Disjoint edges in complete topological graphs

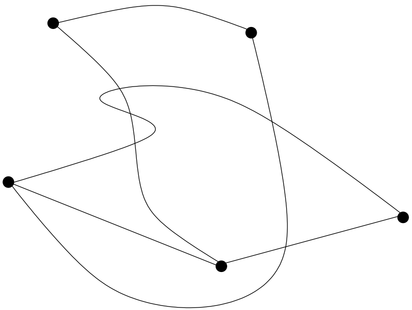
Andrew Suk

June 16, 2012

Problem: Given a complete n-vertex simple topological graph G, what is the size of the largest subset of pairwise disjoint edges.

Definition

A *topological graph* is a graph drawn in the plane with vertices represented by points and edges represented by curves connecting the corresponding points. A topological graph is *simple* if every pair of its edges intersect at most once.



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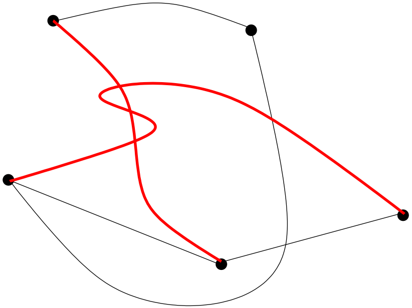
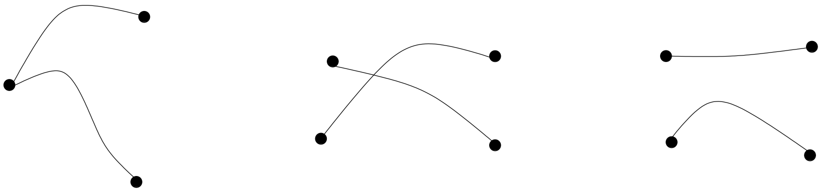


We will only consider *simple* topological graphs.



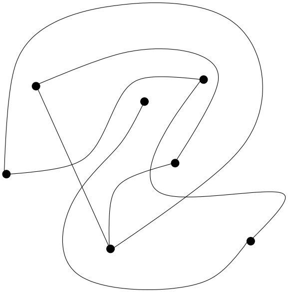
Conjecture (Conway)

Every n-vertex simple topological graph with no two disjoint edges, has at most n edges.

Theorem (Lovász, Pach, Szegedy, 1997)

Every n-vertex simple topological graph with no two disjoint edges, has at most 2n edges.

Best known $1.43n$ by Fulek and Pach, 2010.



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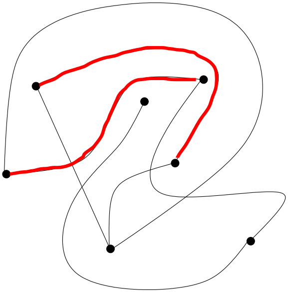


Generalization.

Theorem (Pach and Tóth, 2005)

Every n-vertex simple topological graph with no k pairwise disjoint edges, has at most $C_k n \log^{5k-10} n$ edges.

Conjecture to be at most $O(n)$ (for fixed k). By solving for k in $C_k n \log^{5k-10} n = \binom{n}{2}$.

Corollary (Pach and Tóth, 2005)

Every complete n-vertex simple topological graph has at least $\Omega(\log n / \log\log n)$ pairwise disjoint edges.

Conjecture (Pach and Tóth)

There exists a constant δ, such that every complete n-vertex simple topological graph has at least $\Omega(n^{\delta})$ pairwise disjoint edges.

History

Pairwise disjoint edges in complete n-vertex simple topological graphs:

1. $\Omega(\log^{1/6} n)$, Pach, Solymosi, Tóth, 2001.
2. $\Omega(\log n/\log\log n)$, Pach and Tóth, 2005.
3. $\Omega(\log^{1+\epsilon} n)$, Fox and Sudakov, 2008.

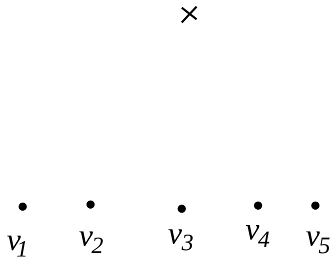
Note $\epsilon \approx 1/50$. All results are slightly stronger statements.

Main result

Theorem (Suk, 2011)

Every complete n-vertex simple topological graph has at least $\Omega(n^{1/3})$ pairwise disjoint edges.

Clearly the simple condition is required.



