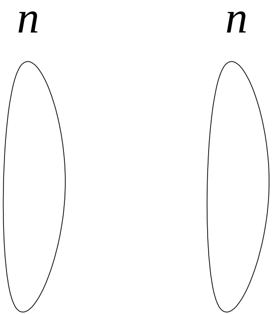
# Density and Regularity theorems for semi-algebraic hypergraphs

Jacob Fox (MIT), Janos Pach (EPFL), Andrew Suk (UIC)

January 3, 2015

## An old Ramsey-type result, Kövári, Sós, and Turán and Erdős

Bipartite graph $G$, edge set $E$.



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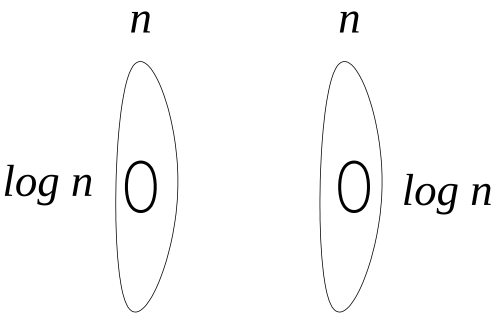

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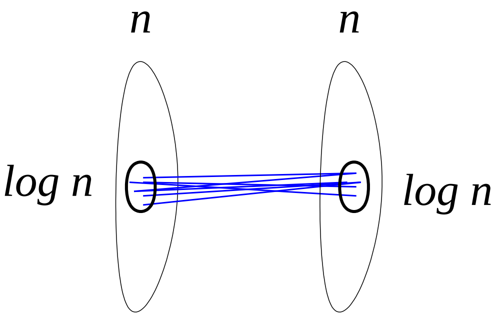

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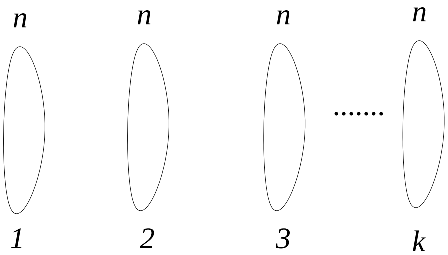
$k$-partite $k$-uniform hypergraph $H$, edge set $E$.



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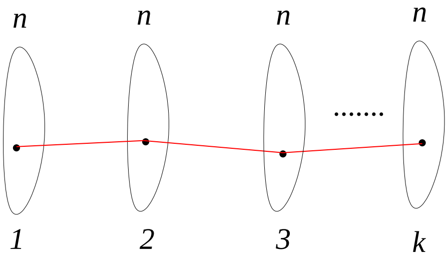

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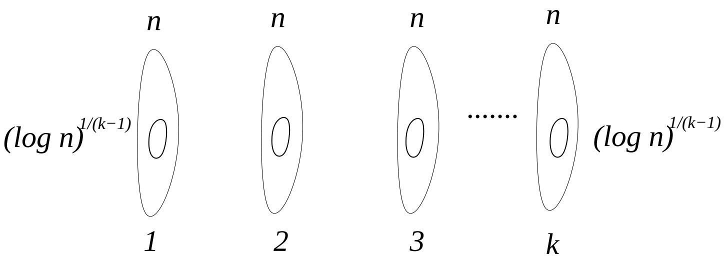

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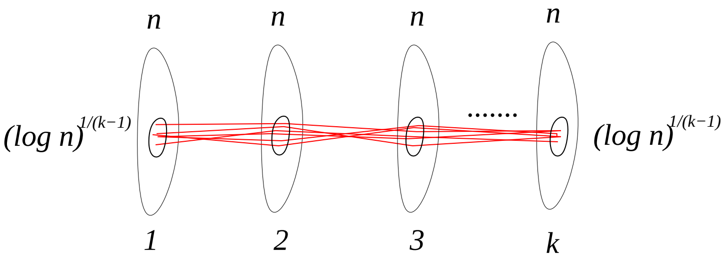


These results are tight.

# Generalization, Erdős

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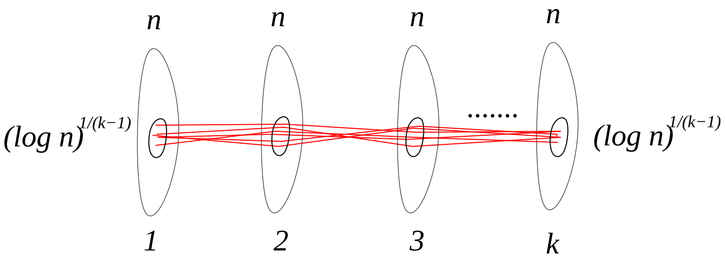



