Properties of Artin L-Functions

Copy from Lang, *Algebraic Number Theory*

1) L(u,1,Y/X) = ζ(u,X)
= Ihara zeta function of X
(our analogue of the Dedekind zeta function, also Selberg zeta function)

2)

$$\zeta(u,Y) = \prod_{\rho \in \hat{G}} L(u,\rho,Y/X)^{d_\rho}$$

product over all irred. reps of G,
d_ρ=degree ρ

3) You can prove ζ(u,X)$^{-1}$ divides ζ(u,Y)$^{-1}$ directly and you don't need to assume Y/X Galois.

Thus the analog of the Dedekind conjecture for zetas of algebraic number fields is proved easily for graph zetas.

Ihara Theorem for L-Functions

$$L(u, r, Y/X)^{-1}$$

$$= (1-u^2)^{(r-1)d_r} \det(I' - A'_r u + Q' u^2)$$

r=rank fundamental group of X = |E|-|V|+1

r **= representation of G = Gal(Y/X), d = d**$_r$ **= degree** r

Definitions. **nd´nd matrices A′, Q′, I′, n=|X|**

nxn matrix A(g), g Î **Gal(Y/X),**

has entry for a,b Î **X given by**

(A(g))$_{a,b}$ **= # { edges in Y from (**a**,e) to (**b**,g) }**

Here e=identity of G.

$$A'_r = \sum_{g \in G} A(g) \otimes r(g)$$

Q = diagonal matrix, jth diagonal entry

= q$_j$ **= (degree of jth vertex in X)-1,**

Q′ = QÄI$_d$ **,** **I′ = I**$_{nd}$ **= identity matrix.**

Proof can be found in Stark and Terras, *Advances in Math.*, Vol. 154 (2000)

NOTES FOR REGULAR GRAPHS mostly

- **Another proof uses Selberg trace formula on tree to prove Ihara's theorem. For case of trivial representation, see A.T., *Fourier Analysis on Finite Groups & Applics;* for general case, see and Venkov & Nikitin, *St. Petersberg Math. J.*, 5 (1994)**

- $\left(\frac{1}{z_X}\right)^{(r)}(0)=(-1)^{r+1}2^r(r-1)k(X)$**, where k(X)=the number of spanning trees of X, the complexity**

- **Ihara zeta has functional equations relating value at u and 1/(qu), q=degree - 1**

- **Riemann Hypothesis, for case of trivial representation (poles), means graph is <u>Ramanujan</u> i.e., non-trivial spectrum of adjacency matrix is contained in the spectrum for the universal covering tree which is the interval (-2Öq, 2Öq) [see Lubotzky, Phillips & Sarnak, *Combinatorica*, 8 (1988)]**

- **RH is true for most graphs but it can be false**

- **Hashimoto [Adv. Stud. Pure Math., 15 (1989)] proves Ihara z for certain graphs is essentially the z function of a Shimura curve over a finite field**

The Prime Number Theorem

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Let $p_X(m)$ denote the number of prime path equivalence classes [C] in X where the length of C is m. Assume X is finite connected (q+1)-regular. Since 1/q is the absolute value of the closest pole(s) of z(u,X) to 0, then

$p_X(m)$ ~ q^m/m as m ® ¥.

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The proof comes from the method of generating functions (See Wilf, *generatingfunctionology*) and the fact that (as in Stark & Terras, *Advances in Math*, 121 & 154):

$$u\frac{d}{du}\log z(u,X)=\sum_{m=1}^{\infty} n_X(m)u^m$$

Here $n_X(m)$ is the number of closed paths C in X of length m without backtracking or tails.

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Note: When X is not regular, we could define q to be the reciprocal of the absolute value of the closest pole(s) of zeta to 0.

