

Applying VC-dimension arguments in geometry

Andrew Suk (UC San Diego)

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Part I: Simple applications (Incidence geometry)

Vapnik-Chervonenkis dimension



Applications of VC-dimensions in geometry.

- 1 Epsilon nets
- 2 Range searching
- 3 Incidence geometry
- 4 Enumeration of intersection graphs
- 5 Graph Drawing

Definition: VC-dimension

Set system $\mathcal{F} \subset 2^V$, $|V| = n$.

Definition

A set $S \subset V$ is **shattered** by $\mathcal{F}$ if for all $X \subset S$, there is an $A \in \mathcal{F}$ such that $S \cap A = X$.

$\mathcal{F} =$

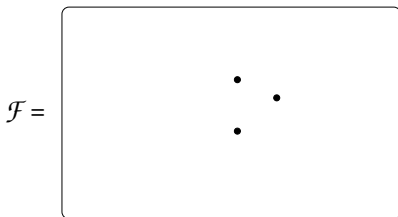


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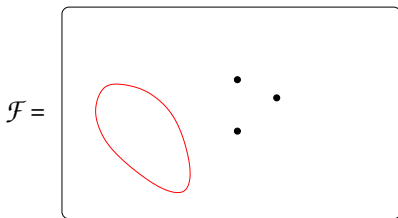


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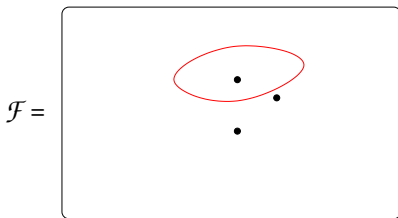


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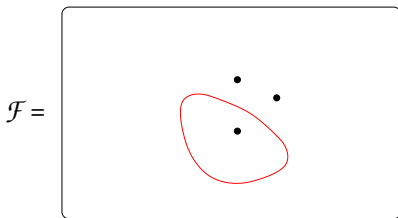


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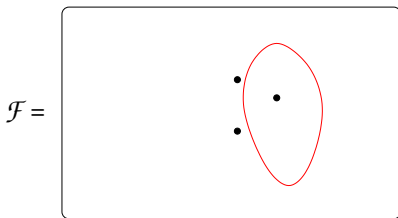


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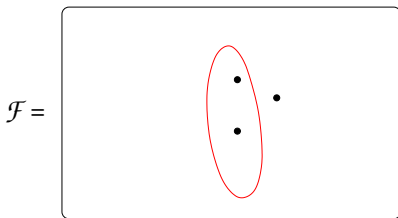


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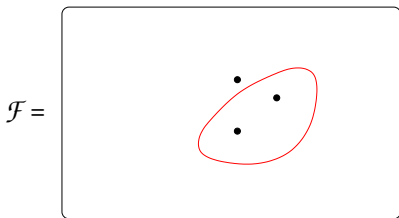


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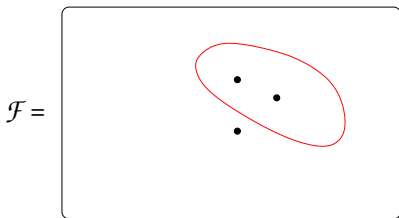


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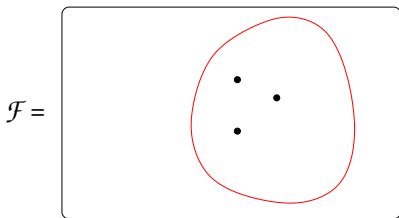


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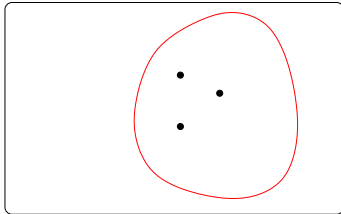
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Definition

The **VC-dimension** of $\mathcal{F}$ is the size of the largest subset $S \subset V$ that is shattered by $\mathcal{F}$.

$\mathcal{F} =$



Theorem (Sauer, Shelah 1972)

If $\mathcal{F}$ is a set-system on n vertices with VC-dimensions d , then

$$|\mathcal{F}| \leq \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{d}.$$

Bound is tight. $\mathcal{F}$ = all sets of size at most d .

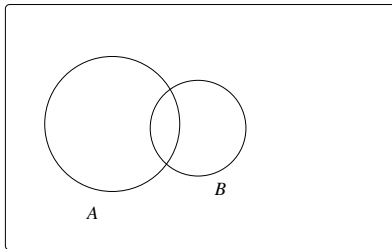
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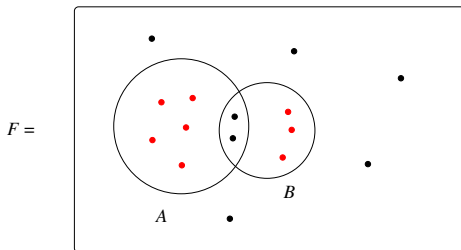


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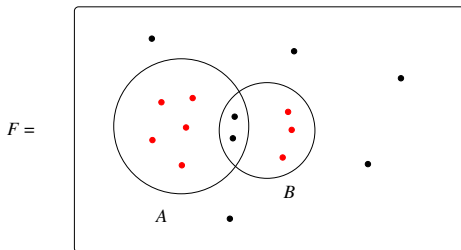


