ZETA FUNCTIONS OF HEISENBERG GRAPHS OVER FINITE RINGS

Michelle DeDeo, María Martínez, Archie Medrano,
Marvin Minei, Harold Stark, and Audrey Terras
Department of Mathematics
University of California, San Diego
La Jolla, CA 92092-0112
aterras@ucsd.edu

Abstract We investigate Ihara-Selberg zeta functions of Cayley graphs for the Heisenberg group over finite rings $\mathbb{Z}/p^n\mathbb{Z}$, where p is a prime. In order to do this, we must compute the Galois group of the covering obtained by reducing coordinates in $\mathbb{Z}/p^{n+1}\mathbb{Z}$ modulo p^n. The Ihara-Selberg zeta functions of the Heisenberg graph mod p^{n+1} factor as a product of Artin L-functions corresponding to the irreducible representations of the Galois group of the covering. Emphasis is on graphs of degree four. These zeta functions are compared with zeta functions of finite torus graphs which are Cayley graphs for the abelian groups $(\mathbb{Z}/p^n\mathbb{Z})^r$.

1. Introduction

The aim of this paper is to study the special functions known as Ihara-Selberg zeta functions for Cayley graphs of finite Heisenberg groups as well as their factorizations into products of Artin-Ihara L-functions. The **Heisenberg group** $H(R)$ over a ring R consists of upper triangular 3×3 matrices with entries in R and ones on the diagonal. The Ihara-Selberg zeta function is analogous to the Riemann zeta function with primes replaced by certain closed paths in a graph. This paper is a continuation of (DeDeo et al., 2004) where we presented a study of the statistics of the spectra of adjacency matrices of finite Heisenberg graphs.

When R is the field of real numbers $\mathbb{R}$, the group is well known for its connection with the uncertainty principle in quantum physics. When the ring R is $\mathbb{Z}$, the ring of integers, there are degree 4 and 6 Cayley graphs (see the next paragraph) associated to $H(\mathbb{Z})$ whose spectra (i.e., eigenvalues of the adjacency matrix) have been much studied starting with D. R. Hofstadter's work on energy levels of Bloch electrons (Hofstadter, 1976) which includes a picture of the Hofstadter butterfly. This subject also goes under the name of the spectrum of the almost Mathieu operator or the Harper operator. See (DeDeo et al., 2004) and (Terras, 1999) for more information on the Heisenberg group. See also (Kotani and Sunada, 2000).

If S is a subset of a finite group G, the **Cayley graph** $X(G, S)$ has as its vertex set the set G. Edges connect vertices $g \in G$ and gs, for all $s \in S$. Usually we will assume that $s \in S$ implies $s^{-1} \in S$ so that the graph is undirected. And we will normally assume that S is a set of generators of G so that the graph will be connected. It is not hard to see that the spectrum of the adjacency matrix of $X(G, S)$ is contained in the interval $[-k, k]$, if $k = |S|$.

Heisenberg groups over finite fields have provided a examples of random number generators (see (Zack, 1990)) as well Ramanujan graphs (see (Myers, 1995)). **Ramanujan graphs** were defined by (Lubotzky et al., 1988) to be finite connected k-regular graphs such that the eigenvalues λ of the adjacency matrix satisfy $|\lambda| \leq 2\sqrt{k-1}$. Other references are (Diaconis and Saloff-Coste, 1994) and (Terras, 1999). As shown in the last reference, the size of the second largest (in absolute value) eigenvalue of the adjacency matrix governs the speed of convergence to uniform for the standard random walk on a connected regular graph. Ramanujan graphs have the best possible eigenvalue bound for connected regular graphs of fixed degree in an infinite sequence of graphs with number of vertices going to infinity. For such graphs, the random walker gets lost as quickly as possible. Equivalently, this says that such graphs can be used to build efficient communication networks.

There are more reasons to study the Heisenberg group. First, as a nilpotent group (see (Terras, 1999) for the definition), it may be viewed as the closest to abelian. Second, it is an important subgroup of $GL(3, R)$ (the general linear group of matrices x such that x and x^{-1} have entries in the ring R) for those interested in creating a finite model of the symmetric space of the real $GL(3, \mathbb{R})$ analogous to the finite upper half plane model of the Poincaré upper half plane.

Some of our motivation comes from quantum chaoticists who investigate the statistics of various spectra as well as zeros of zeta functions. This MSRI website (`http://www.msri.org/`) has movies and transparencies of many talks from 1999 on the subject. See, for example, the talks of Sarnak from Spring, 1999. Other references are (Sarnak, 1995) and (Terras, 2000; Terras, 2002).

The Ihara-Selberg zeta function is an analogue of the **Riemann zeta function** $\zeta(s)$. The latter is defined for $\mathrm{Re}(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Thanks to this Euler product, the zeros of zeta are important for any work on the statistics of the primes. The earliest results on the location of zeta zeros led to a proof of the prime number theorem which says that the number of primes less than or equal to x is asymptotic to $x/\log x$, as x goes to infinity. Now there is a million dollar prize problem to prove the Riemann hypothesis which says that the non-real zeros of the analytic continuation of $\zeta(s)$ lie on the line $\mathrm{Re}(s) = 1/2$. This would give the best possible error estimate in the prime number theorem. For a report of experimental verification for the first 100 billion zeros, see the web site `http://www.hipilib.de/zeta/index.html`. Quantum chaoticists have experimental evidence that the zeros of zeta behave analogously to the eigenvalues of a random Hermitian matrix. See (Katz and Sarnak, 1999) for a discussion of various zeta functions whose zeros and poles have been studied in the same manner that the physicists study energy levels of physical systems.