EXAMPLE 1. Y=cube, X=tetrahedron

|X| =4, |Y| = 8, r=3, G = {e,g}

representations of G are 1 and r: r(e) = 1, r(g) = -1

$I' = I_4$, $Q' = 2I_4$,

$A(e)_{u,v}$ = #{ length 1 paths u′ to v′ in Y}

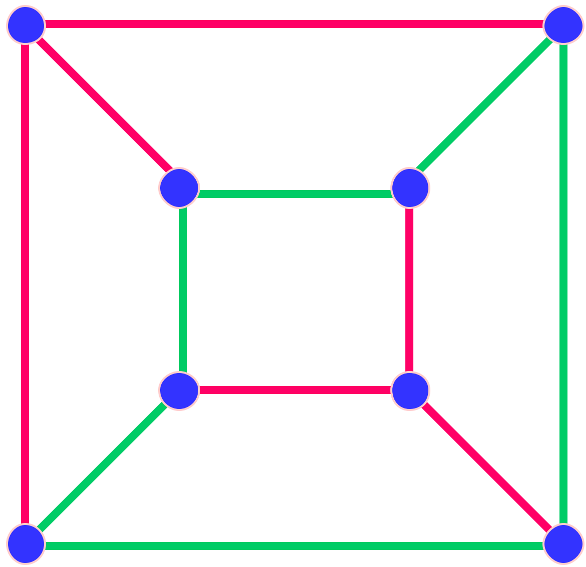
$A(g)_{u,v}$ = #{ length 1 paths u′ to v″ in Y}

$$A(e) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad A(g) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

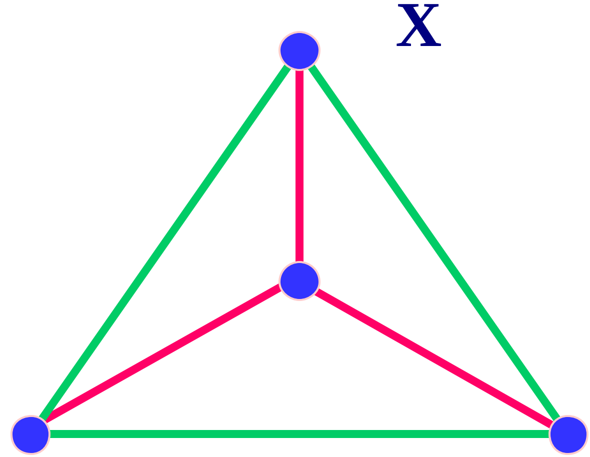
A'_1 = A = adjacency matrix of X

$$A'_r = A(e) - A(g) = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{pmatrix}$$

Zeta and L-Functions of Cube & Tetrahedron



Y



X

- $z(u,Y)^{-1} = L(u,r,Y/X)^{-1}\, z(u,X)^{-1}$
- $L(u, r, Y/X)^{-1} = (1-u^2)(1+u)(1+2u)(1-u+2u^2)^3$
- $z(u,X)^{-1} = (1-u^2)^2(1-u)(1-2u)(1+u+2u^2)^3$
- roots of $z(u,X)^{-1}$ are 1,1,1, ½,r,r,r
  where r=(-1±Ö-7)/4 and |r|=1/Ö2

