive: go around path more than once

EXAMPLES of Primes in a Graph



$$[C] = [e_1 e_2 e_3]$$

$$[D] = [e_4 e_5 e_3]$$

$$[E] = [e_1 e_2 e_3 e_4 e_5 e_3]$$

$$v(C)=3, v(D)=4, v(E)=6$$

$$E=CD$$

another prime $[C^n D]$, $n=2,3,4, \dots$
 infinitely many primes

Ihara Zeta Function

$$\zeta_v(u, X) = \prod_{[C] \text{ primes in } X} (1 - u^{v(c)})^{-1}$$

Ihara's Theorem (Bass, Hashimoto, etc.)

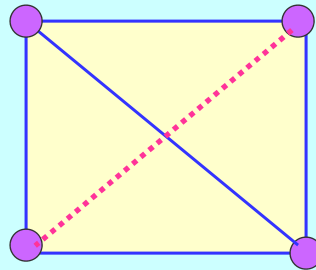
A = adjacency matrix of X

Q = diagonal matrix j th diagonal entry
= degree j th vertex -1 ;

r = rank fundamental group = $|E| - |V| + 1$

$$\zeta(u, X)^{-1} = (1 - u^2)^{r-1} \det(I - Au + Qu^2)$$

2 Examples
 K_4 and
 $X = K_4$ -edge



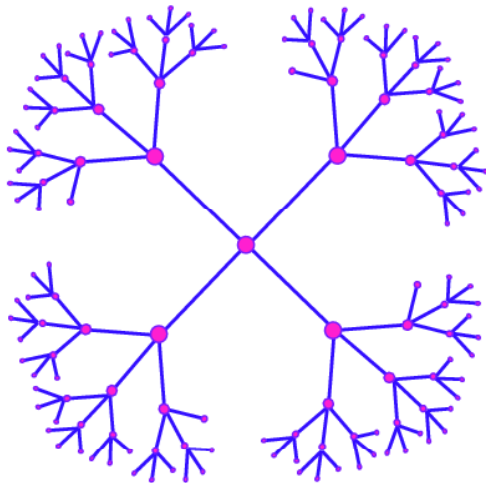
$$\zeta(u, K_4)^{-1} = (1 - u^2)^2 (1 - u)(1 - 2u)(1 + u + 2u^2)^3$$

$$\zeta(u, X)^{-1} = (1 - u^2)(1 - u)(1 + u^2)(1 + u + 2u^2)(1 - u^2 - 2u^3)$$

Remarks

Ihara defined the zeta as a product over p-adic group elements.
 Serre saw the graph theory interpretation.
 Hashimoto and Bass extended the theory.

- Later we may outline Bass's proof of Ihara's theorem. It involves defining an edge zeta function with more variables
- Another proof of the Ihara theorem for regular graphs uses the Selberg trace formula on the universal covering tree. For the trivial representation, see A.T., *Fourier Analysis on Finite Groups & Applies*; for general case, see and Venkov & Nikitin, *St. Petersburg Math. J.*, 5 (1994)



Part of the universal covering tree T_4 of a 4-regular graph.

A tree has no closed paths and is connected.

T_4 is infinite and so I cannot draw it.

It can be identified with the 3-adic quotient $SL(2, \mathbb{Q}_3)/SL(\cdot, \mathbb{Z}_3)$

A finite 4-regular graph is a quotient of this tree T_4 modulo Γ =the fundamental group of the graph X

For $q+1$ - regular graph, meaning that each vertex has $q+1$ edges coming out

$u=q^{-s}$ makes Ihara zeta more like Riemann zeta.

$f(s)=\zeta(q^{-s})$ has a functional equation relating $f(s)$ and $f(1-s)$.

Riemann Hypothesis (RH)

says $\zeta(q^{-s})$ has no poles with $0 < \text{Re } s < 1$ unless $\text{Re } s = \frac{1}{2}$.

