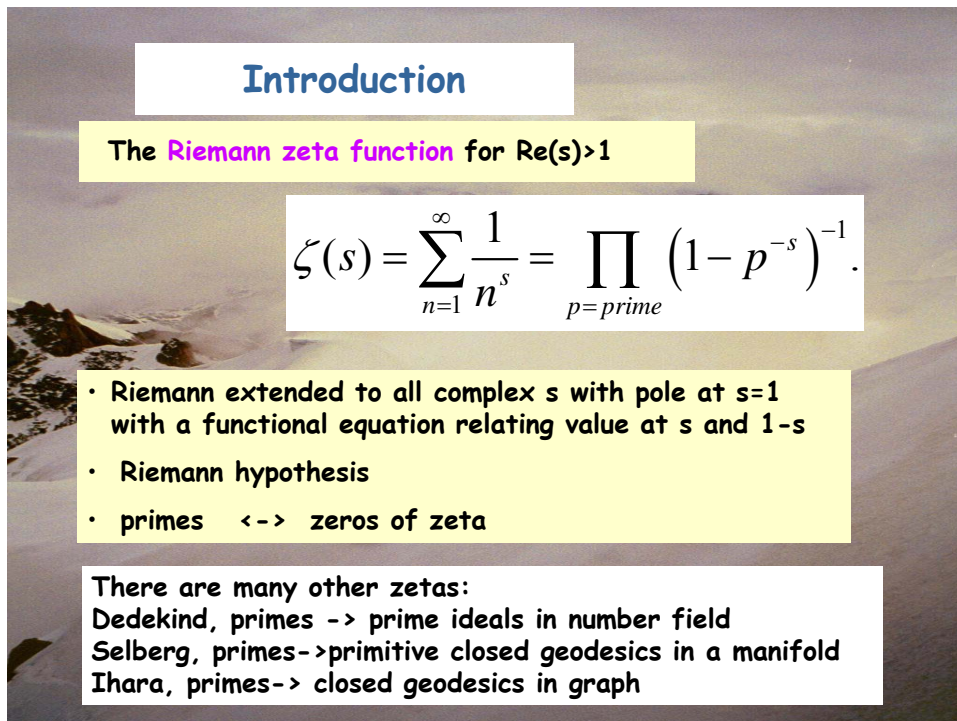


**What is the Riemann Hypothesis for Zeta Functions of Irregular Graphs?**

Audrey Terras  
Banff  
February, 2008

Joint work with H. M. Stark,  
M. D. Horton, etc.



**Introduction**

The **Riemann zeta function** for  $\text{Re}(s) > 1$

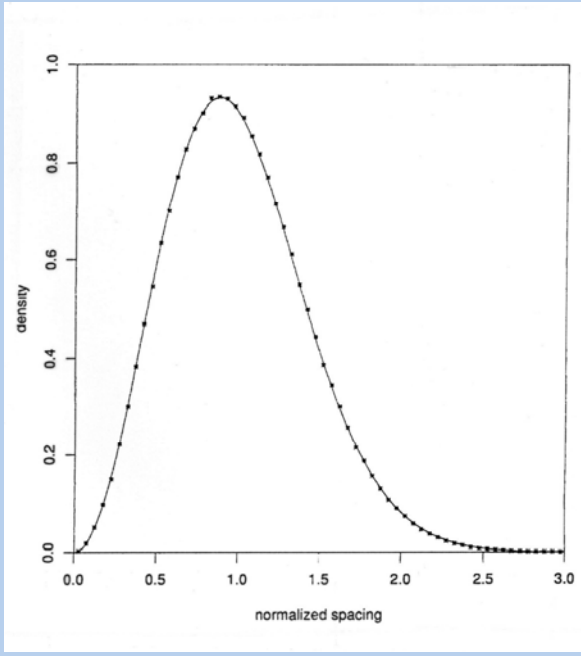
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=\text{prime}} (1 - p^{-s})^{-1}.$$