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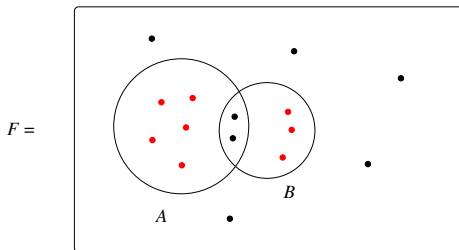
The **distance** between A and B is $|A \Delta B|$,

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$\mathcal{F}$ is δ -**separated** if for any $A, B \in \mathcal{F}$, $|A \triangle B| \geq \delta$,

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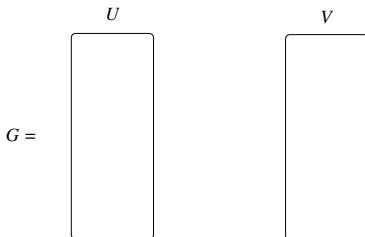
$$|\mathcal{F}| \leq \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{d} = O(n^d).$$

Theorem (Haussler 1995)

If $\mathcal{F}$ is a set-system on n vertices with VC-dimensions d and is δ -separated, then

$$|\mathcal{F}| \leq c_d \left(\frac{n}{\delta} \right)^d.$$

Bipartite graphs with bounded VC-dimension



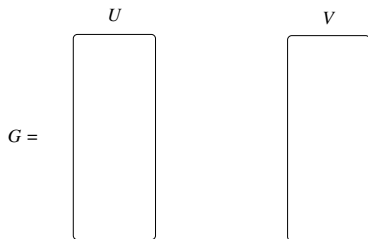
Theorem (Kövari-Sós-Turán)

If G is $K_{t,t}$ -free, then

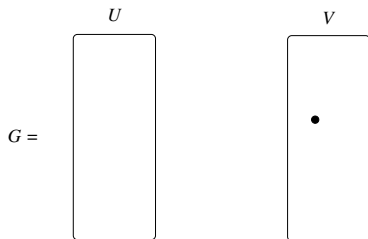
$$|E(G)| \leq O_t(|U||V|^{1-1/t} + |V|).$$

$$|E(G)| \leq O_t(|V||U|^{1-1/t} + |U|).$$

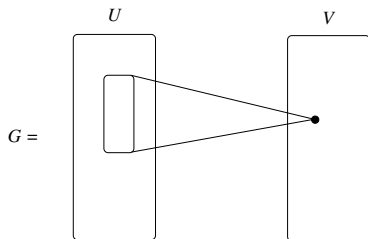
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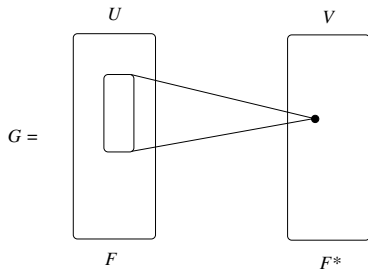
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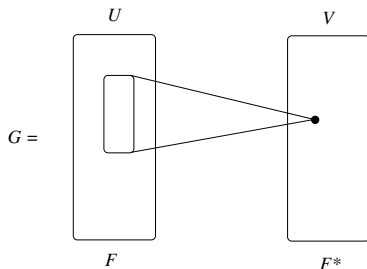
Bipartite graphs with bounded VC-dimension



Bipartite graphs with bounded VC-dimension



Asymmetric bound via VC-dimension



Theorem (Fox-Pach-Sheffer-S.-Zahl 2017)

Let G be a bipartite graph with parts U, V , such that the set system $\mathcal{F}$ and $\mathcal{F}^*$ has VC-dimension at most d . If $\mathcal{F}$ is $K_{t,t}$ -free,

$$|E(G)| \leq O_{d,t}(|U||V|^{1-1/d} + |V|).$$

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Jazner-Pohoata (2020) $|E(G)| \leq o(n^{2-1/d})$.

A more useful parameter

Dual VC-dimension. Let $\mathcal{F}$ be a set-system on a ground set V , $|V| = n$.

Definition

The dual shatter function $\pi_{\mathcal{F}}^*(m)$, is defined to be the maximum number of equivalence classes on V , defined by m sets in $\mathcal{F}$.

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Given sets $A_1, \dots, A_m \in \mathcal{F}$, $x, y \in V$ are equivalent if they both lie in the same sets among $A_1, \dots, A_m$.

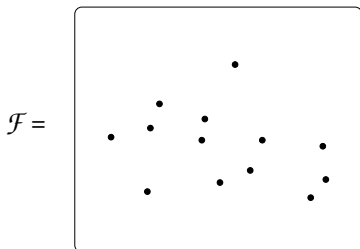
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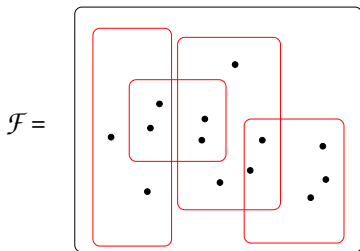
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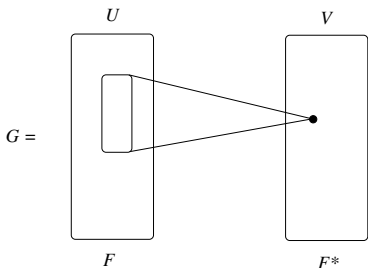
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Asymmetric bound via VC-dimension