To define a graph-theoretical analogue of $\zeta(s)$, we must define "prime" in a graph X. Modelling the idea of the Selberg zeta function of a Riemannian manifold, we use the prime cycles $[C]$ in X. Orient the edges of X, which we assume is a finite connected graph. A **prime** $[C]$ in X is an equivalence class of tailless backtrackless primitive cycles C in X. Here $C = a_1 a_2 \cdots a_s$, where the a_j are oriented edges of X. The **length** of C is $\nu(C) = s$. "Backtrackless" means that $a_{i+1} \neq a_i^{-1}$, for all i. "Tailless" means that $a_s^{-1} \neq a_1$. The "equivalence class" of C is $[C]$ which consists of all cycles $a_i a_{i+1} \cdots a_s a_1 a_2 \cdots a_{i-1}$; i.e., the same path with all possible starting points. We call the class $[C]$ "primitive" if you only go around once; i.e., $C \neq D^m$, for all integers $m \geq 2$ and all paths D in X.

The **Ihara zeta function** of a connected graph X is defined for $u \in \mathbb{C}$, with $|u|$ sufficiently small, by

$$\zeta_X(u) = \prod_{\substack{[C] \text{ prime} \\ \text{cycle in } X}} \left(1 - u^{\nu(C)}\right)^{-1}. \tag{1.1}$$

The connection with the adjacency matrix A of X is given by **Ihara's theorem** which says

$$\zeta_X(u)^{-1} = \left(1 - u^2\right)^{r-1} \det\left(I - Au + Qu^2\right), \tag{1.2}$$

where $r = |E|-|V|-1$ = rank of fundamental group of X and Q is the diagonal matrix whose jth diagonal entry is Q_{jj} =(-1+ degree of jth vertex). Proofs of the Ihara theorem can be found in (Stark and Terras, 1996; Stark and Terras, 2000), (Terras, 1999). In the first two papers, edge and path zeta functions of more than one variable are also discussed. The most elementary proof of formula (1.2) was found by Bass and involves the edge zeta function associated to more than one variable for which the analogous determinant formula is easy to prove. See (Stark and Terras, 2000) pages 168 and 172.

Remark 1.1. *The Ihara zeta function is related to walk generating functions of graphs, in particular, that for reduced walks considered by (Godsil, 1993, p. 72), but it is not the same. Differences come from not counting tails and the fact that a prime can pass through a given vertex more than once. Related generating functions have also been considered by probabilists studying first passage times for random walks but again they are different. See (Kemperman, 1961).*

We believe that it is worth singling out this special function associated to graphs for several reasons. First, for number theorists, it provides a new analogue of the Riemann zeta function which is easier to experiment on than the zeta functions of number or function fields. Secondly, it connects the zeta functions from many disparate areas such as number theory, differential geometry, and dynamical systems. Thirdly, this zeta function has a generalization to analogues of Artin L-functions. See the definition in formula (2.7)*. Thus we can make use of the Galois theory of normal covering graphs to obtain factorizations of the zeta function.*

It follows from formula (1.2) that there is an analogue of the prime number theorem for primes in a graph. This says that if X is a connected $(q+1)$-regular graph and $\pi(n)$ is the number of prime paths $[C]$ of length n in X, then

$$\pi(n) \sim \frac{q^n}{n}, \text{ as } n \to \infty. \tag{1.3}$$

The proof is easy. One can simply imitate the proof of the analogous result for function fields over finite fields in (Rosen, 2002), pp. 56–57.

From (1.2), we know that these zeta functions are reciprocals of polynomials. When the graph is connected and $(q+1)$-regular, one sees that it is a Ramanujan graph if (and only if) the Ihara-Selberg zeta function satisfies **the Riemann hypothesis** in the sense that the zeros of the polynomial satisfy $|u| = q^{-1/2}$. See (Terras, 1999, p. 418). When the zeta function satisfies the Riemann hypothesis, the error estimate in the prime number theorem (1.3) is best possible. While the Ihara zeta function of a random regular graph may satisfy the Riemann hypothesis, the zeta functions that we encounter here in the study of finite Heisenberg graphs are not Ramanujan in general. See (DeDeo et al., 2004), where it is shown that the spectrum of the adjacency matrix of the degree 4 Heisenberg graph over a finite ring with q elements approaches the interval $[-4, 4]$ as q approaches infinity.

Special values or residues of the Ihara-Selberg zeta function give graph theoretic constants such as the number of spanning trees. There are connections with famous polynomials such as the Alexander polynomials of knots. See (Lin and Wang, 2001).

Here we consider Cayley graphs $\mathcal{H}_S(q) = X(G, S)$ with vertex set the **Heisenberg group** $G = \text{Heis}(\mathbb{Z}/q\mathbb{Z})$ consisting of matrices $(x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$, where $x, y, z \in \mathbb{Z}/q\mathbb{Z}$, $q = p^n$ and p is prime. The edge set S is chosen to have 4 elements $S = \left\{ X^{\pm 1}, A^{\pm 1} \right\}$, where $X = (x, y, z)$ and $A = (a, b, c)$. We assume that $ay \not\equiv bx \pmod{p}$ to insure that the graph is connected (see (DeDeo et al., 2004)). For p odd, all these graphs are isomorphic. When $p = 2$, there are only two isomorphism classes. These facts are proved in (DeDeo et al., 2004). Define the **degree 4 Heisenberg graph**

$$\mathcal{H}(q) = X(\text{Heis}(\mathbb{Z}/q\mathbb{Z}), \{(\pm 1, 0, 0), (0, \pm 1, 0)\}). \tag{1.4}$$

When $p = 2$, define a second Cayley graph

$$\mathcal{H}(2^n)' = X\left(\text{Heis}(\mathbb{Z}/2^n\mathbb{Z}), \left\{(1, 1, 0)^{\pm 1}, (\pm 1, 0, 0)\right\}\right).$$

Histograms of the spectra of the degree 4 Heisenberg graphs were studied in (DeDeo et al., 2004). These figures were made using the representations of the Heisenberg group to block diagonalize the adjacency matrix of $\mathcal{H}_S(q)$. This changes the size of the eigenvalue problem from a $p^{3n} \times p^{3n}$ matrix problem to a collection of $p^n \times p^n$ matrix problems. The histograms were compared with those for the **finite torus graphs**

$$\mathcal{T}^{(n)}(q) = X\left((\mathbb{Z}/q\mathbb{Z})^n, \{\pm e_1, \pm e_2, \ldots, \pm e_n\}\right), \tag{1.5}$$

where e_i denotes a unit vector with ith component 1 and the rest 0.