**In this talk:** We can do much better if $H$ is a semi-algebraic $k$-uniform hypergraph.

We say that $H = (V, E)$ is a **semi-algebraic $k$-uniform hypergraph in $d$-space** if

$V = \{n \text{ points in } \mathbb{R}^d\}$

$E$ defined by polynomials $f_1, ..., f_t$ and a Boolean formula $\Phi$ such that

$$(p_{i_1}, ..., p_{i_k}) \in E$$

$$\Leftrightarrow \Phi(f_1(p_{i_1}, ..., p_{i_k}) \geq 0, ..., f_t(p_{i_1}, ..., p_{i_k}) \geq 0) = \text{yes}$$

# Complexity of relation $E$

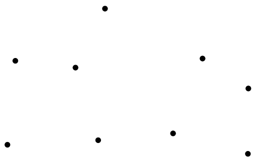
$x_i \in \mathbb{R}^d$

$E$ has complexity $(t, D)$

1. described by polynomials $f_1, ..., f_t$,
2. and the degree of ALL $kt$ $d$-variate polynomials
   $f_i(x_1, ..., x_{k-1}, x_k), f_i(x_1, ..., x_{k-2}, x_{k-1}, x_k), ..., f_i(x_1, x_2, ..., x_k)$,
   for $i = 1...t$, is at most $D$.

Note. $f_i$ has degree at most $Dk$.

# Example



$V = \{n \text{ points in the plane}\}$,
$E = \{\text{triples having a clockwise orientation}\}$.

$H = (V, E)$ semi-algebraic 3-uniform hypergraph in the plane $(d = 2)$

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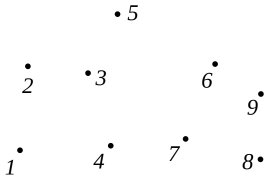


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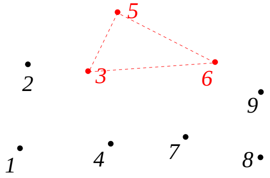


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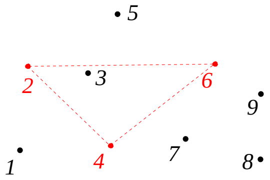


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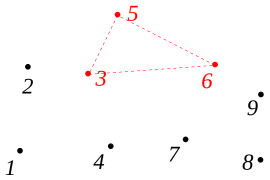
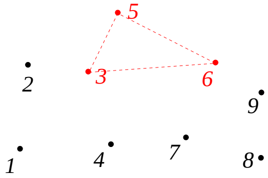



$E = \{$triples having a clockwise orientation$\}$.

$$det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} > 0.$$

# Example

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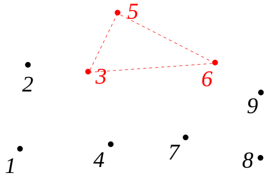


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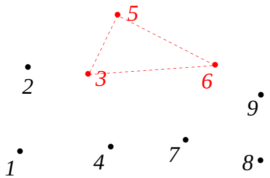


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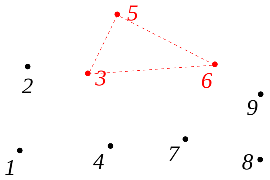


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$E = (d+1)$-tuples with a positive orientation, complexity $(t, D) = (1, 1)$.

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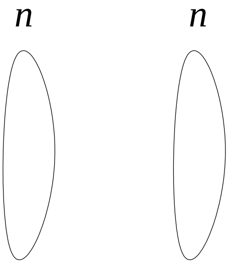
# Previous results

**Theorem (Alon, Pach, Pinchasi, Radoicic, Sharir 2005)**

*Let $H = (V_1, V_2, E)$ be a bipartite semi-algebraic graph $(k = 2)$ in $d$-space, where $|V_1| = |V_2| = n$ and $E$ has complexity $(t, D)$. Then there are subsets $V_1', V_2' \subset V$ such that $|V_i'| \geq \epsilon n$ and either $(V_1', V_2') \subset E$ or $(V_1', V_2') \subset \overline{E}$, and $\epsilon = \epsilon(d, t, D)$.*



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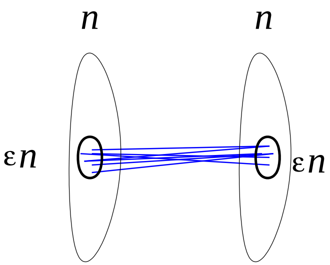

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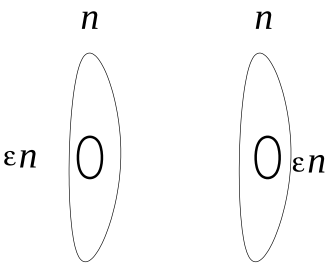

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Including an argument of Komlos:

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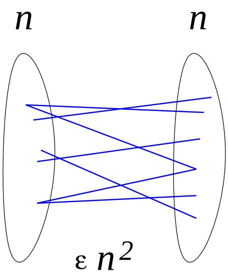

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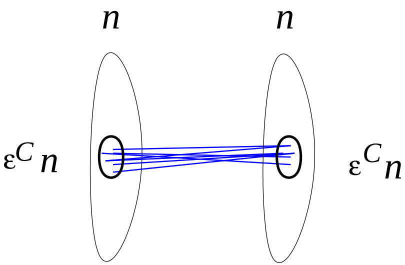

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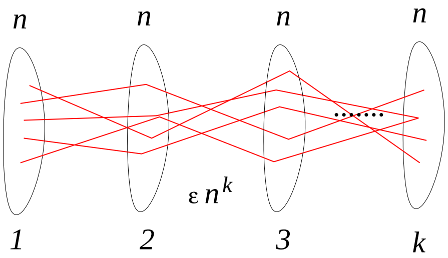

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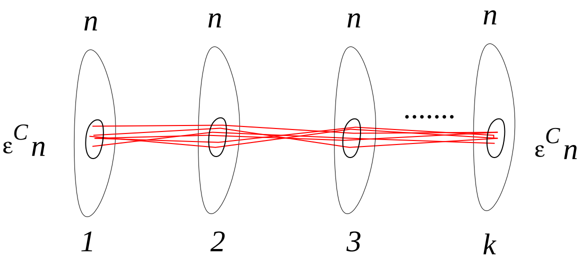


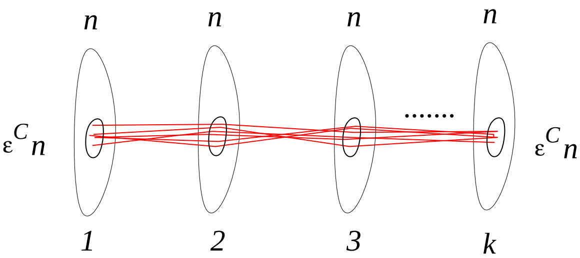
# Generalization

$C(k,d,t,D)$: Dependency on uniformity $k$ is very bad.

Fox, Gromov, Lafforgue, Naor, Pach: $C(k,d,t,D) \sim \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^d}}}}}}_{k}$

(tower-type)

Bukh-Hubard: $C(k,d,t,D) \sim 2^{2^{k+d}}$, double exponential in $k$.




Bukh-Hubard: Set sizes decay triple exponentially in $k$

# New results

For simplicity, complexity $(t, D)$ is fixed.

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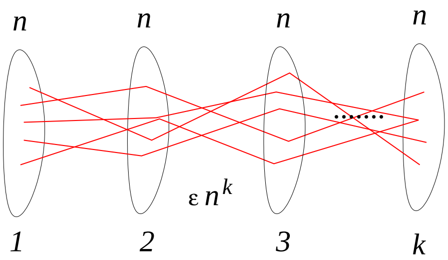

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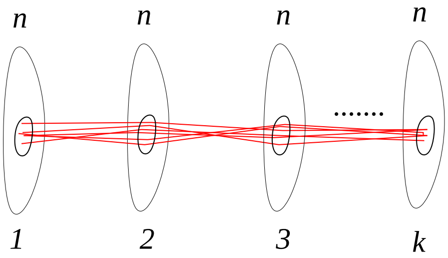

# New results, $D = 1$

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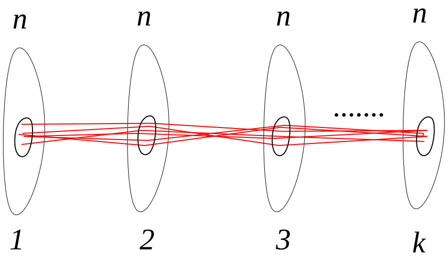


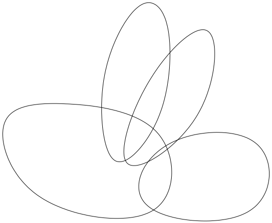
# Applications, Tverberg-type result

**Theorem (Pach, 1998)**

*Let $P_1, P_2, \ldots, P_{d+1} \subset \mathbb{R}^d$ be disjoint n-element point sets with $P_1 \cup \cdots \cup P_{d+1}$ in general position. Then there is a point $q \in \mathbb{R}^d$ and subsets $P'_1 \subset P_1, \ldots, P'_{d+1} \subset P_{d+1}$, with*

$$|P'_i| \geq 2^{-2^{2^{O(d)}}} n,$$

*such that all closed rainbow simplices contains q.*



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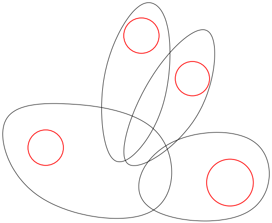