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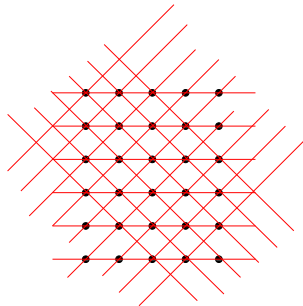
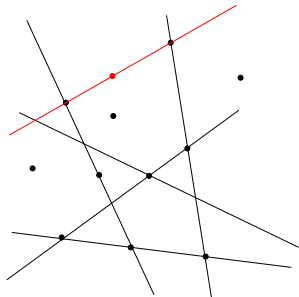
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Incidences between points and lines

Theorem (Szemerédi-Trotter)

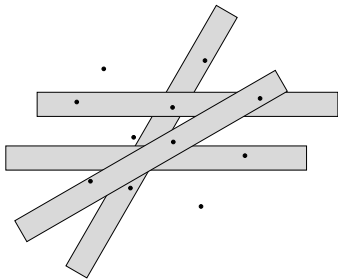
Let P be a set of n points in the plane and L a set of n lines in the plane. Then $|I(P, L)| \leq O(n^{4/3})$.



Incidences between points and strips

Problem (Pach-Tardos)

Let P be a set of n points in the plane and S a set of n strips in the plane of unit width. If $I(P, S)$ is $K_{t,t}$ -free, what is the maximum number of incidences between P and S ?



Incidences between points and lines

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Proofs.

- 1 Cutting lemma + Kövári-Sós-Turán.
- 2 Crossing lemma
- 3 Polynomial partition + Kövári-Sós-Turán.

Theorem (Kövári-Sós-Turán)

$$|I(P, L)| \leq O(|P||L|^{1/2} + |L|).$$

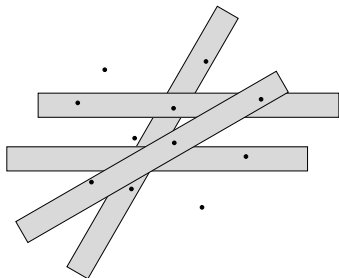
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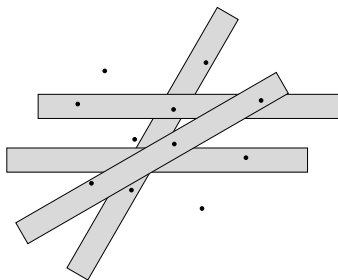
Cutting lemma + VC-dimension **will work**.

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$$|I(P, S)| = O_t(n^{4/3}).$$



Cutting lemma + Asymmetric VC-dimension bound

Theorem (S.-White 2025+)

One can select $n^{\frac{3}{d+1}-o(1)}$ points from $[n]^d$ with no $d+2$ on a sphere or a hyperplane.

Previous bound: $\Omega(n^{\frac{1}{d+1}})$, Thiele 1995

Other applications

Cutting lemma + Asymmetric VC-dimension bound

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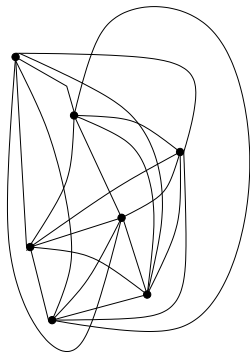
Theorem (Fox-Pach-S. 2025+)

The number of labelled n -vertex intersection graphs of x -monotone pseudo-segments in the plane is at most $2^{O(n^{4/3} \log^2 n)}$.

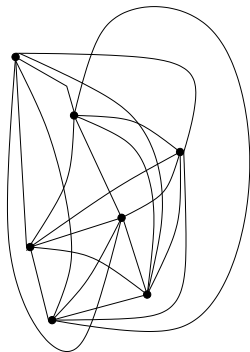
Previous bound: $2^{O(n^{3/2})}$, Kyncl 2013

Part II: More complicated applications (Graph Drawing)

Drawings of the complete graph K_n

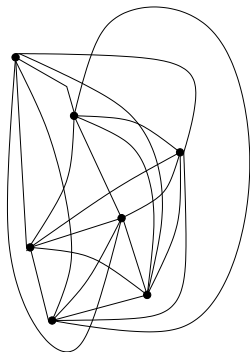


Drawings of the complete graph K_n



Question: Can we always find a “nice” planar subconfiguration?

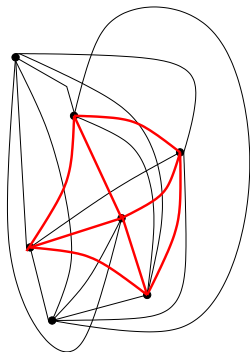
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Examples: noncrossing subgraph,