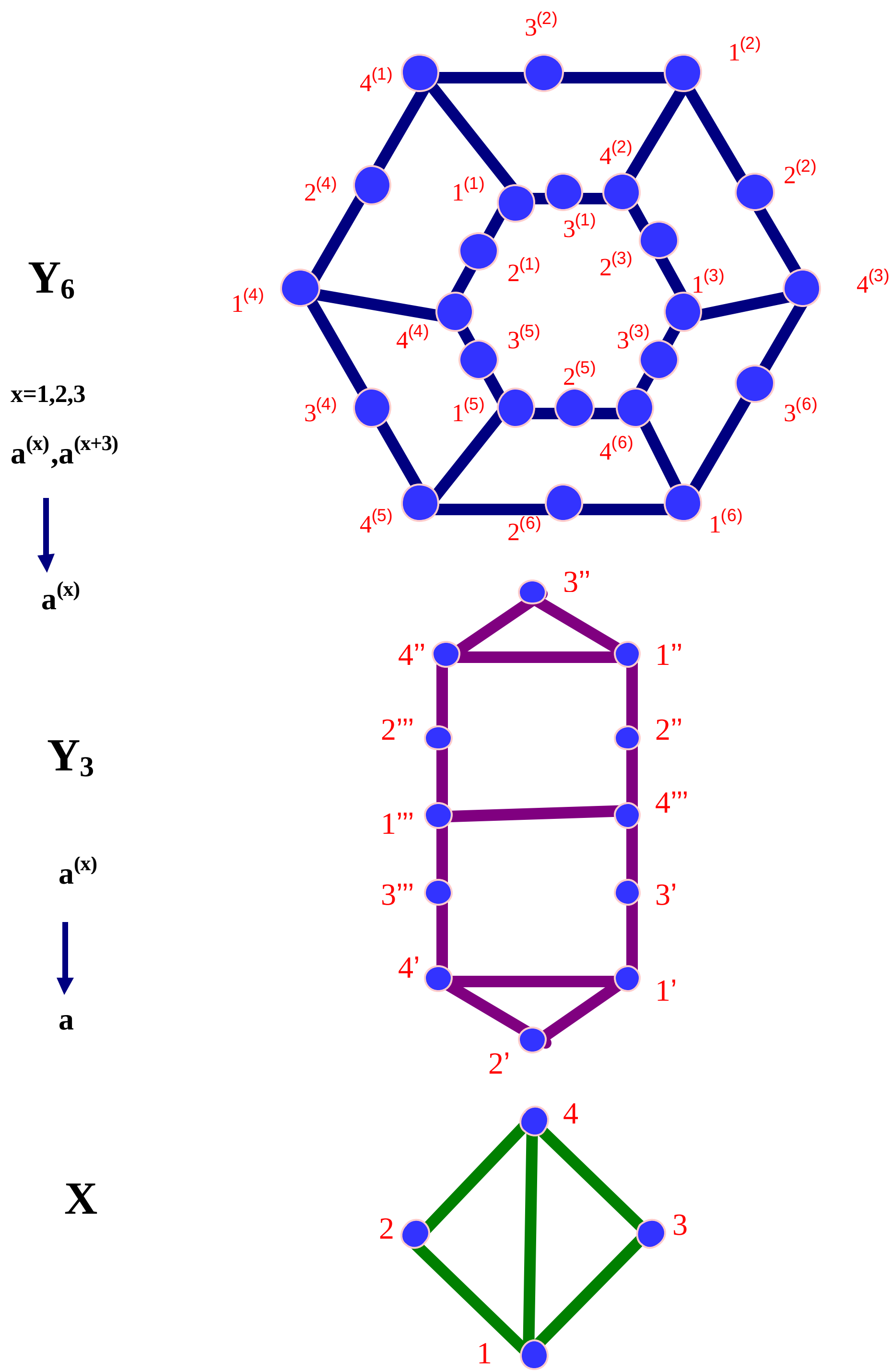
- The pole of $z(u,X)$ closest to 0 governs the prime number theorem discussed a few pages back. It is 1/q=1/2. The coefficients of the following generating function are the numbers of closed paths without backtracking or tails of the indicated length

$$u\frac{d}{du}\log z(u,X) = 24u^3 + 24u^4 + 96u^6 + 168u^7 + 168u^8 + 528u^9 + 1200u^{10} + 1848u^{11} + O(u^{12})$$

So there are 8 primes of length 3 in X, for example.

Example 2. Galois Cover of Non-Normal Cubic

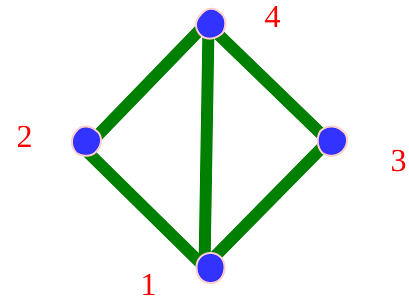
This example is analogous to example 2 in part 1.