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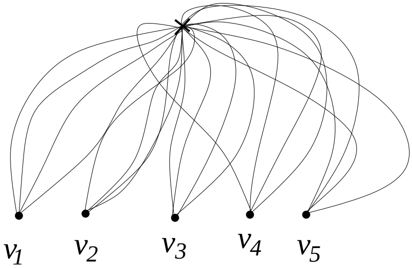

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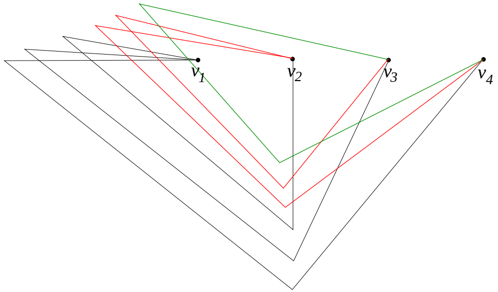
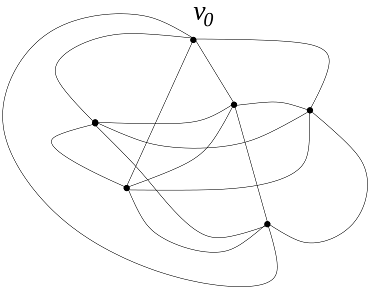


Sketch of proof

Theorem (Suk, 2011)

Every complete n-vertex simple topological graph has at least $\Omega(n^{1/3})$ pairwise disjoint edges.

K_{n+1}



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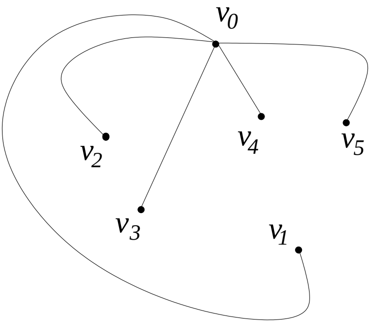
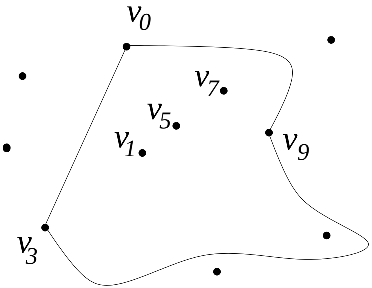


Define $\mathcal{F}_1 = \bigcup_{1 \le i < j \le n} \{S_{i,j}\}$, where $S_{i,j}$ is the set of vertices inside triangle v_0, v_i, v_j.



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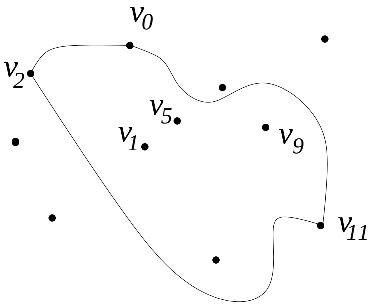

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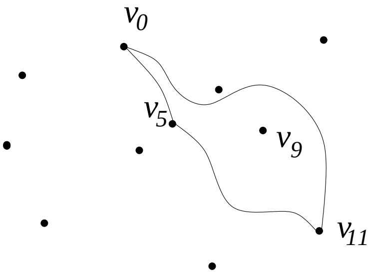


$S_{3,9} = \{v_1, v_5, v_7\}$, $S_{2,11} = \{v_1, v_5, v_9\}$, $S_{5,11} = \{v_9\}$.

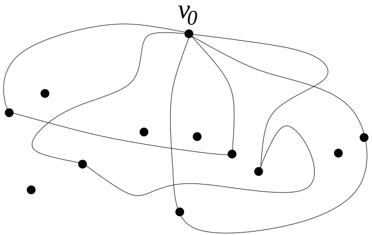
$\mathcal{F}_1$ is not "complicated".

Lemma

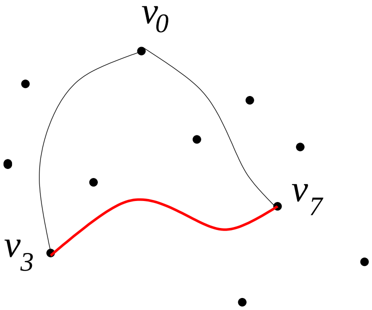
Any m sets in $\mathcal{F}_1$, $S_1, ..., S_m$, partitions the ground set X into $O(m^2)$ equivalence classes.

Vertices $x \sim y$, if both x, y belong to the exact same sets among $S_1, ..., S_m$. In other words, no set among $S_1, ..., S_m$ contains x and not y (and vice versa).

Proof: m triangles partitions the plane into $O(m^2)$ cells.



Define set system $\mathcal{F}_2 = \bigcup_{1 \le i < j \le n} \{S'_{i,j}\}$, where $v_k \in S'_{i,j}$ if topological edges $v_0 v_k$ and $v_i v_j$ cross.