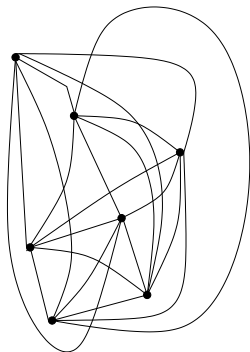
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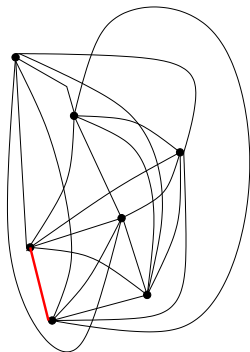
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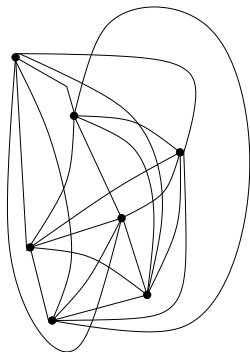
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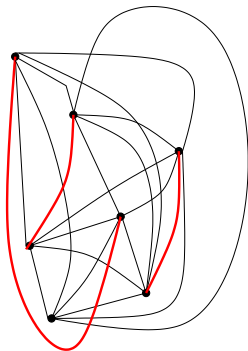
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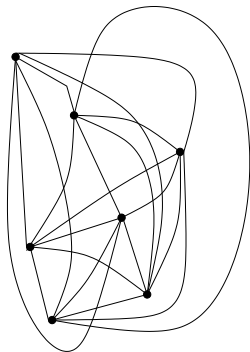
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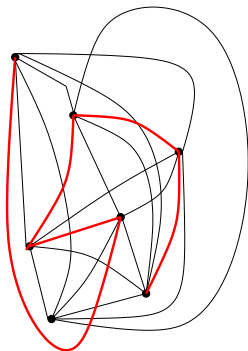
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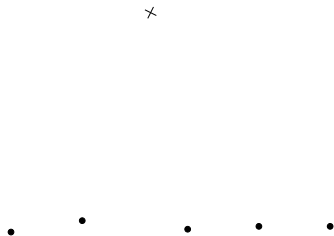
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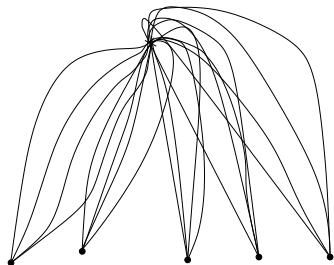
Drawings of K_n with many crossings



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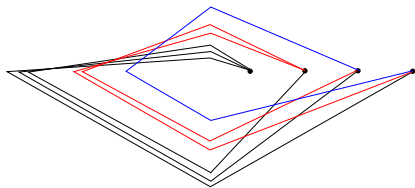
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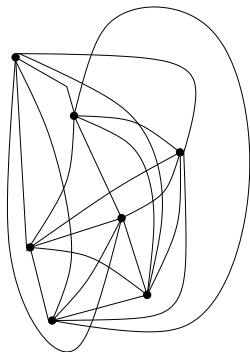
Every pair of edges cross.

Drawings of K_n with many crossings

Pach-Tóth 2010. Every pair of edges cross, every pair of edges cross at most twice.



Simple condition is necessary



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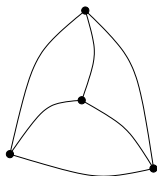
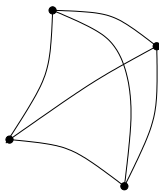
Every pair of edges cross at most once.

Simple drawing of G

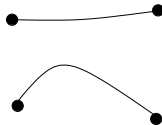
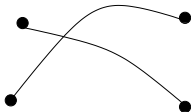
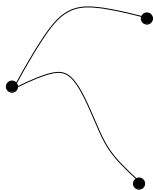
V = points in the plane.

E = curves connecting the corresponding points (vertices).

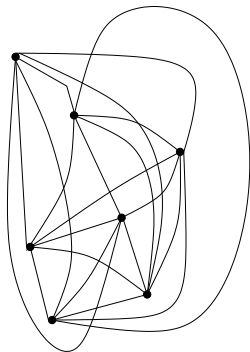
Every pair of edges have at most 1 point in common.



We will only consider simple drawings of a graph G .



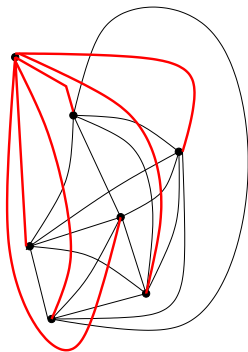
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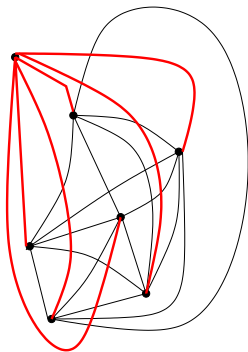
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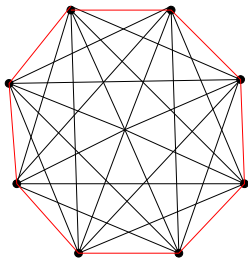


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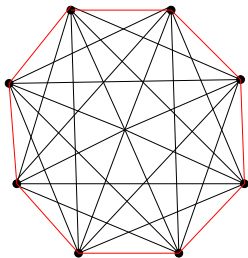
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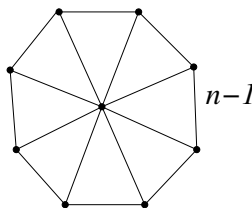


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Theorem (Harborth and Mengersen, 1994)

There are simple drawings of K_n such that every edge crosses at least $(\frac{3}{4} + o(1))n$ other edges.

Finding an edge that crosses few other edges

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Conjecture (Brass, Moser, Pach, 2005)

$$h(n) = o(n^2).$$

Informal definition. An edge is short if it crosses at most $o(n^2)$ other edges.

Short edges always exist in simple drawings of K_n

Theorem (Jan Kynčl, Pavel Valtr, 2009)

$$\Omega(n^{3/2}) < h(n) < O\left(\frac{n^2}{\log^{1/4} n}\right).$$

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Key ideas

- 1 VC-dimension theory
- 2 Minimality argument

Let $\mathcal{F}$ be a set-system on a ground set V , $|V| = n$,
 $\pi_{\mathcal{F}}^*(m) = O(m^d)$.