G=S_3, H={(1),(23)} fixes Y_3. $a^{(1)}$=(a,(1)), $a^{(2)}$=(a,(13)), $a^{(3)}$=(a,(132), $a^{(4)}$=(a,(23)),$a^{(5)}$=(a,(123)),$a^{(6)}$=(a,(23))

Here we use the standard cycle notation for elements of the symmetric group.

3 classes of primes in base graph X from preceding page



- **Class C1** path in X (list vertices) 14312412431

  f=1, g=3 3 lifts to Y_3

  1'4'3'''1'''2'''4''1''2''4'''3'1'

  1''4''3''1''2''4'''1'''2'''4''3''1''

  1'''4'''3'1'2'4'1'2'4'3'''1'''

  Frobenius trivial ⇒ density 1/6

- **Class C2** path in X (list vertices) 1241

  2 lifts to Y_3

  1'2'4'1' f=1

  1''2''4'''1'''2'''4''1'' f=2

  Frobenius order 2 ⇒ density 1/2

- **Class C3** path in X (list vertices) 12431

  f=3 1 lift to Y_3

  1'2'4'3'''1'''2'''4''3''1''2''4'''3'1'

  Frobenius order 3 ⇒ density 1/3

Ihara Zeta Functions

- $z(u,X)^{-1}=(1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3)$

- $z(u,Y_3)^{-1}=z(u,X)^{-1}\ (1-u^2)^2(1-u-u^3+2u^4)$
  ´ $(1-u+2u^2-u^3+2u^4)(1+u+u^3+2u^4)$
  ´ $(1+u+2u^2+u^3+2u^4)$

- $z(u,Y_6)^{-1}=z(u,Y_3)^{-1}\ (1-u^2)^8(1+u)(1+u^2)(1-u+2u^2)$
  $\times\ (1-u^2+2u^3)\ (1-u-u^3+2u^4)\ (1-u+2u^2-u^3+2u^4)$
  $\times\ (1+u+u^3+2u^4)(1+u+2u^2+u^3+2u^4)$

It follows that, as in the number theory analog,

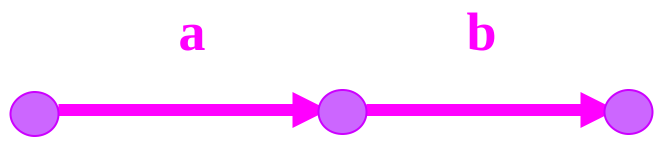
$$z(u,X)^2\ z(u,Y_6) = z(u,Y_2)\ z(u,Y_3)^2$$

Here Y_2 is an intermediate quadratic extension between Y_6 and X. See Stark and Terras, *Adv. in Math.*, 154 (2000), Figure 13, for a discussion.

The poles of $z(u,X)$ are u=1,1,-1, ±i,(-1±Ö7i)/4,w,w,1/q Where w,1/q are roots of the cubic. The closest pole to 0 is 1/q. And q is approximately 1.5214. So the prime number theorem gives a considerably smaller main term, q^m/m, for this graph X than for K_4, where q=2.

Multiedge Artin L-Functions

Orient the edges of the graph. Multiedge matrix W has ab entry $w(a,b)=w_{ab}$ in $\mathbb{C}$, if the edges a and b look like



Otherwise set $w_{ab}=0$ Define for closed path $C=a_1a_2\ldots a_s$,

$$N_E(C)=w(a_s,a_1)w(a_1,a_2)\ldots w(a_{s-1},a_s)$$

$$L_E(W,r,Y/X)=\prod_{[C]}\left(1-r\left(\frac{Y/X}{D}\right)N_E(C)\right)^{-1}$$