Here we investigate the Ihara-Selberg zeta functions of these Heisenberg graphs. Taking $S = \{\pm(1, 0, 0), \pm(0, 1, 0)\}$, the graph $\mathcal{H}_S\left(p^{n+1}\right)$ covers the graph $\mathcal{H}_S\left(p^n\right)$ in the usual sense of covering spaces in topology. See Theorem 2.2. The covering is unramified and normal or Galois with Abelian Galois group isomorphic to the subgroup of (x, y, z) in $\text{Heis}\left(\mathbb{Z}/p^{n+1}\mathbb{Z}\right)$ such that x, y, and z are all congruent to 0 modulo p^n. This implies that the spectrum of the adjacency operator on

$\mathcal{H}_S\left(p^n\right)$ is contained in that of $\mathcal{H}_S\left(p^{n+1}\right)$ and that $\zeta_{\mathcal{H}(p^n)}(u)^{-1}$ divides $\zeta_{\mathcal{H}(p^{n+1})}(u)^{-1}$. Moreover it says that the adjacency matrix of $\mathcal{H}_S\left(p^{n+1}\right)$ can be block diagonalized with blocks the size of the adjacency matrix of $\mathcal{H}_S\left(p^n\right)$ associated to the characters of the Galois group. See Proposition 2.1.

The same result that implies Proposition 2.1 implies that the Ihara zeta function of $\mathcal{H}_S\left(p^{n+1}\right)$ factors as a product of Artin-Ihara L-functions $L(u,\chi)$ corresponding to the characters χ of irreducible representations ρ of the Galois group of the covering. See (Hashimoto, 1990) or (Stark and Terras, 2000). We use this factorization to compute the Ihara-Selberg zeta function explicitly for the smallest Heisenberg graphs. See formulas (2.11) and (2.12). Contour maps of (powers of) the absolute value of $\zeta_{\mathcal{H}(2)}(u)^{-1}$ and $\zeta_{\mathcal{H}(4)}(u)^{-1}$ can be found in Figures 3 and 4.

The last part of this paper concerns comparisons of zeta functions for Cayley graphs of the Heisenberg group with analogous Cayley graphs for finite torus groups. We find, for example, that the zeta functions of the smallest degree four Heisenberg and torus graphs can be compared using the following formula

$$\begin{aligned} \zeta_{\mathcal{H}(4)}(u)^{-1}/\zeta_{\mathcal{T}^{(2)}(4)}(u)^{-1} &= \left(1-u^2\right)^{48}\left(3u^2+1\right)^{20} \\ &\times\left(3u^2-2u+1\right)^4\left(3u^2+2u+1\right)^4\left(9u^4-2u^2+1\right)^{10}. \end{aligned} \tag{1.6}$$

2. Ihara-Selberg Zeta Functions

We say that Y is an **unramified finite covering** of a finite graph X if there is a covering map $\pi : Y \to X$ which is an onto graph map (i.e., taking adjacent vertices to adjacent vertices) such that for every $x \in X$ and for every $y \in \pi^{-1}(x)$, the set of points adjacent to y in Y is mapped by π one-to-one, onto the points in X which are adjacent to x. Note that when graphs have loops and multiple edges, one must be a bit more careful with this definition if one wants Galois theory to work properly. See (Stark and Terras, 2000, p. 137). A d-sheeted covering is a **normal** covering iff there are d graph automorphisms $\sigma : Y \to Y$ such that $\pi(\sigma(y)) = \pi(y)$ for all $y \in Y$. These automorphisms form the Galois group $G(Y/X)$. See (Stark and Terras, 1996; Stark and Terras, 2000) for examples of normal and non-normal coverings and the factorization of their zeta functions.

Take a spanning tree T in X. View Y as $|G|$ sheets, where each sheet is a copy of T labeled by the elements of the Galois group G. So the points of Y are (x, g), with $x \in X$ and $g \in G$. Then an element $a \in G$ acts on the cover by $a(x, g) = (x, ag)$.

Suppose the graph X has m vertices. Define the $m \times m$ matrix $A(g)$ for $g \in G$ by defining the i, j entry to be

$$A(g)_{i,j} = \text{the number of edges in } Y \text{ between } (i, e) \text{ and } (j, g), \tag{2.1}$$

where e denotes the identity in G. Using these $m \times m$ matrices, we can find a block diagonalization of the adjacency matrix of Y as follows.

Proposition 2.1. *If Y is a normal d-sheeted covering of X with Galois group G, then the adjacency matrix of Y can be block diagonalized where the blocks are of the form*

$$M_\rho = \sum_{g \in G}^{\oplus} A(g) \otimes \rho(g),$$

each taken $d_\rho =$ degree of ρ times, as the representations ρ run through $\widehat{G}$. Here $A(g)$ is defined in formula (2.1).

Proof. The adjacency matrix A_Y of Y has the $(i, g), (j, h)$ entry for $i, j \in X$ and $g, h \in G$ given by

$$(A_Y)_{(i,a),(j,b)} = \text{the number of edges between } (i, a) \text{ and } (j, b). \tag{2.2}$$

and this is the same as the number of edges between (i, e) and $\left(j, a^{-1}b\right)$, if e is the identity of G.

Also define the $|G| \times |G|$ matrix $\sigma(g)$ indexed by elements $a, b \in G$:

$$(\sigma(g))_{a,b} = \begin{cases} 1, \text{ if } a^{-1}b = g, \\ 0, \text{ otherwise.} \end{cases} \tag{2.3}$$

Note that σ is essentially the matrix of the right regular representation of G, since if δ_a is the vector with 1 in the a position and 0 everywhere else, we have $\sigma(g)\delta_a = \delta_{ag^{-1}}$.