Stabbing

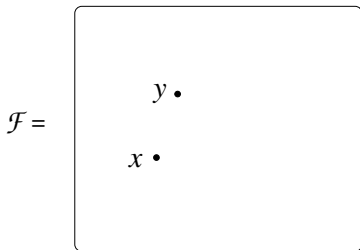
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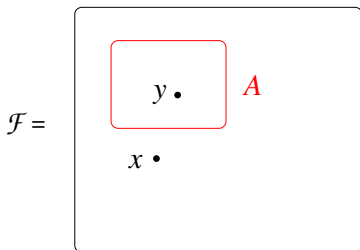
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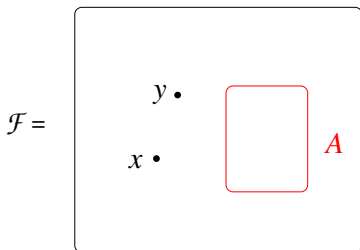
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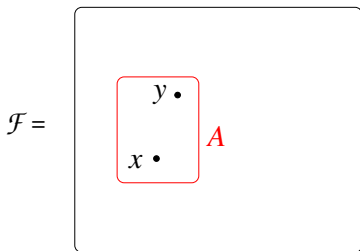
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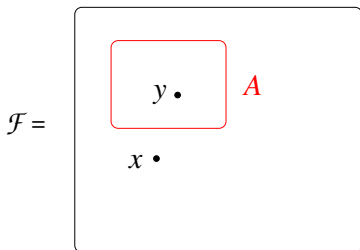
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Short edge lemma

Theorem (Chazelle-Welzl 1989)

$\mathcal{F}$ is a set system on a ground set V with $\pi_{\mathcal{F}}^*(m) = O(m^d)$. Then there is a pair of vertices $x, y \in V$ such that $\{x, y\}$ is stabbed by at most $c|\mathcal{F}|/n^{1/d}$.

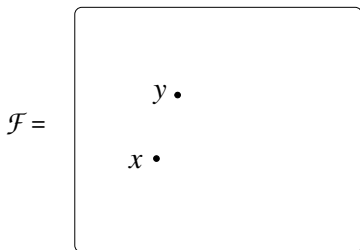
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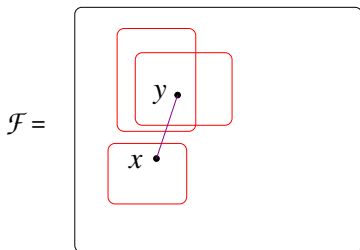
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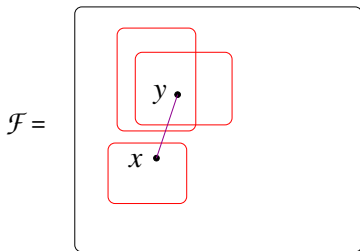
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Together with an iterative re-weighting technique

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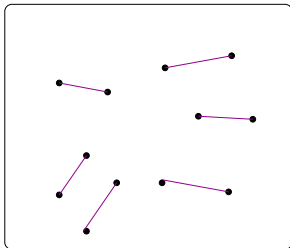


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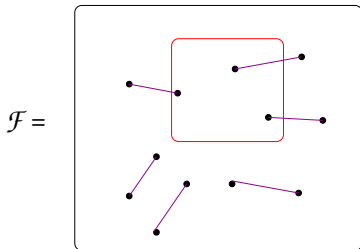
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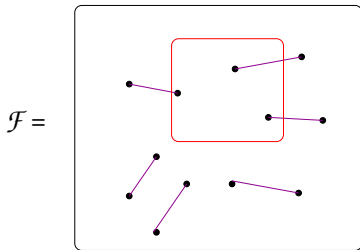
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Combining Haussler's packing lemma + iterative re-weighting + triangle inequality

Lemma (S. 2023)

$\mathcal{F}$ is a set system on V with $|V| = n$ and $\pi_{\mathcal{F}}^*(m) = O(m^d)$. Then there is a subset $X \subset V$, $|X| \leq O(n^{1/2+1/(2d)})$, and a perfect matching M on $V \setminus X$ such that

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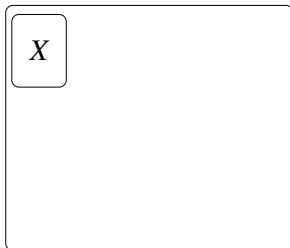


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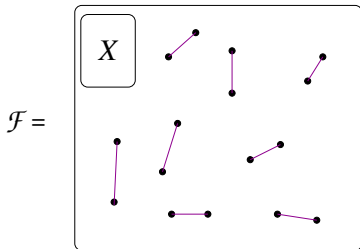
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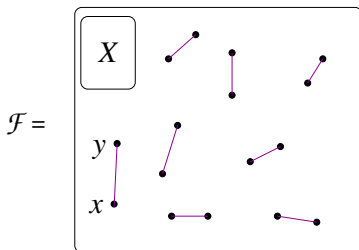
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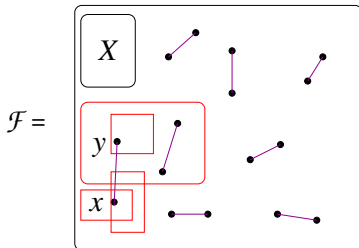
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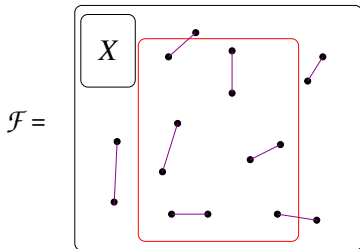
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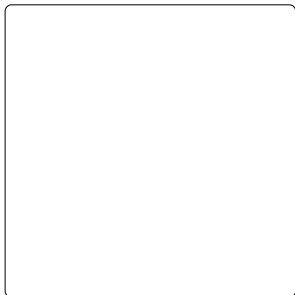
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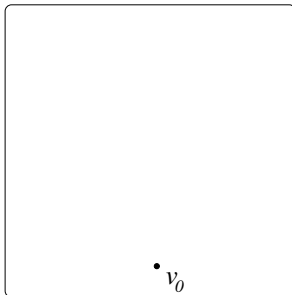
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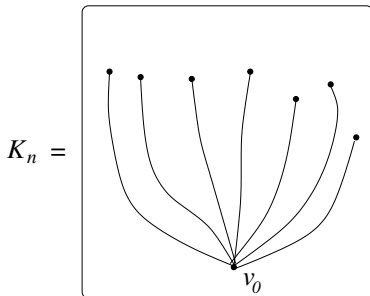