- Riemann extended to all complex  $s$  with pole at  $s=1$  with a functional equation relating value at  $s$  and  $1-s$
- Riemann hypothesis
- primes  $\leftrightarrow$  zeros of zeta

There are many other zetas:  
 Dedekind, primes  $\rightarrow$  prime ideals in number field  
 Selberg, primes  $\rightarrow$  primitive closed geodesics in a manifold  
 Ihara, primes  $\rightarrow$  closed geodesics in graph

Odlyzko's Comparison  
 of Spacings of  
 Imaginary Parts of  
 Zeros of Zeta and  
 Eigenvalues of **Random  
 Hermitian Matrix.**

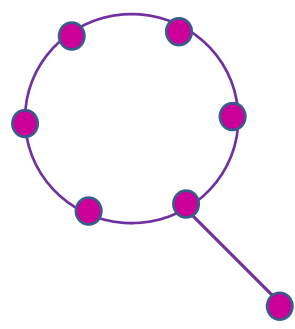
See  
 B. Cipra, *What's  
 Happening in the  
 Mathematical  
 Sciences, 1998-1999,*  
*A.M.S., 1999.*



### Finite Graphs as a Toy Model for RMT

We want to investigate spectral statistics for matrices coming from finite undirected connected graphs. Here mostly spectra and not spacings.

Usually we assume:  
 graph is not a cycle or  
 a cycle with degree 1  
 vertices



A Bad Graph is

It is OK to be irregular (vertices do not all have same degree) and to have multiple edges and loops.  
 You can add physics to it by making the graph a quantum graph.  
 Brian Winn's talk should do this.

# Primes in Graphs

(correspond to geodesics in compact manifolds)  
 are equivalence classes  $[C]$  of closed backtrackless  
 tailless primitive paths  $C$

**DEFINITIONS**

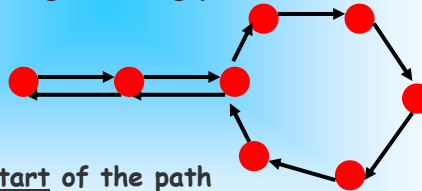
backtrack



equivalence class: change starting point

tail

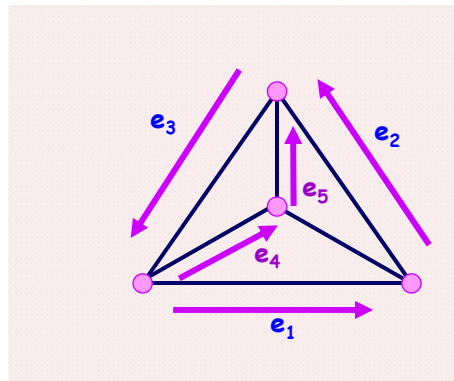
$\alpha$



Here  $\alpha$  is the start of the path

non-primitive: go around path more than once

## EXAMPLES of Primes in a Graph



$$[C] = [e_1 e_2 e_3]$$

$$[D] = [e_4 e_5 e_3]$$

$$[E] = [e_1 e_2 e_3 e_4 e_5 e_3]$$

$$v(C)=3, v(D)=4, v(E)=6$$

$$E=CD$$

another prime  $[C^n D]$ ,  $n=2,3,4, \dots$   
 infinitely many primes

## Ihara Zeta Function

$$\zeta(u, X) = \prod_{\substack{[C] \\ \text{prime}}} (1 - u^{v(C)})^{-1}$$

$v(C)$  = # edges in  $C$   
converges for  $u$  complex,  $|u|$  small

$$\zeta(u, X)^{-1} = (1 - u^2)^{r-1} \det(I - Au + Qu^2)$$

$A$  = adjacency matrix,  $Q + I$  = diagonal matrix of degrees,  $r$  = rank fundamental group  
Can prove via Selberg trace formula for trees in regular graph case. There are simpler proofs.

For  $q+1$  - regular graph, meaning that each vertex has  $q+1$  edges coming out

$u = q^{-s}$  makes Ihara zeta more like Riemann zeta.

$f(s) = \zeta(q^{-s})$  has a functional equation relating  $f(s)$  and  $f(1-s)$ .