It follows from (2.1), (2.2), and (2.3) that

$$A_Y = \sum_{g \in G} A(g) \otimes \sigma(g). \tag{2.4}$$

One of the fundamental theorems of representation theory (see (Terras, 1999, p. 256)) says that

$$\sigma(g) \cong \sum_{\rho \in \widehat{G}}^{\oplus} d_\rho \rho(g). \tag{2.5}$$

It follows that $A_Y \cong \sum_{\rho\in\widehat{G}}^{\oplus} d_\rho M_\rho$. This completes the proof of Proposition 2.1. $\square$

Theorem 2.2. *Assume p is odd. $\mathcal{H}(p^{n+1})$ is an unramified graph covering of $\mathcal{H}(p^n)$. Moreover it is a normal covering with abelian Galois group*

$$\begin{aligned} &\mathrm{Gal}\left(\mathcal{H}\left(p^{n+1}\right)/\mathcal{H}\left(p^{n}\right)\right) \cong \Gamma \\ &\cong \left\{(a,b,c) \in \mathrm{Heis}\left(\mathbb{Z}/p^{n+1}\mathbb{Z}\right) \,|(a,b,c) \equiv 0 \pmod{p}^n \right\}. \end{aligned}$$

Proof. The projection $\pi : \mathcal{H}\left(p^{n+1}\right) \to \mathcal{H}(p^n)$ is just the reduction of the coordinates $\bmod p^{n+1}$ to coordinates $\bmod p^n$. Clearly this preserves adjacency. Moreover, given $g \in \mathcal{H}\left(p^n\right)$, if we take a point $g' \in \mathcal{H}\left(p^{n+1}\right)$ in $\pi^{-1}g$, we see that the points in $\mathcal{H}\left(p^{n+1}\right)$ adjacent to g' have the form $g's$, for $s \in S_0 = \left\{\ (\pm 1,0,0),(0,\pm 1,0)\ \right\}$. The points adjacent to g in $\mathcal{H}\left(p^n\right)$ are of the same form except computed $\bmod p^n$. And π maps these adjacent points in $\mathcal{H}\left(p^{n+1}\right)$ one-to-one, onto those in $\mathcal{H}\left(p^n\right)$.

If $(a,b,c) \in \Gamma$ defined in the statement of Theorem 2.2, we define the Galois group element

$$\gamma_{(a,b,c)}\left((x,y,z) \bmod p^{n+1}\right) = (a,b,c)(x,y,z) \bmod p^{n+1}.$$

It follows that $\pi\circ\gamma = \pi$, since $(a,b,c) \equiv 0 \,(\bmod p^n)$ and π reduces things mod p^n. Moreover, it is easy to see that Γ is abelian since if (a,b,c) and (u,v,w) are both $\equiv 0 \,(\bmod p^n)$, then $(a,b,c)(u,v,w) = (a+u, b+v, c+w+av)$ and p^n divides both a and v so that $av \equiv 0 \left(\bmod p^{n+1}\right)$. $\square$

Corollary 2.3. *The spectrum of $\mathcal{H}\left(p^n\right)$ is contained in the spectrum of $\mathcal{H}\left(p^{n+1}\right)$. Moreover $\zeta_{\mathcal{H}(p^n)}(u)^{-1}$ divides $\zeta_{\mathcal{H}(p^{n+1})}(u)^{-1}$.*

Proof. Use Proposition 2.1 or see (Stark and Terras, 1996, p. 131). $\square$

Example. The last Theorem and Corollary also work if $p = 2$, except that then the graph at the bottom of the cover can be a multi-graph when $n = 1$, as in Figure 1. Consider the covering $\mathcal{H}(4)$ over $\mathcal{H}(2)$. Note that $\mathcal{H}(2)$ is a multigraph with 2 edges between any vertices that are adjacent, because $1 \equiv -1 \pmod 2$ and we want the graph to have degree 4. So the graph of $\mathcal{H}(2)$ is a cycle graph as in Figure 1. We label the vertices using the following table.

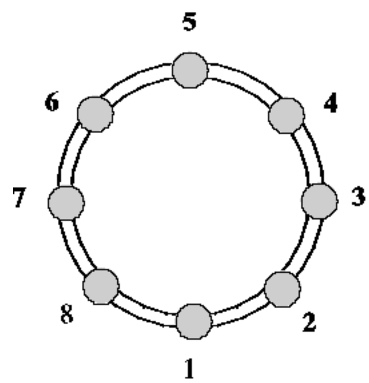

Figure 1. The Cayley Graph $\mathcal{H}(2) = X(\mathrm{Heis}(\mathbb{Z}/2\mathbb{Z}), \{(\pm 1, 0, 0), (0, \pm 1, 0)\})$.

Table 1. Vertex Labeling for $\mathcal{H}(2)$.

label	1	2	3	4	5	6	7	8
vertex	$(0,0,0)$	$(1,0,0)$	$(1,1,1)$	$(0,1,1)$	$(0,0,1)$	$(1,0,1)$	$(1,1,0)$	$(0,1,0)$

We obtain a spanning tree for $\mathcal{H}(2)$ by cutting one of each pair of double edges and then cutting both edges between vertices 6 and 7. This really gives a line graph but we will draw it as a circle cut between vertices 6 and 7. So we draw the covering graph $\mathcal{H}(4)$ by placing 8 copies of the cut circle which is the spanning tree of $\mathcal{H}(2)$ and labeling each with a group element from $\mathrm{Gal}(\mathcal{H}(4)/\mathcal{H}(2))$. We know that this can be identified with the subgroup of $\mathrm{Heis}(\mathbb{Z}/4\mathbb{Z})$ consisting of (u, v, w) where u, v, w are all even. We label the Galois group elements using the following table.

The covering graph $\mathcal{H}(4)/\mathcal{H}(2)$ has 8 sheets and each sheet is a copy of the spanning tree of $\mathcal{H}(2)$. So every point on $\mathcal{H}(4)$ has a label (n, v), where $1 \leq n \leq 8$ and $v \in \{a, b, c, d, e, f, g, h\}$. We will just write nv. See Figure 2 for a picture of the tree with connections between level a and the rest. You can use the action of the Galois group to find all the edges of $\mathcal{H}(4)$. It makes a pretty complicated figure. The following table

Table 2. Galois Group Labeling for Gal($\mathcal{H}(4)/\mathcal{H}(2)$). In this labeling, a not e is the identity of the group.

label	a	b	c	d	e	f	g	h
Galois group element	$(0,0,0)$	$(2,0,0)$	$(2,2,2)$	$(0,2,2)$	$(0,0,2)$	$(2,0,2)$	$(2,2,0)$	$(0,2,0)$

shows which connections are made in Figure 2. This table allows one to compute the matrices $A(g), g \in G = \text{Gal}(\mathcal{H}(4)/\mathcal{H}(2))$.