RH means graph is Ramanujan i.e., non-trivial spectrum of adjacency matrix is contained in the spectrum for the universal covering tree which is the interval $(-2\sqrt{q}, 2\sqrt{q})$

[see Lubotzky, Phillips & Sarnak, *Combinatorica*, 8 (1988)].

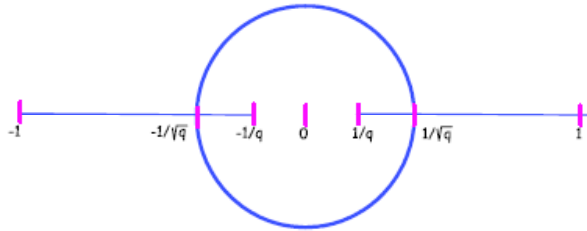
**Ramanujan graph is a good expander
(good gossip network)**

What is an expander graph X?

4 Ideas

- 1) **spectral property** of some matrix associated to our finite graph X
Choose one of 3:
 - ❖ Adjacency matrix A,
 - ❖ Laplacian $D-A$, or $I-D^{-1/2}AD^{-1/2}$, D =diagonal matrix of degrees
 - ❖ edge matrix W_1 for X (to be defined)Lubotzky: **Spectrum for X SHOULD BE INSIDE spectrum of analogous operator on universal covering tree for X.**
- 2) X behaves like a **random graph**.
- 3) **Information is passed quickly in the gossip network based on X.**
- 4) **Random walker** on X gets lost **FAST**.

Possible Locations of Poles u of $\zeta(u)$ for $q+1$ Regular Graph



$1/q$ always the closest pole to 0 in absolute value.

Circle of radius $1/\sqrt{q}$ from the RH poles.

Real poles ($\neq \pm q^{-1/2}, \pm 1$) correspond to non-RH poles.

Alon conjecture for regular graphs says RH \cong true for "most" regular graphs.

See Joel Friedman's web site for proof (www.math.ubc.ca/~jf)

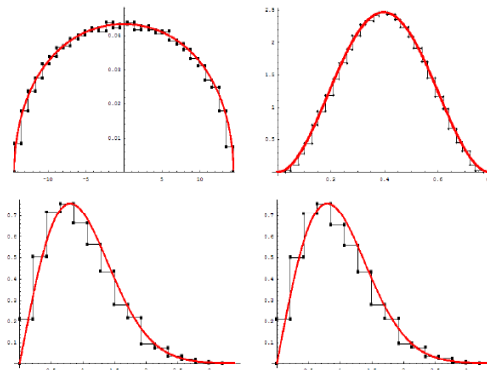
See Steven J. Miller's web site:

(www.math.brown.edu/~sjmiller) for a talk on experiments leading to conjecture that the percent of regular graphs satisfying RH approaches 27% as # vertices $\rightarrow \infty$, via Tracy-Widom distribution.

Derek Newland's Experiments

Graph analog of Odlyzko experiments for Riemann zeta

Mathematica experiment with random 53-regular graph - 2000 vertices



Spectrum adjacency matrix $\zeta(52^{-s})$ as a function of s

Top row = distributions for eigenvalues of A on left and imaginary parts of the zeta poles on right $s = \frac{1}{2} + it$.

Bottom row = their respective normalized level spacings.

Red line on bottom: Wigner surmise GOE, $\gamma = (\pi x/2) \exp(-\pi x^2/4)$.

What is the meaning of the RH for irregular graphs?

For irregular graph, natural change of variables is $u=R^s$, where
 R = radius of convergence of Dirichlet series for Ihara zeta.

Note: R is closest pole of zeta to 0. No functional equation.

Then the critical strip is $0 \leq \text{Re}s \leq 1$ and translating back to u -
variable. In the $q+1$ -regular case, $R=1/q$.

Graph theory RH:

$\zeta(u)$ is pole free in $R < |u| < \sqrt{R}$

To investigate this, we need to define
the edge matrix W_1 . See Lecture 2.