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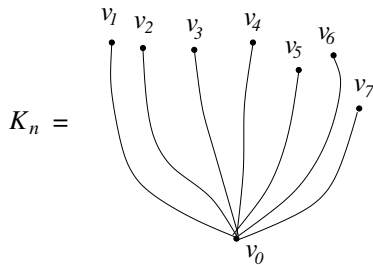
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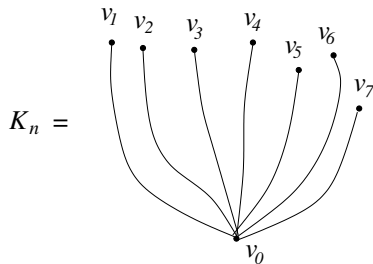
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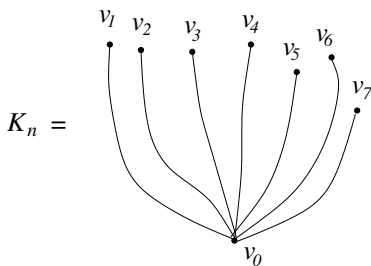


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Proof. Ground set $V = \{v_1, v_2, \dots, v_{n-1}\}$.

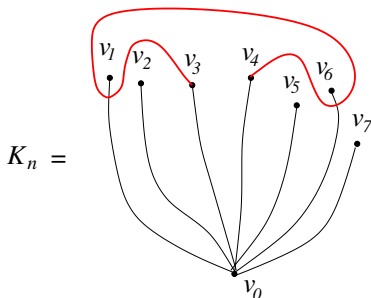
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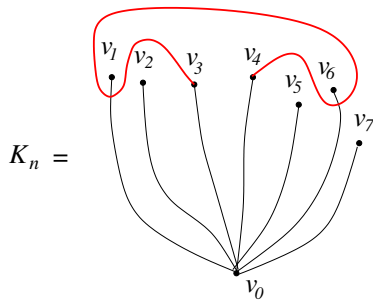


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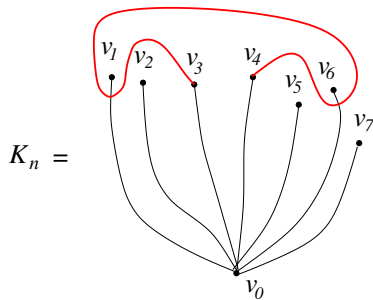
Example: $T_{3,4} = \{v_1, v_6\}$.

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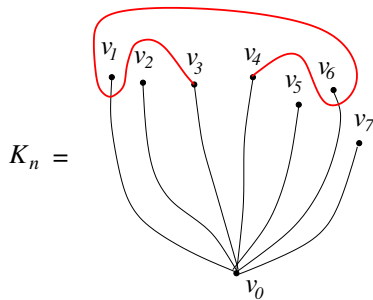
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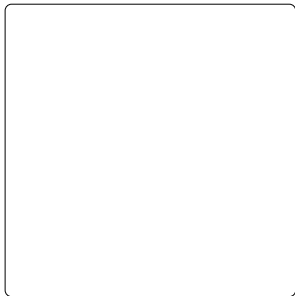
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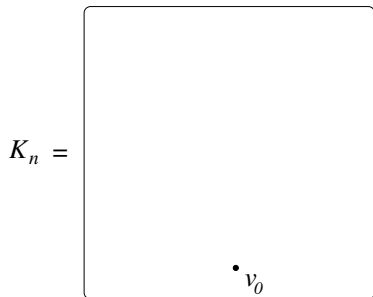
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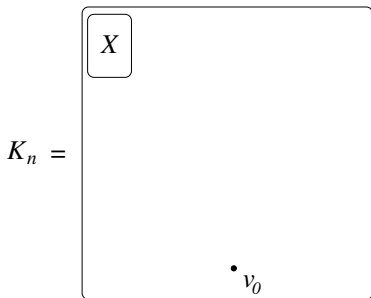


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Apply the matching with low stabbing number lemma