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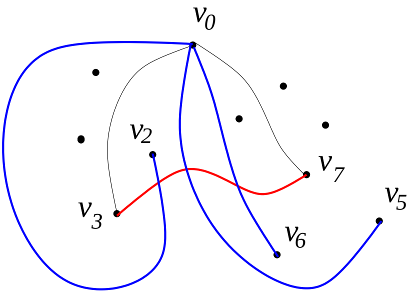

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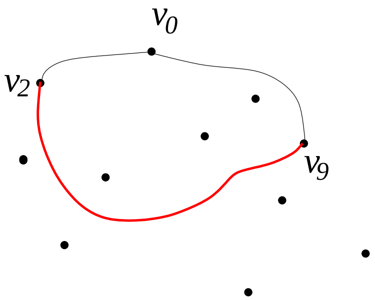

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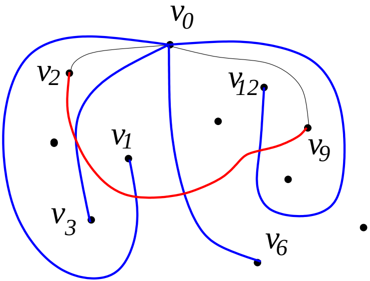


$S'_{3,7} = \{v_2, v_6, v_5\}, S'_{2,9} = \{v_1, v_3, v_6, v_{12}\}.$

Again, $\mathcal{F}_2$ is not "complicated". Set $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. One can show

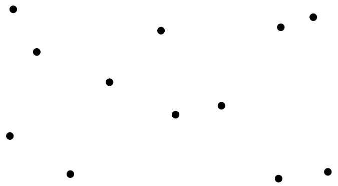
Lemma

Any m sets in $\mathcal{F}$ partitions X into at most $O(m^3)$ equivalence classes.

Main tool

Theorem (Matching theorem, Chazelle and Welzl, 1989)

Let $\mathcal{F}$ be a set system on an n element point set X (n is even), such that any m sets in $\mathcal{F}$ partitions X into at most $O(m^3)$ equivalence classes. Then there exists a perfect matching M on X such that each set in $\mathcal{F}$ stabs at most $O(n^{2/3})$ members in M.



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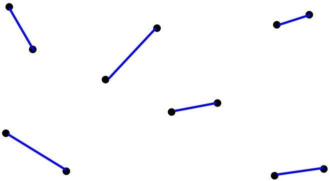

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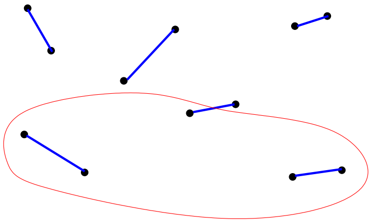

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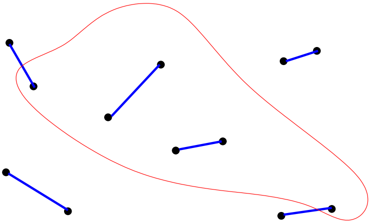

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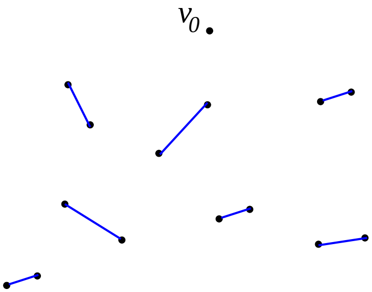
Auxiliary graph G, where $V(G) = M$ and $v_i v_j \to v_k v_l$ if $S_{i,j}$ or $S'_{i,j}$ stabs $v_k v_l$.



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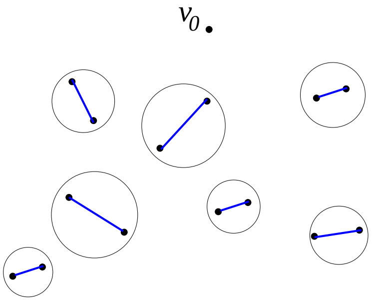

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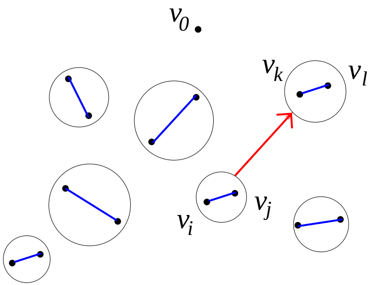

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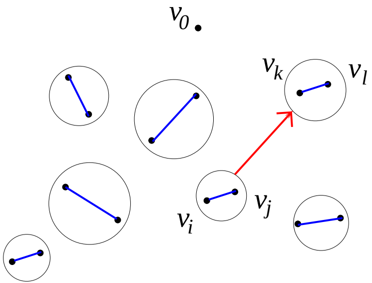
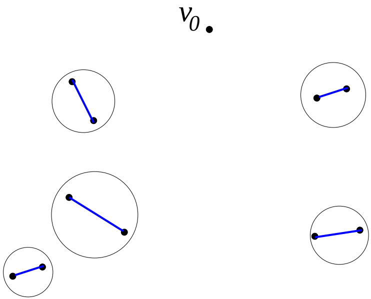


$S_{i,j}$ and $S'_{i,j}$ stabs (in total) at most $O(n^{2/3})$ members in $M = V(G)$. $|E(G)| \leq O(n^{5/3})$.