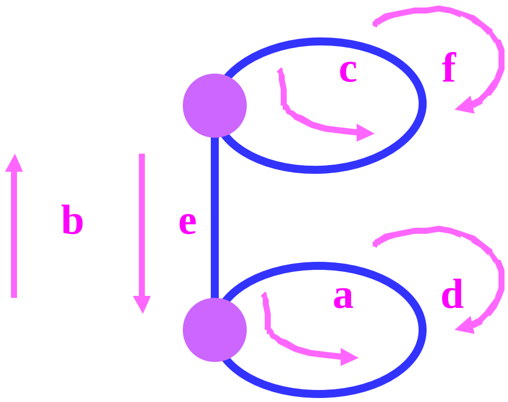
where the product is over primes [C] of X and [D] is any prime of Y over [C]

Properties

- **$L_E(W,1,Y/X)=\zeta_E(W,X)$, the edge zeta function**
- **$L_E(W,r)^{-1}=\det(I-W_r)$, where W_r is a $2|E|\times 2|E|$ block matrix with ij block given by $(w_{ij}\, r(Frob(e_i)))$**
- **Induction property**
- **Factorization of edge zeta as a product of edge L-functions**
- **specialize all wij=u and get the Artin-Ihara vertex L function**

EXAMPLE.

X=Dumbbell Graph and Fission of an Edge

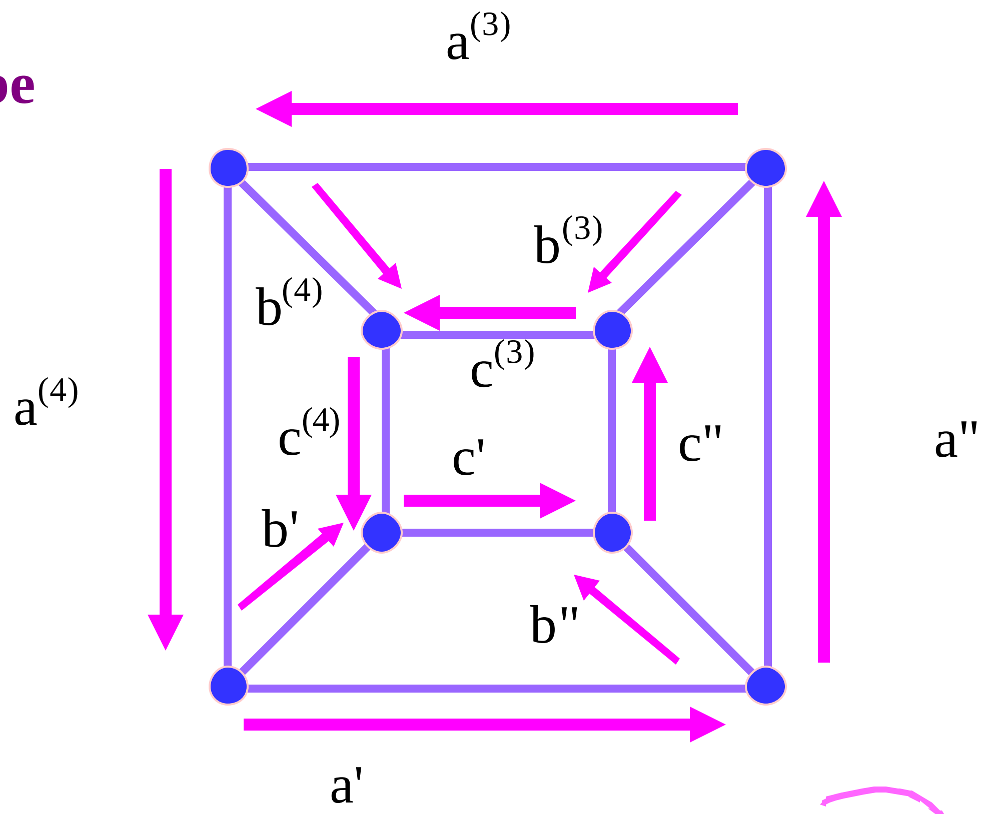


$$z_E(W,X)^{-1} = \det\begin{pmatrix} w_{aa}-1 & w_{ab} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{bc} & 0 & 0 & w_{bf} \\ 0 & 0 & w_{cc}-1 & 0 & w_{ce} & 0 \\ 0 & w_{db} & 0 & w_{dd}-1 & 0 & 0 \\ w_{ea} & 0 & 0 & w_{ed} & -1 & 0 \\ 0 & 0 & 0 & 0 & w_{fe} & w_{ff}-1 \end{pmatrix}$$