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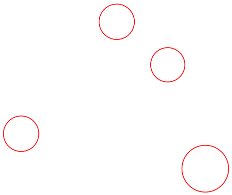

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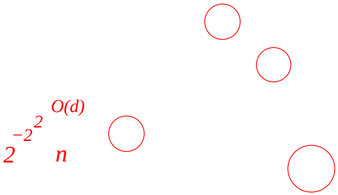

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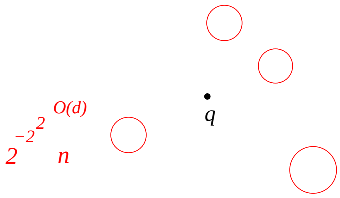

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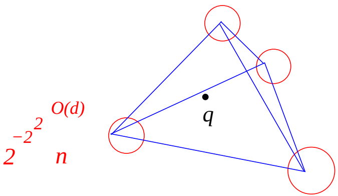

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*such that all closed rainbow simplices contains q.*

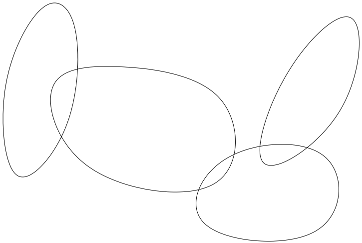
# Applications, Same-type Lemma

**Theorem (Bárány and Valtr, 1998)**

*Let $P_1, \ldots, P_k$ be n-element point sets in $\mathbb{R}^d$ such that $P_1 \cup \cdots \cup P_k$ is in general position. Then there are subsets $P_1' \subset P_1, \ldots, P_k' \subset P_k$ such that the k-tuple $(P_1', \ldots, P_k')$ has same-type transversals and*

$$|P_i'| \geq 2^{-k^{O(d)}} n,$$

*for all $i$.*



# Applications, Same-type Lemma

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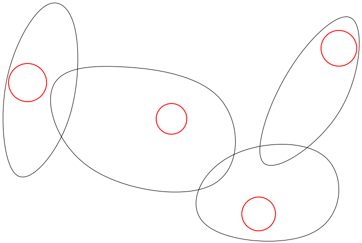

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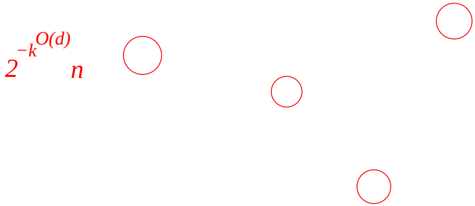

# Applications, Same-type Lemma

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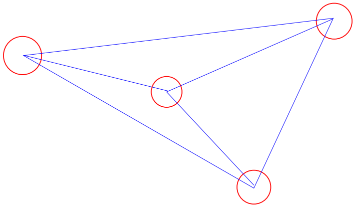
$2^{-k^{O(d)}} n$



# Applications, Same-type Lemma

**Theorem (Bárány and Valtr, 1998)**

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**Theorem (Fox, Pach, S., 2014)**

*Let $P_1, \ldots, P_k$ be n-element point sets in $\mathbb{R}^d$ such that $P_1 \cup \cdots \cup P_k$ is in general position. Then there are subsets $P_1' \subset P_1, \ldots, P_k' \subset P_k$ such that the k-tuple $(P_1', \ldots, P_k')$ has same-type transversals and*

$$|P_i'| \geq 2^{-O(d^3 k \log k)} n,$$

*for all $i$.*

Regularity lemma: $H = (P, E)$ semi-algebraic $k$-uniform hypergraph in $\mathbb{R}^d$.

**Theorem (Fox, Pach, S., 2014)**

*For any $\epsilon > 0$, we can partition $P$ into at most $M(\epsilon)$ parts, such that almost all $k$-tuples of parts are* **complete or empty**. *Moreover $M(\epsilon) < (1/\epsilon)^c$, where $c$ depends only on $k, d, E$.*

Usual regularity: almost all $k$-tuples of parts are "**random**". $M(\epsilon)$ is huge:

- $k = 2$, $M(\epsilon) \leq tower(1/\epsilon) = 2^{2^{2^{\cdots 2}}}$
- $k = 3$, $M(\epsilon) \leq wowzer(1/\epsilon) = tower(tower(...(tower(2))))$
- $k = 4$, $M(\epsilon) \leq wowzer(wowzer(...(wowzer(2))))$.

# Future work and open problems

1. Find more applications.
2. Extend results to more complicated relations, i.e., Semi-Pfaffian, o-minimal, etc.

**Thank you!**