Table 3. Table of Connections Between Sheet a in $\mathcal{H}(4)$ and the other sheets.

vertex	adjacent vertices in $\mathcal{H}(4)$
$1a$	$2b, 8h, 2a, 8a$
$2a$	$1b, 3d, 1a, 3a$
$3a$	$2d, 4f, 2a, 4a$
$4a$	$3f, 5h, 3a, 5a$
$5a$	$4h, 6b, 4a, 6a$
$6a$	$5b, 7e, 7h, 5a$
$7a$	$6e, 6h, 8f, 8a$
$8a$	$1h, 7f, 7a, 1a$

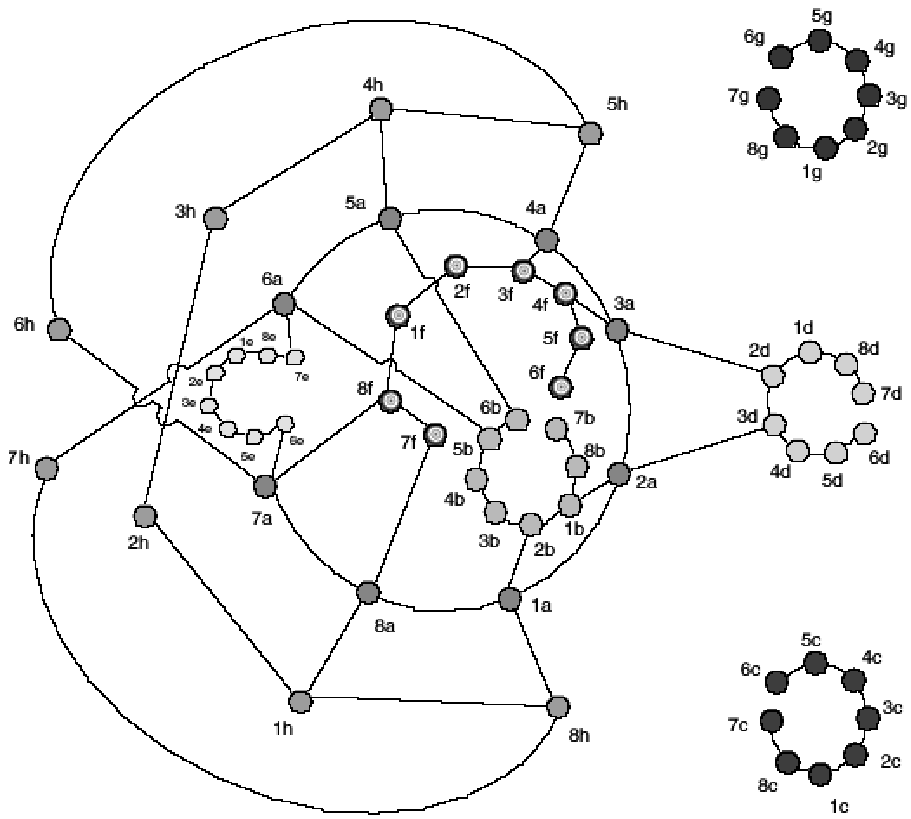

Figure 2. Connections Between Level a and the Rest of the Cayley Graph $\mathcal{H}(4) = X(\text{Heis}(\mathbb{Z}/4\mathbb{Z}), \{(\pm 1, 0, 0), (0, \pm 1, 0\})$

The representations of the abelian Galois group have the form $\chi_{r,s,t}(a, b, c) = \exp\left(\frac{2\pi i(ra+sb+tc)}{4}\right)$, for $r, s, t \pmod 2$. Then one must compute the matrices $M_{\chi_{r,s,t}}$ appearing in Proposition 2.1. For example

$M_{\chi_{0,0,0}}$ is the adjacency matrix of $\mathcal{H}(2)$ and

$$M_{\chi_{0,1,1}} = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
$$M_{\chi_{1,0,0}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.6}$$

The eigenvalues of the M_χ are to be found in the following table.

Table 4. Eigenvalues of $M_{r,s,t} = M_{\chi_{r,s,t}}$.

(r, s, t)	Eigenvalues of $M_{r,s,t}$
$(0,0,0)$	$-4, 0, 0, 4, -2\sqrt{2}, -2\sqrt{2}, 2\sqrt{2}, 2\sqrt{2}$
$(1,0,0)$ and $(0,1,0)$	$-2, -2, -2, -2, 2, 2, 2, 2$
$(1,1,0)$	$0, 0, 0, 0, 0, 0, 0, 0$
$(1,1,1), (0,1,1), (0,0,1)$, and $(1,0,1)$	$0, 0, 0, 0, -2\sqrt{2}, -2\sqrt{2}, 2\sqrt{2}, 2\sqrt{2}$

So we see that the spectrum of $\mathcal{H}(4)$ for $p = 2$ is given in Table 5.

Table 5. Spectrum of $X(\mathrm{Heis}(\mathbb{Z}/4\mathbb{Z})), \{(\pm 1, 0, 0), (0, \pm 1, 0)\}$.

eigenvalue	multiplicity
± 4	1
0	26
± 2	8
$\pm 2\sqrt{2}$	10

The **Artin L-function** associated to the representation ρ of $G = \mathrm{Gal}(Y/X)$ can be defined by a product over prime cycles in X as

$$L(u,\rho,Y/X) = \prod_{[C] \text{ prime in } X} \det\left(I - \rho\left(\mathrm{Frob}\left(\tilde{C}\right)\right) u^{\nu(C)}\right)^{-1}, \quad (2.7)$$

where $\tilde{C}$ denotes any lift of C to Y and $\mathrm{Frob}\left(\tilde{C}\right)$ denotes the Frobenius automorphism defined by

$$\mathrm{Frob}\left(\tilde{C}\right) = ji^{-1},$$

if $\tilde{C}$ starts on Y-sheet labeled by $i \in G$ and ends on Y-sheet labeled by $j \in G$. As in Proposition 2.1, define

$$M_\rho = \sum_{g \in G} A(g) \otimes \rho(g). \quad (2.8)$$

Then, setting $Q_\rho = Q \otimes I_{d_\rho}$, with $d_\rho = d = \deg \rho$, we have the following analogue of formula (1.2):

$$L(u,\rho,Y/X)^{-1} = \left(1-u^2\right)^{(r-1)d_\rho} \det\left(I - M_\rho u + Q_\rho u^2\right). \quad (2.9)$$

See (Stark and Terras, 1996) for an elementary proof and more information.