**Riemann Hypothesis (RH)**

says  $\zeta(q^{-s})$  has no poles with  $0 < \text{Re } s < 1$  unless  $\text{Re } s = \frac{1}{2}$ .

RH means graph is Ramanujan i.e., non-trivial spectrum of adjacency matrix is contained in the spectrum for the universal covering tree which is the interval  $(-2\sqrt{q}, 2\sqrt{q})$

[see Lubotzky, Phillips & Sarnak, *Combinatorica*, 8 (1988)].

**Ramanujan graph is a good expander**  
**(good gossip network)**

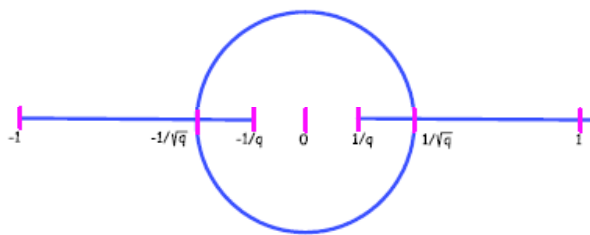
## What is an expander graph X?

### 4 Ideas

- 1) **spectral property** of some matrix associated to our finite graph X  
Choose on of 3:
  - ❖ Adjacency matrix A,
  - ❖ Laplacian  $D-A$ , or  $I-D^{-1/2}AD^{-1/2}$ ,  $D$ =diagonal matrix of degrees
  - ❖ edge matrix  $W$  for X (to be defined)
 Lubotzky: **Spectrum for X SHOULD BE INSIDE** spectrum of analogous operator on universal covering tree for X.
- 2) X behaves like a **random graph**.
- 3) **Information is passed quickly** in the gossip network based on X.
- 4) **Random walker** on X gets lost FAST.



### Possible Locations of Poles $u$ of $\zeta(u)$ for $q+1$ Regular Graph



**Alon conjecture** for regular graphs says RH  $\cong$  true for "most" regular graphs.

See Joel Friedman's web site for proof  
([www.math.ubc.ca/~jf](http://www.math.ubc.ca/~jf))

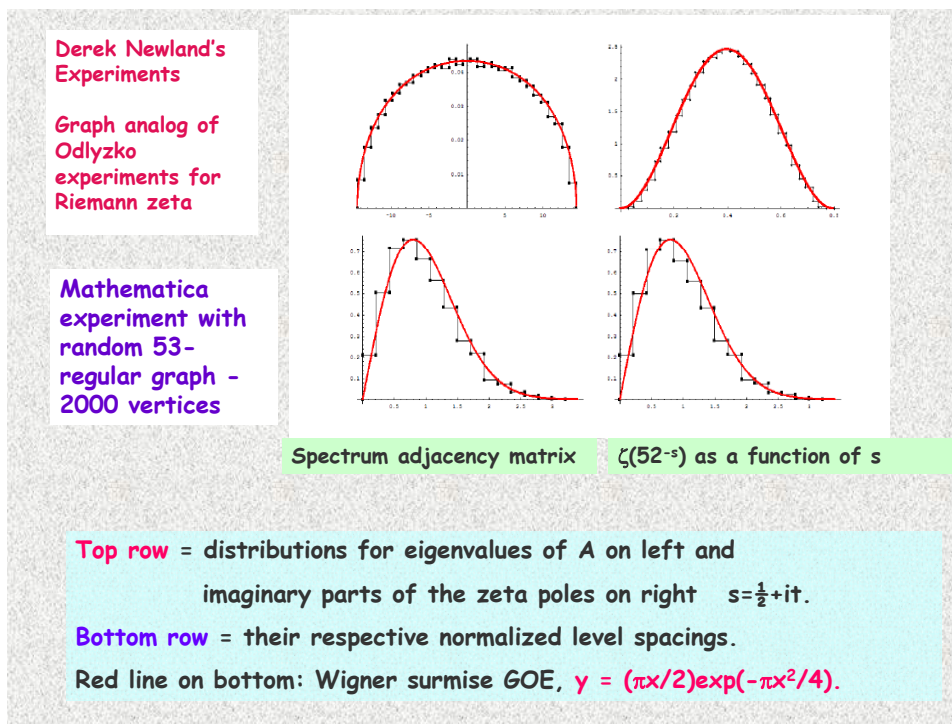
See Steven J. Miller's web site:

([www.math.brown.edu/~sjmiller](http://www.math.brown.edu/~sjmiller)) for a talk on experiments leading to conjecture that the percent of regular graphs satisfying RH approaches 52% as  $\#$  vertices  $\rightarrow \infty$ , via Tracy-Widom distribution.

$1/q$  always the closest pole to 0 in absolute value.

**Circle** of radius  $1/\sqrt{q}$  from the RH poles.

Real poles ( $\neq \pm q^{-1/2}, \pm 1$ ) correspond to non-RH poles.



## What is the meaning of the RH for irregular graphs?

For irregular graph, natural change of variables is  $u = R^s$ , where

$R$  = radius of convergence of Dirichlet series for Ihara zeta.

Note:  $R$  is closest pole of zeta to 0. No functional equation.

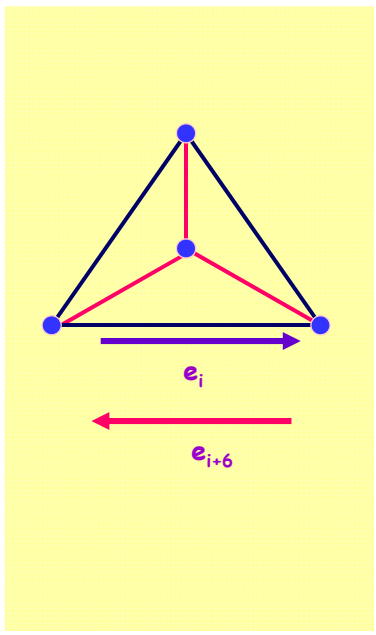
Then the critical strip is  $0 \leq \text{Re } s \leq 1$  and translating back to  $u$ -variable:

**Graph theory RH:**

$$\zeta(u) \text{ is pole free in } R < |u| < \sqrt{R}$$

To investigate this, we need to define the edge matrix  $W$ .

# Labeling Edges of Graphs

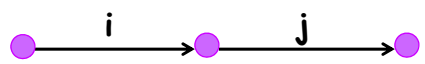


$X$  = finite undirected graph  
 Orient or direct the  $m$  edges arbitrarily.  
 Label them as in picture where  $m=6$ .

With this labeling, we have the properties of the edge matrix on the next slides.

## The Edge Matrix $W$

Define  $W$  to be the  $2|E| \times 2|E|$  matrix with  $i, j$  entry 1 if edge  $i$  feeds into edge  $j$ , (end vertex of  $i$  is start vertex of  $j$ ) provided  $i \neq$  opposite of  $j$ , otherwise the  $i, j$  entry is 0.



**Theorem.**  $\zeta(u, X)^{-1} = \det(I - Wu)$ .

Proof uses  $N_m = \#$  tailless backtrackless oriented closed paths of length  $m$  is  $\text{Tr}(W^m)$

**Corollary.** The poles of Ihara zeta are the reciprocals of the eigenvalues of  $W$ .

The pole  $R$  of zeta is:

$$R = 1 / \text{Perron-Frobenius eigenvalue of } W.$$

**Properties of W**

1)  $W = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}$ , **B and C symmetric**

2) Row sums of entries are  $q_j+1$ =degree vertex which is start of edge j.