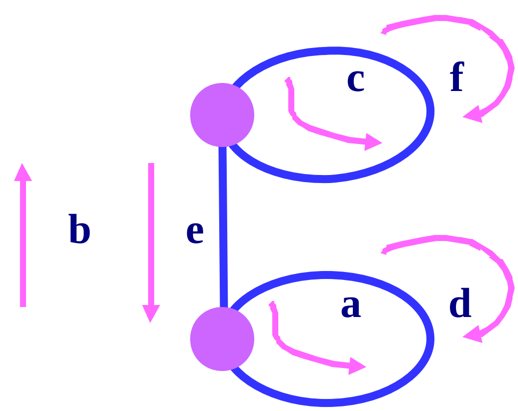
Here b and e are the vertical edges.
Specialize all variables with b and e to be 0 and get zeta function of subgraph with vertical edge removed - Fision
This gives the graph with just 2 disconnected loops.

Example 3. Cube Covering Dumbbell

Y=Cube



X=Dumbbell



Gal(Y/X) = $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\} \approx \mathbb{Z}/4\mathbb{Z}$.

Identification sends σ_j to $j - 1 \pmod 4$)

The representations are 1-dimensional: $\pi_a(b)=i^{a(b-1)}$.

Galois group elements associated to edges a,b,c are

Frob(a) = σ_2, Frob(b) = σ_1, Frob(c) = σ_2.

Edge L-Functions for Example 3.

$$z(W,X)^{-1}=L(W,1,Y/X)^{-1}=\det\begin{pmatrix} w_{aa}-1 & w_{ab} & 0 & 0 & 0 & 0\\ 0 & -1 & w_{bc} & 0 & 0 & w_{bf}\\ 0 & 0 & w_{cc}-1 & 0 & w_{ce} & 0\\ 0 & w_{db} & 0 & w_{dd}-1 & 0 & 0\\ w_{ea} & 0 & 0 & w_{ed} & -1 & 0\\ 0 & 0 & 0 & 0 & w_{fe} & w_{ff}-1 \end{pmatrix}$$

$$L_E(W,p_1,Y/X)^{-1}=\det\begin{pmatrix} iw_{aa}-1 & iw_{ab} & 0 & 0 & 0 & 0\\ 0 & -1 & w_{bc} & 0 & 0 & w_{bf}\\ 0 & 0 & iw_{cc}-1 & 0 & iw_{ce} & 0\\ 0 & -iw_{db} & 0 & -iw_{dd}-1 & 0 & 0\\ w_{ea} & 0 & 0 & w_{ed} & -1 & 0\\ 0 & 0 & 0 & 0 & -iw_{fe} & -iw_{ff}-1 \end{pmatrix}$$

$$L(W,p_2,Y/X)^{-1}=\det\begin{pmatrix} -w_{aa}-1 & -w_{ab} & 0 & 0 & 0 & 0\\ 0 & -1 & w_{bc} & 0 & 0 & w_{bf}\\ 0 & 0 & -w_{cc}-1 & 0 & -w_{ce} & 0\\ 0 & -w_{db} & 0 & -w_{dd}-1 & 0 & 0\\ w_{ea} & 0 & 0 & w_{ed} & -1 & 0\\ 0 & 0 & 0 & 0 & -w_{fe} & -w_{ff}-1 \end{pmatrix}$$

$$L(W,p_3,Y/X)^{-1}=\det\begin{pmatrix} -iw_{aa}-1 & -iw_{ab} & 0 & 0 & 0 & 0\\ 0 & -1 & w_{bc} & 0 & 0 & w_{bf}\\ 0 & 0 & -iw_{cc}-1 & 0 & -iw_{ce} & 0\\ 0 & iw_{db} & 0 & iw_{dd}-1 & 0 & 0\\ w_{ea} & 0 & 0 & w_{ed} & -1 & 0\\ 0 & 0 & 0 & 0 & iw_{fe} & iw_{ff}-1 \end{pmatrix}$$