Formula (2.5) implies that the zeta function of Y factors as follows

$$\zeta_X(u) = \prod_{\rho \in \widehat{G}} L(u,\rho,Y/X)^{d_\rho}. \quad (2.10)$$

See (Stark and Terras, 2000).

For our example, the Galois group is abelian and all degrees are 1. We obtain a factorization of the Ihara-Selberg zeta function of $\mathcal{H}(4)$ as a product of Artin L-functions of the Galois group of $\mathcal{H}(4)/\mathcal{H}(2)$. We use definition (2.8)) and Table 3 to compute the matrices $M_{\chi_{r,s,t}}$ as in formula (2.6). Then formula (2.9) gives the following list of L-functions. Here $Q = 3I_8$, $r = 9$.

Reciprocals of L-functions for $H(4)/H(2)$.

1) For $\chi = \chi_{0,0,0}$, $A =$ adjacency matrix of $\mathcal{H}(2)$, and

$$\begin{aligned} \zeta_{\mathcal{H}(2)}(u)^{-1} = L(u,1)^{-1} &= \left(1-u^2\right)^8 (u-1)(u+1) \\ &\times (3u-1)(3u+1)\left(3u^2+1\right)^2\left(9u^4-2u^2+1\right)^2. \end{aligned} \quad (2.11)$$

2) $L\left(u,\chi_{1,0,0}\right)^{-1} = L\left(u,\chi_{0,1,0}\right)^{-1}$
$= \left(1-u^2\right)^8 \left(3u^2+2u+1\right)^4 \left(3u^2-2u+1\right)^4.$

3) $L\left(u,\chi_{1,1,1}\right)^{-1} = L\left(u,\chi_{0,1,1}\right)^{-1} = L\left(u,\chi_{0,0,1}\right)^{-1} = L\left(u,\chi_{1,0,1}\right)^{-1}$
$= \left(1-u^2\right)^8 \left(9u^4-2u^2+1\right)^2 \left(3u^2+1\right)^4.$

4) When $\rho = \chi_{1,1,0}$ we find that $M_{\chi_{1,1,0}} = 0$, so that

$$\begin{aligned} L\left(u,\chi_{1,1,0}\right)^{-1} &= \left(1-u^2\right)^{(r-1)d} \det\left(I + Q_\rho u^2\right) \\ &= \left(1-u^2\right)^8 \left(1+3u^2\right)^8. \end{aligned}$$

It follows from these computations and (2.10) that the Ihara zeta function of $\mathcal{H}(4)$ is

$$\begin{aligned} \zeta_{\mathcal{H}(4)}(u)^{-1} = & -\left(1-u^2\right)^{65} \left(9u^2-1\right) \left(3u^2+1\right)^{26} \\ & \times \left(9u^4-2u^2+1\right)^{10} \left(3u^2+2u+1\right)^8 \left(3u^2-2u+1\right)^8. \end{aligned} \tag{2.12}$$

Consider the torus graphs

$$\mathcal{T}^{(n)}(q) = X\left((\mathbb{Z}/q\mathbb{Z})^n, \{\pm e_1, \pm e_2, \cdots, \pm e_n\}\right),$$

where e_i denotes the vector with 1 in the ith coordinate and 0 elsewhere. Because the torus groups $(\mathbb{Z}/q\mathbb{Z})^n$ are abelian, it is relatively easy to generate spectra. In fact, the eigenvalues of the adjacency matrix of $\mathcal{T}^{(n)}(q)$ are

$$\lambda_a = 2\left(\cos\left(\frac{2\pi i a_1 b_1}{q}\right) + \cos\left(\frac{2\pi i a_2 b_2}{q}\right) + \cdots + \cos\left(\frac{2\pi i a_n b_n}{q}\right)\right),$$

for $a, b \in (\mathbb{Z}/q\mathbb{Z})^n$. Note that, by a result of our earlier paper (DeDeo et al., 2004), the part of the spectrum of the degree 4 Heisenberg graph $\mathcal{H}(4)$ corresponding to 1-dimensional representations of $\mathcal{H}(4)$ contains the spectrum of $\mathcal{T}^{(2)}(4)$. One obtains a second proof of this fact by noting that $\mathcal{H}(4)$ is actually a covering graph of $\mathcal{T}^{(2)}(4)$, via the covering map sending (x,y,z) to (x,y).

We can easily compute the Selberg-Ihara zeta functions of the small torus graphs using covering graph theory. As in Theorem 2.2, the Galois group of $\mathcal{T}^{(n)}\left(p^{r+1}\right)/\mathcal{T}^{(n)}\left(p^r\right)$ is

$$\Gamma \cong \left\{x \in \left(\mathbb{Z}/p^{r+1}\mathbb{Z}\right)^n \mid x \equiv 0 \pmod{p})^r\right\}.$$

Since the 1-dimensional graphs are cycles, we know that

$$\zeta_{\mathcal{T}^{(1)}(q)}(u)^{-1} = \left(1-u^q\right)^2, \text{ for all } q.$$