**Poles Ihara Zeta** are in region  $q^{-1} \leq |u| \leq 1$ ,  
 $q+1$ =maximum degree of vertices of X.

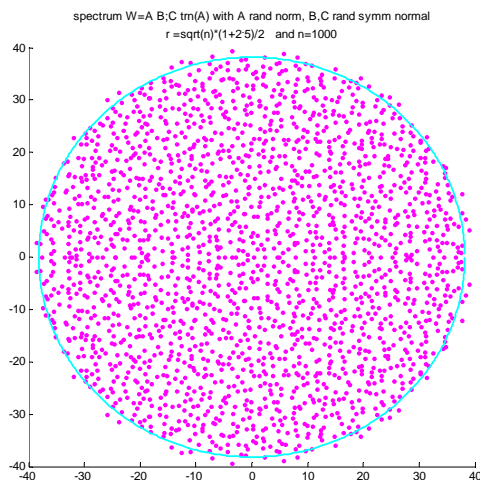
**Theorem of Kotani and Sunada**

If  $p+1$ =min vertex degree, and  $q+1$ =maximum vertex degree,  
 non-real poles u of zeta satisfy

$$\frac{1}{\sqrt{q}} \leq |u| \leq \frac{1}{\sqrt{p}}$$

Kotani & Sunada, *J. Math. Soc. U. Tokyo*, 7 (2000)

**Spectrum of Random Matrix with Properties of W-matrix**



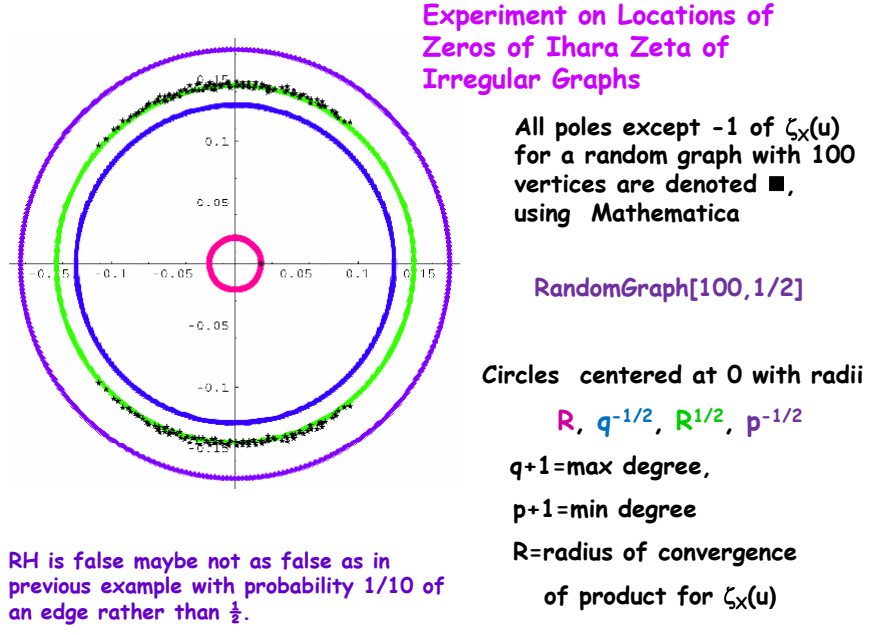
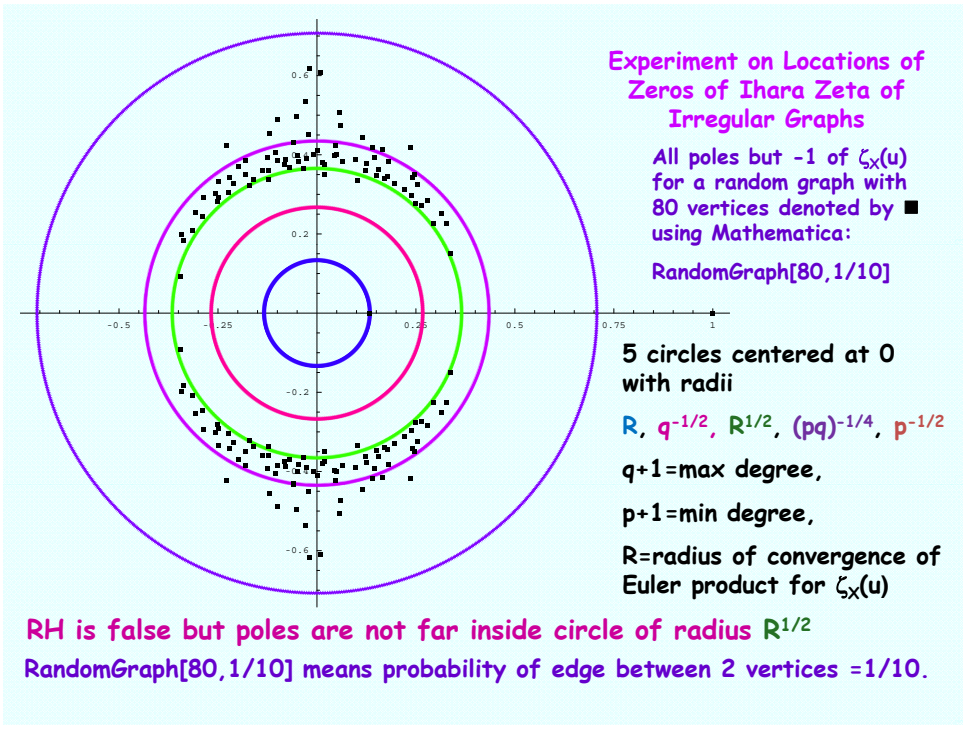
$$W = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}$$