Note that the product of these 6x6 determinants is the 24x24 determinant whose reciprocal is the multiedge zeta function of Y, the cube.

Path L-Functions

Here we discuss a new kind of L-function with smaller sized matrix determinants.

Fundamental Group of X can be identified with group generated by edges left out of a spanning tree

$$e_1,\ldots e_r, e_1^{-1},\ldots, e_r^{-1}$$

2r ×2r **multipath matrix** Z has ij entry

z_{ij} in C if $e_j \neq e_i^{-1}$ and $z_{ij}=0$, otherwise.

Imitate the definition of the edge Artin L-functions.

Write a prime path as a reduced word in a conjugacy class

$$C = a_1 \cdots a_s,\ \ where\ \ a_j \in \{e_1^{\pm 1},\ldots,e_r^{\pm 1}\}$$

and define the **path norm**

$$N_P(C) = z(a_s, a_1)\prod_{i=1}^{s-1} z(a_i, a_{i+1})$$

where $z(e_i,e_j)=z_{ij}$.

Define the **path zeta L-function**

$$L_P(Z,p,Y/X) = \prod_{[C]} \det\left(1 - p\left(\frac{Y/X}{D}\right) N_P(C)\right)^{-1}$$

Product is over prime cycles [C] in X

[D] is any prime of Y over [C]

Specializing Path L-Functions to Edge L-Functions

The path L-functions have analogous properties to the edge L-functions.

❈They are reciprocals of polynomials.

❈They provide factorizations of the path zeta functions.

❈ The most important property is that of

Specialization to Path L-functions.

- edges left out of a spanning tree T of X: $e_1,\ldots e_r$
  generate fundamental group of X
- inverse edges are $e_{r+1} = e_1^{-1},\ldots, e_{2r} = e_r^{-1}$
- edges of the spanning tree T are $t_1,\ldots, t_{|X|-1}$
- with inverse edges $t_{|X|},\ldots t_{2|X|-2}$

If $e_i \neq e_j^{-1}$, write the part of the path between e_i and e_j as the (unique) product $t_{k_1}\cdots t_{k_n}$

C is 1st a product of e_j (generators of the fundamental group), then a product of actual edges e_j and t_k.

Specialize the multipath matrix Z to Z(W) with entries

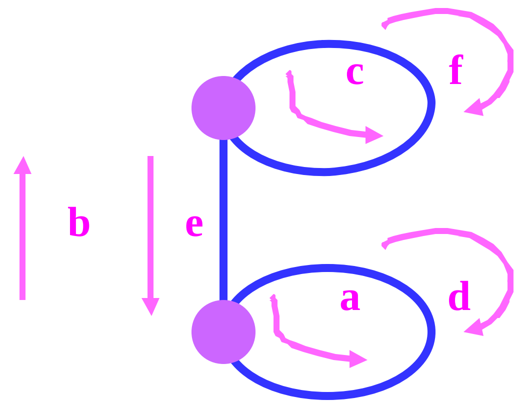
$$z_{ij} = w(e_i,t_{k_1})w(t_{k_n},e_j)\prod_{n=1}^{n-1} w(t_{k_n},t_{k_{n+1}})$$

Then

$$L_P(Z(W),X) = L_E(W,X)$$

Example - the Dumbbell

Recall the edge zeta was a 6x6 determinant. The specialized path zeta is only 4x4. Maple computes it much faster than the 6x6.