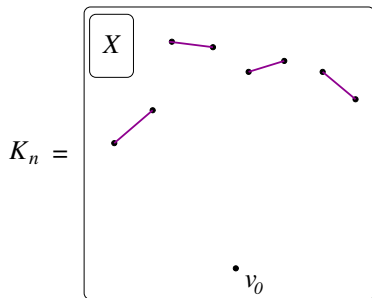


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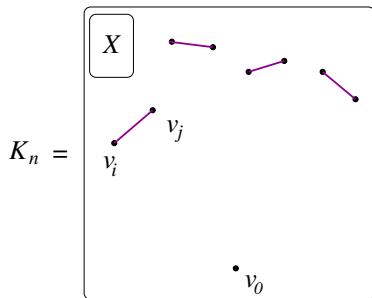
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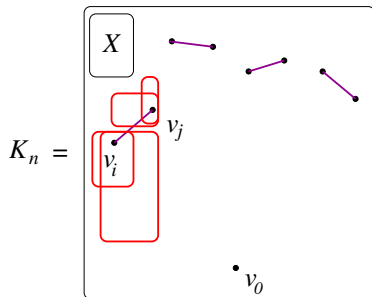
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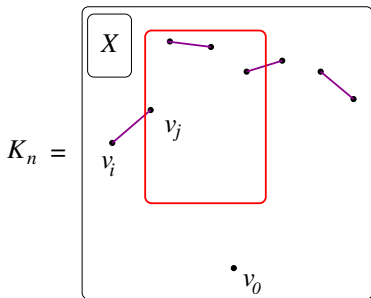
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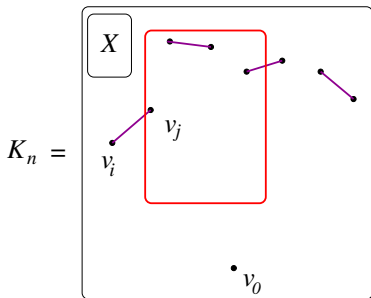


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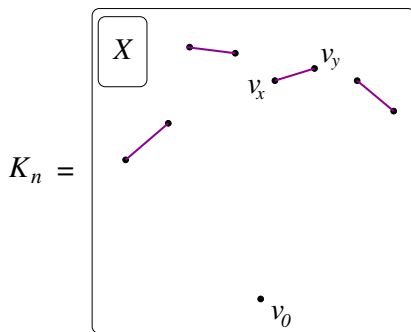


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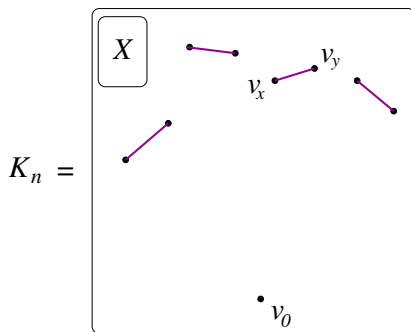
Each set $T_{i,j} \in \mathcal{F}$ stabs at most $O(n^{1/2})$ matchings.

Apply the matching with low stabbing number lemma



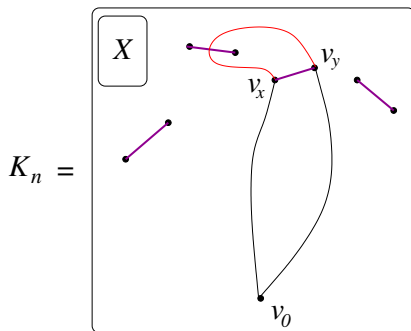
Let $\{v_x, v_y\}$ be the matching such that the triangle $T_{x,y} = (v_0, v_x, v_y)$ contains the fewest matchings.

Apply the matching with low stabbing number lemma



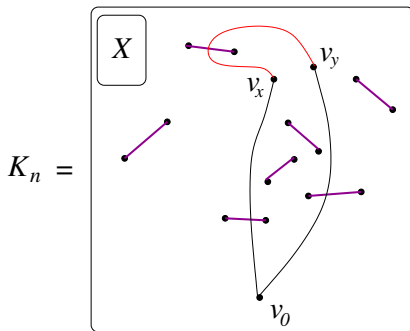
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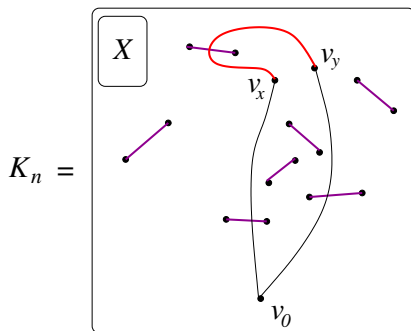
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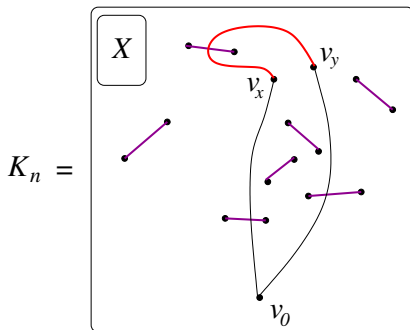
Note. At most $O(n^{1/2})$ matchings stab $T_{x,y}$.

Apply the matching with low stabbing number lemma



Claim. Edge $v_x v_y$ crosses at most $O(n^{7/4})$ other edges.

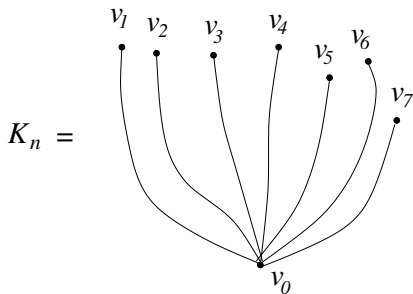
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Claim. Edge $v_x v_y$ crosses at most $O(n^{7/4})$ other edges.

Cheat. $|x - y| < n^{3/4}$.