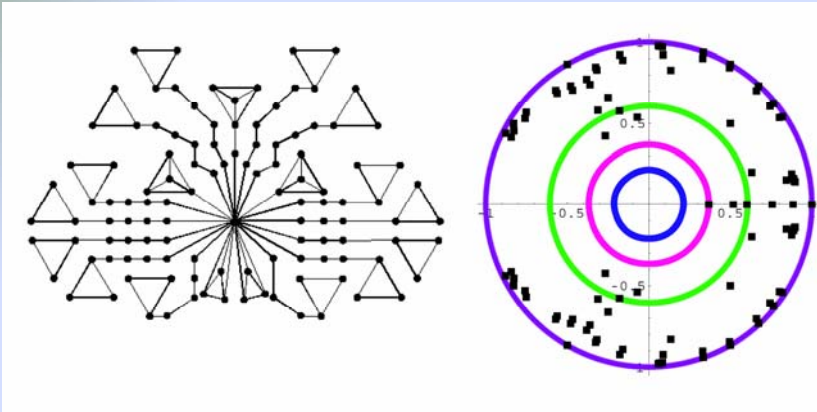
**B and C symmetric**

**Girko circle law with a symmetry with respect to real axis since our matrix is real.**

**We used Matlab command randn(1000) to get A,B,C matrices with random normally distributed entries mean 0 std dev 1.**

We seem to see level repulsion if it means looking at histogram of distances between nearest neighbors. See P. LeBoeuf, *Random matrices, random polynomials, and Coulomb systems*. Wigner surmise is  $4\Gamma\left(\frac{5}{4}\right) s^3 e^{-\left(\frac{5}{4}\right)s^4}$  Ginebre ensembles should be mentioned here.

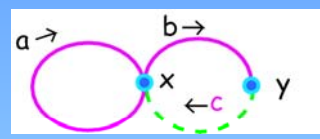
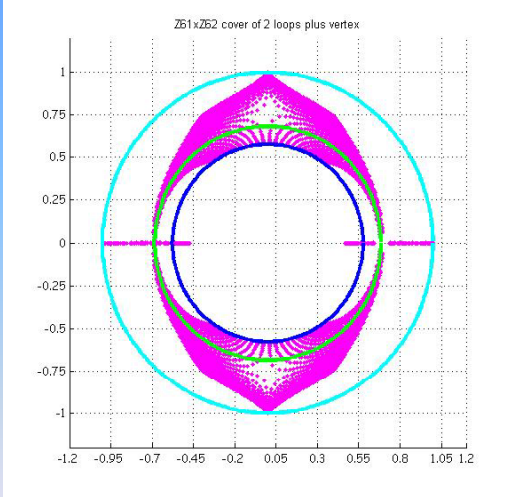