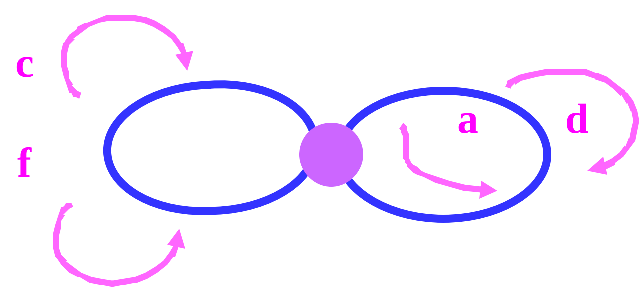


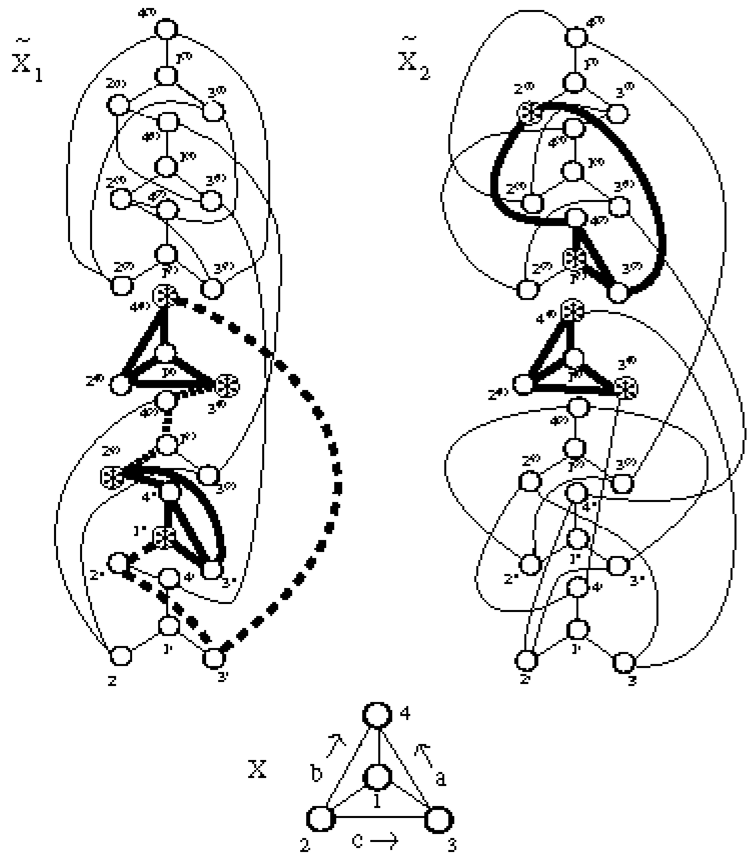
$$z_E(W,X)^{-1} = \det\begin{pmatrix} w_{aa}-1 & w_{ab}w_{bc} & 0 & w_{ab}w_{bf} \\ w_{ce}w_{ea} & w_{cc}-1 & w_{ce}w_{ed} & 0 \\ 0 & w_{db}w_{bc} & w_{dd}-1 & w_{db}w_{bf} \\ w_{fe}w_{ea} & 0 & w_{fe}w_{ed} & w_{ff}-1 \end{pmatrix}$$

Fusion of an edge is now easy to do in the path zeta.

To obtain edge zeta of graph obtained from dumbbell graph, by fusing edges b and e,

Replace $w_{xb}w_{by}$ with w_{xy}
Replace $w_{xe}w_{ey}$ with w_{xy}





Application of Galois Theory of Graph Coverings. You can't hear the shape of a graph.

Find 2 regular graphs (without loops and multiple edges) which are isospectral but not isomorphic.

See A.T. & Stark in *Adv. in Math.*, Vol. 154 (2000) for the details. The method goes back to algebraic number theorists who found number fields K_i which are non isomorphic but have the same Dedekind zeta. See Perlis, *J. Number Theory*, 9 (1977).

THE END