In 2-dimensions, we consider only the smallest values of q (namely $q = 2$ and $q = 4$) and find that if $\Gamma = \mathrm{Gal}\left(\mathcal{T}^{(2)}(4)/\mathcal{T}^{(2)}(2)\right)$, the representations of Γ have the form $\chi_{r,s}(x,y) = \exp\left(\frac{2\pi i(rx+sy)}{4}\right)$, for $(x,y) \in \Gamma$, $(r,s) \in (\mathbb{Z}/2\mathbb{Z})^2$. Therefore $(x,y) \equiv 0 \pmod 2$. It follows that

$$\begin{aligned}\zeta_{\mathcal{T}^{(2)}(4)}(u)^{-1} &= \zeta_{\mathcal{T}^{(2)}(2)}(u)^{-1} L\left(u,\chi_{0,1}\right)^{-1} L\left(u,\chi_{1,1}\right)^{-1} L\left(u,\chi_{1,0}\right)^{-1} \\ &= -\left(1-u^2\right)^{17}\left(9u^2-1\right)\left(3u^2+1\right)^{6}\left(3u^2-2u+1\right)^{4}\left(3u^2+2u+1\right)^{4}.\end{aligned}$$

Here $\zeta_{\mathcal{T}^{(2)}(2)}(u)^{-1} = -\left(1-u^2\right)^5\left(9u^2-1\right)\left(3u^2+1\right)^2$.

From these results plus (2.11) and (2.12) we see that

$$\begin{aligned}\zeta_{\mathcal{H}(4)}(u)^{-1}/\zeta_{\mathcal{T}^{(2)}(4)}(u)^{-1} &= \left(1-u^2\right)^{48}\left(3u^2+1\right)^{20} \\ &\times\left(3u^2-2u+1\right)^{4}\left(3u^2+2u+1\right)^{4}\left(9u^4-2u^2+1\right)^{10}\end{aligned} \tag{2.13}$$

and

$$\zeta_{\mathcal{H}(2)}(u)^{-1}/\zeta_{\mathcal{T}^{(2)}(2)}(u)^{-1} = \left(1-u^2\right)^{4}\left(9u^4-2u^2+1\right)^{2}. \tag{2.14}$$

Figure 3 shows a contour plot of the absolute value of $\zeta_{\mathcal{H}(2)}(x+iy)^{-1}$ made using the Mathematica command Plot3D. It should be compared with Figure 4 showing a contour plot of the 1/10 power of the absolute value of $\zeta_{\mathcal{H}(4)}(x+iy)^{-1}$. The roots of $\zeta_{\mathcal{H}(4)}(x+iy)^{-1}$ (not counting multiplicity) are approximately the following 14 numbers:

$$\begin{gathered}-1, -0.333333, 0.57735i, -0.471405-0.333333i, \\ -0.471405+0.333333i, -0.333333-0.471405i, \\ -0.333333+0.471405i, 0.333333-0.471405i, \\ 0.333333+0.471405i, 0.471405-0.333333i, \\ 0.471405+0.333333i, -0.57735i, 0.333333, 1.\end{gathered}$$

Future Work. There are many other questions one can ask in this context. One should study the zeros of Ihara-Selberg zeta functions $\mathcal{H}(q)$ for large q. One should consider these questions for Cayley graphs of other finite groups and even for irregular graphs for which there is no obvious relation between the spectrum of the adjacency matrix and the zeros of the Ihara-Selberg zeta function.

Can such zeta functions be used to recognize groups involved in Cayley graphs? In particular, one wonders whether you can see the shape of a group by staring at zeros of the zeta function of Cayley graphs associated to the group? This is an analogous question to that of Mark Kac about hearing the shape of a drum (as the Dirichlet spectrum of

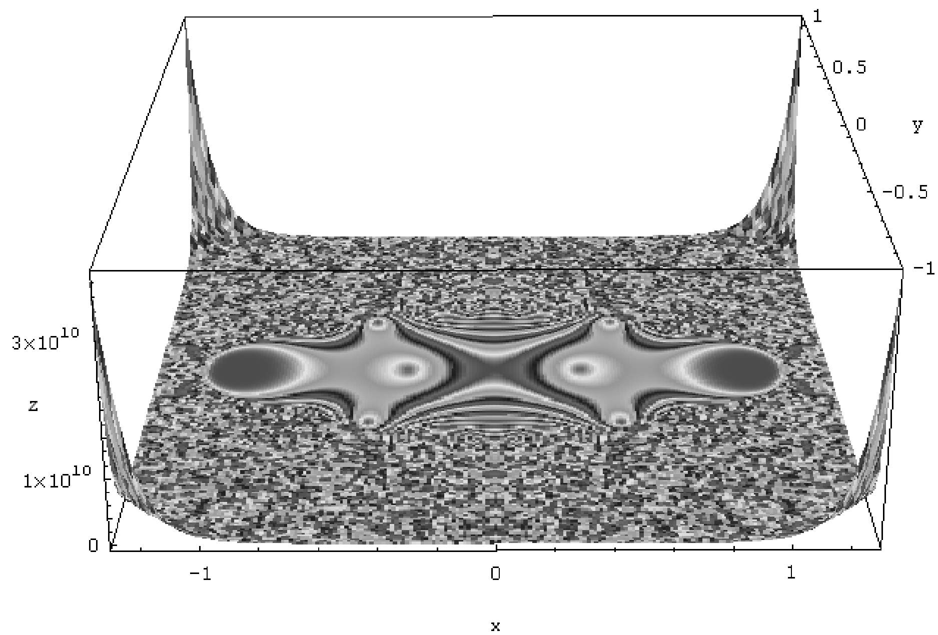

Figure 3. A contour plot of the absolute value of $\zeta_{\mathcal{H}(2)}(x+iy)^{-1}$.