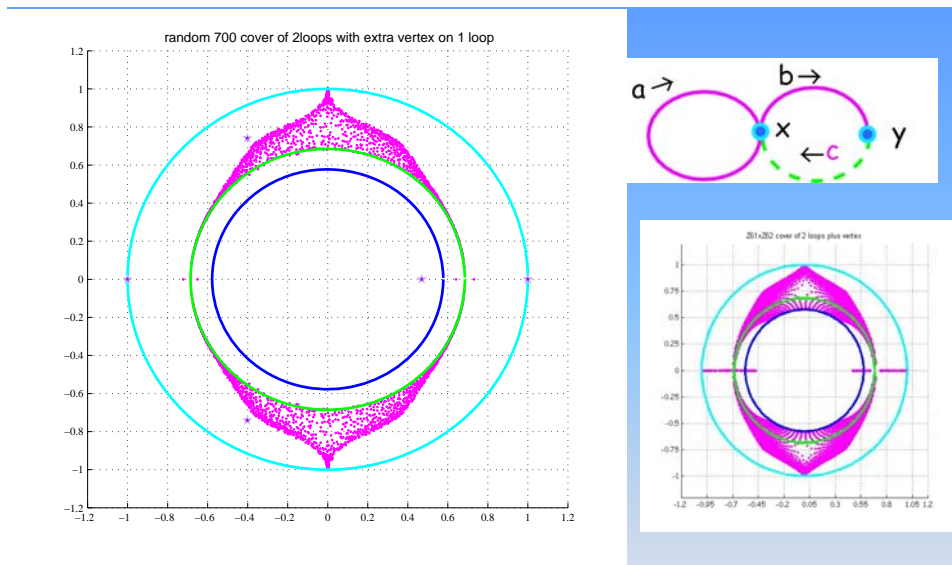
Matthew Horton's Graph has  $1/R \cong e$  to 7 digits.  
 Poles of Ihara zeta are boxes on right. Circles have radii  $R, q^{-\frac{1}{2}}, R^{\frac{1}{2}}, p^{-\frac{1}{2}}$ , if  $q+1 = \text{max deg}$ ,  $p+1 = \text{min deg}$ . Here

The RH is false. Poles more spread out over plane.

Poles of Ihara Zeta for a  $\mathbb{Z}_{61} \times \mathbb{Z}_{62}$ -Cover of 2 Loops + Extra Vertex are pink dots



Circles Centers (0,0); Radii:  $3^{-1/2}, R^{1/2}, 1$ ;  $R \cong .47$   
 RH very False



$Z$  is random 700 cover of 2 loops plus vertex graph in picture.

The pink dots are at poles of  $\zeta_Z$ . Circles have radii  $q^{-1/2}$ ,  $R^{1/2}$ ,  $p^{-1/2}$ , with  $q=3$ ,  $p=1$ ,  $R \cong .4694$ . RH approximately True.

References: 3 papers with Harold Stark in *Advances in Math*.

- ❖ Paper with Matthew Horton & Harold Stark in Snowbird Proceedings, Contemporary Mathematics, Volume 415 (2006)  
Quantum Graphs and Their Applications, *Contemporary Mathematics*, v. 415, AMS, Providence, RI 2006.
- ❖ See my draft of a book:  
[www.math.ucsd.edu/~aterras/newbook.pdf](http://www.math.ucsd.edu/~aterras/newbook.pdf)
- ❖ Draft of new paper joint with Horton & Stark: also on my website  
[www.math.ucsd.edu/~aterras/cambridge.pdf](http://www.math.ucsd.edu/~aterras/cambridge.pdf)
- ❖ There was a graph zetas special session of this AMS meeting - many interesting papers some on my website.
- ❖ For work on directed graphs, see Matthew Horton, Ihara zeta functions of digraphs, *Linear Algebra and its Applications*, 425 (2007) 130-142.
- ❖ work of Angel, Friedman and Hoory giving analog of Alon conjecture for irregular graphs, implying our  $R$  (see Joel Friedman's website: [www.math.ubc.ca/~jf](http://www.math.ubc.ca/~jf))