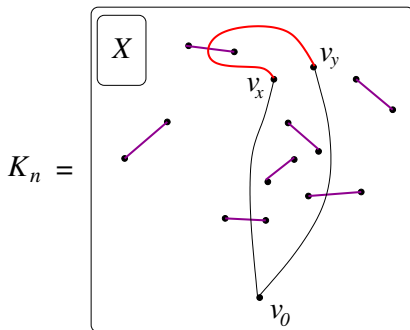
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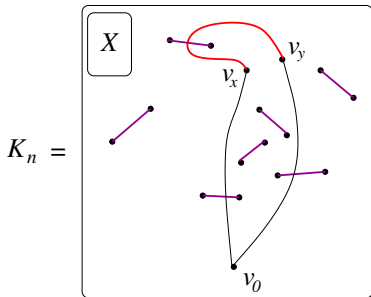
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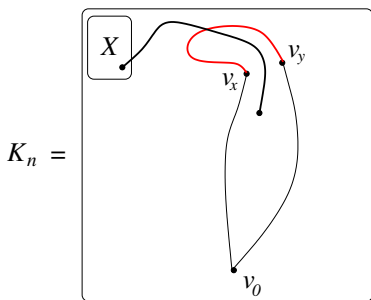
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Counting edges crossing $v_x v_y$



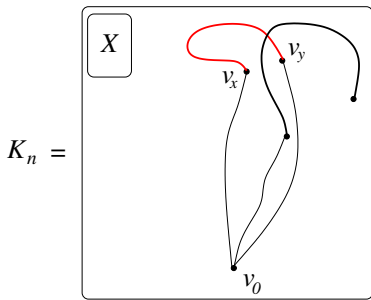
- E_0 edges incident to v_0 . $|E_0| < n$.

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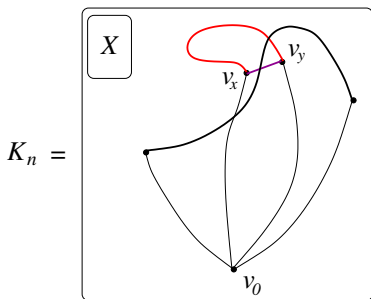
- E_0 edges incident to v_0 . $|E_0| < n$.
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Counting edges crossing $v_x v_y$



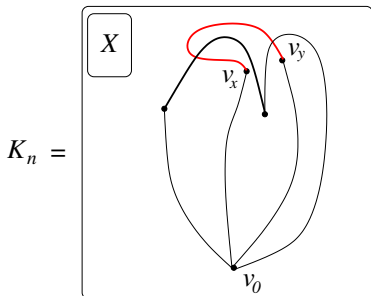
- E_0 edges incident to v_0 . $|E_0| < n$.
- E_1 edges incident to X , $|X| < O(n^{3/4})$. $|E_1| = O(n^{7/4})$.
- E_2 edges with endpoint between v_x, v_y . $|E_2| = O(n^{7/4})$.

Counting edges crossing $v_x v_y$



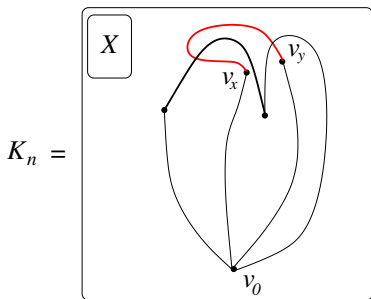
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- E_2 edges with endpoint between v_x, v_y . $|E_2| = O(n^{7/4})$.
- E_3 both ends lie outside (inside) of $T_{x,y}$. $|E_3| = O(n^{7/4})$.

Counting edges crossing $v_x v_y$



- E_0 edges incident to v_0 . $|E_0| < n$.
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- E_2 edges with endpoint between v_x, v_y . $|E_2| = O(n^{7/4})$.
- E_3 edges whose triangle stabs $v_x v_y$. $|E_3| = O(n^{7/4})$.
- E_4 edges not above that crosses $v_x v_y$.

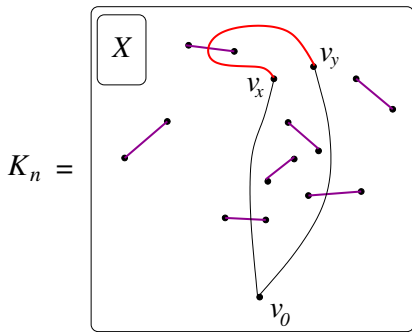
Counting edges crossing $v_x v_y$



Edges in E_4 .

Using minimality. $|E_4| = O(n^{7/4})$.

Putting it all together



Edge $v_x v_y$ crosses at most

$$|E_0| + |E_1| + |E_2| + |E_3| + |E_4| = O(n^{7/4})$$

other edges.

□.

$$\Omega(n^{3/2}) < h(n) < O(n^{7/4})$$

Conjecture (Kynčl-Valtr 2009, S. 2023)

$$h(n) = \Theta(n^{3/2}).$$

Hopeful open problems

Theorem (S.-White 2025)

One can select $n^{\frac{3}{d+1}-o(1)}$ points from $[n]^d$ with no $d+2$ on a sphere or a hyperplane.

Previous bound: $\Omega(n^{\frac{1}{d+1}})$, Thiele 1995

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Theorem (Marcus-Tardos 2006)

The maximum number of incidences between n points and n pseudo-circles in the plane is at most $O(n^{3/2} \log n)$.

Thank you!