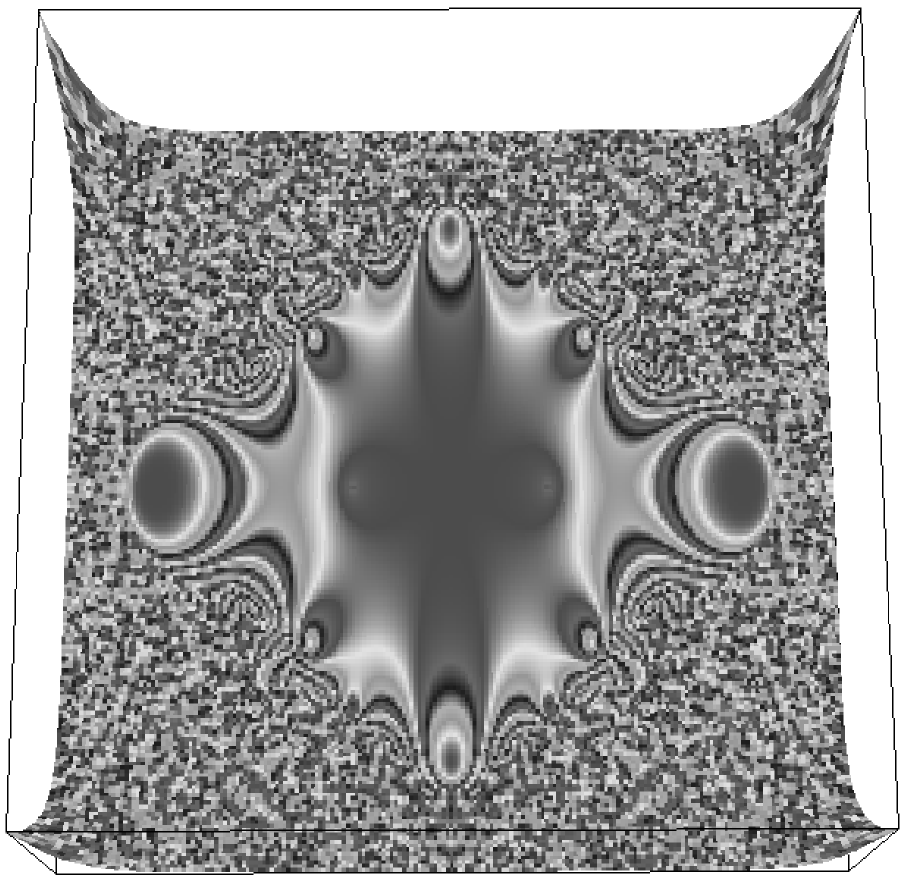

Figure 4. A contour plot of 1/10 power of the absolute value of $\zeta_{\mathcal{H}(4)}(x+iy)^{-1}$.

the Laplace operator on a plane drum determines the fundamental frequencies of vibration). Here we wonder if one can somehow recognize groups from properties of the zero set of zeta functions of associated Cayley graphs with some sort of condition on the generating sets S. Instead of hearing the drum in its spectrum, we are trying to see it. Of course, it is known that there are graphs with the same zeta function that are not isomorphic. See (Stark and Terras, 2000) for examples that are connected, regular, without loops or multiple edges.

References

DeDeo, M., Martínez, M., Medrano, A., Minei, M., Stark, H., and Terras, A. (2004). Spectra of Heisenberg graphs over finite rings. In Feng, W., Hu, S., and Lin, X., editors, *Discrete and Continuous Dynamical Systems, 2003 Supplement Volume*, pages 213–222. Proc. of 4th Internatl. Conf. on Dynamical Systems and Differential Equations.

Diaconis, P. and Saloff-Coste, L. (1994). Moderate growth and random walk on finite groups. *Geom. Funct. Anal.*, 4:1–36.

Godsil, C. D. (1993). *Algebraic Combinatorics*. Chapman and Hall, New York.

Hashimoto, K. (1990). On Zeta and L-functions of finite graphs. *Intl. J. Math.*, 1(4):381–396.

Hofstadter, D. R. (1976). Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields. *Physical Review B*, 14:2239–2249.

Katz, N. and Sarnak, P. (1999). Zeros of zeta functions and symmetry. *Bull. Amer. Math. Soc.*, 36(1):1–26.

Kemperman, J. H. B. (1961). *The Passage Problem for a Stationary Markov Chain*. Univ. of Chicago Press, Chicago, IL.

Kotani, M. and Sunada, T. (2000). Spectral geometry of crystal lattices. Preprint.

Lin, X.-S. and Wang, Z. (2001). Random walk on knot diagrams, colored Jones polynomial and Ihara-Selberg zeta function. In Gillman, J., Menasco, W., and Lin, X.-S., editors, *Knots, Braids, and Mapping Class Groups–Papers dedicated to Joan S. Birman (New York, 1998)*, volume 24 of *AMS/IP Studies in Adv. Math.*, pages 107–121. Amer. Math. Soc., Providence, RI.

Lubotzky, A., Phillips, R., and Sarnak, P. (1988). Ramanujan graphs. *Combinatorica*, 8:261–277.

Myers, P. (1995). *Euclidean and Heisenberg Graphs: Spectral Properties and Applications*. PhD thesis, Univ. of California, San Diego.

Rosen, M. (2002). *Number Theory in Function Fields*, volume 210 of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

Sarnak, P. (1995). Arithmetic quantum chaos. In *The Schur lectures (1992) (Tel Aviv)*, volume 8 of *Israel Math. Soc. Conf. Proc.*, pages 183–236. Bar-Ilan Univ., Ramat-Gan, Israel.

Stark, H. M. and Terras, A. (1996). Zeta functions of finite graphs and coverings. *Adv. in Math.*, 121:124–165.

Stark, H. M. and Terras, A. (2000). Zeta functions of finite graphs and coverings. II. *Adv. in Math.*, 154(1):132–195.

Terras, A. (1999). *Fourier Analysis on Finite Groups and Applications*. Cambridge Univ. Press, Cambridge, UK.

Terras, A. (2000). Statistics of graph spectra for some finite matrix groups: finite quantum chaos. In Dunkl, C., Ismail, M., and Wong, R., editors, *Special Functions (Hong Kong, 1999)*, pages 351–374. World Scientific, Singapore.

Terras, A. (2002). Finite quantum chaos. *Amer. Math. Monthly*, 109(2):121–139.

Zack, M. (1990). Measuring randomness and evaluating random number generators using the finite Heisenberg group. In *Limit Theorems in Probability and Statistics (Pécs, 1989)*, volume 57 of *Colloq. Math. Soc. János Bolyai*, pages 537–544. North-Holland, Amsterdam.