$|E(G)| \leq O(n^{5/3})$, by Turán, G contains an independent set of size $\Omega(n^{1/3})$.



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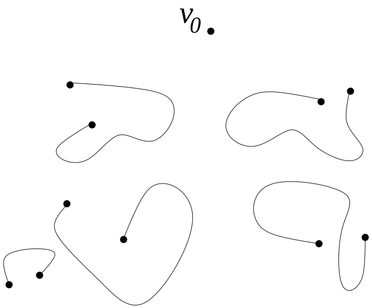

Claim!

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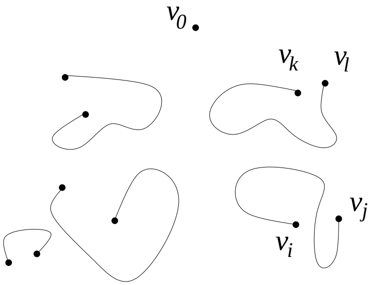
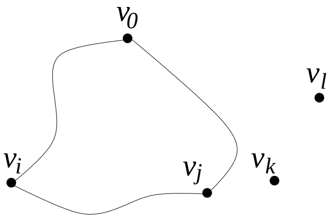
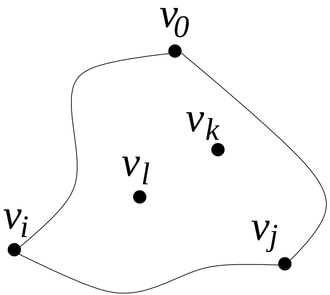
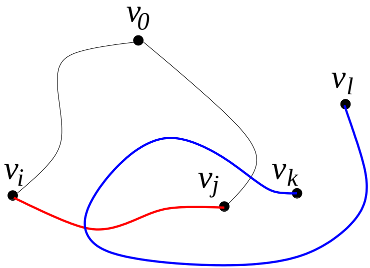


Claim!

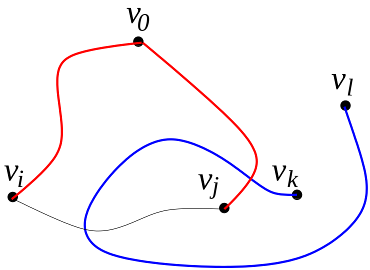
Since $S_{i,j}$ does NOT stab $v_k v_l$



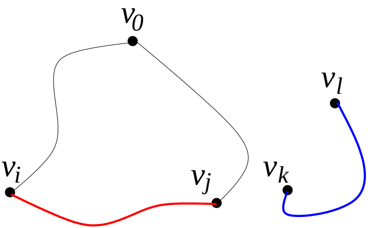




Assume edges cross.

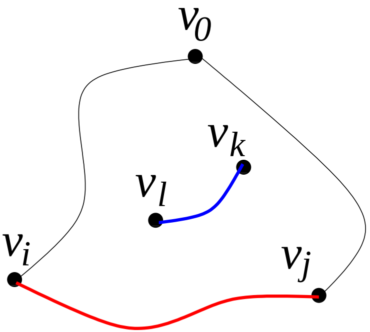


$S'_{k,l}$ stabs $v_i v_j$, which is a contradiction.

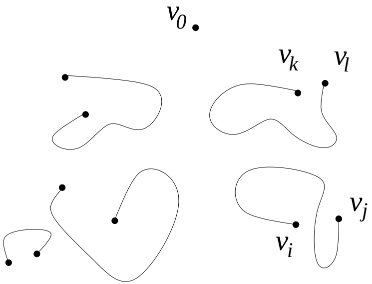


Two edges must be disjoint.

Same argument shows



$\Omega(n^{1/3})$ pairwise disjoint edges in K_{n+1}.



Open Problems.

1. Can the $\Omega(n^{1/3})$ bound be improved? Perhaps to $\Omega(n^{1/2})$?
2. Note that Géza Tóth show that $\pi^*_{\mathcal{F}}(m) = \Theta(m^3)$.
3. Best known upper bound construction: $O(n)$ pairwise disjoint edges.
4. Find $\Omega(n^{\delta})$ pairwise disjoint edges in dense simple topological graphs.

Thank you!