Disjoint edges in complete topological graphs

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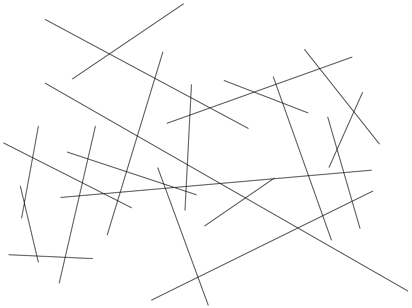
Main problem

Problem: Given a collection of objects $\mathcal{C}$ in the plane, what is the size of the largest subcollection of pairwise disjoint objects?



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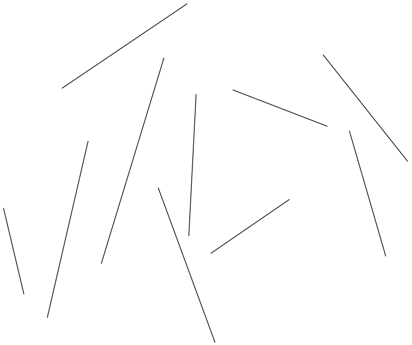

Motivation

Map labeling example. Computer contains the names of every street, river, park, shop, etc.

Map near NYU

Starbucks
La Guardia PLace
Schwartz Plaza
Greene Street
Bobst Library
Washington Square Park
Courant Institute
Think Coffee
Campus Eatery
Dog Park
NYU Stern
Student Services
3rd Street

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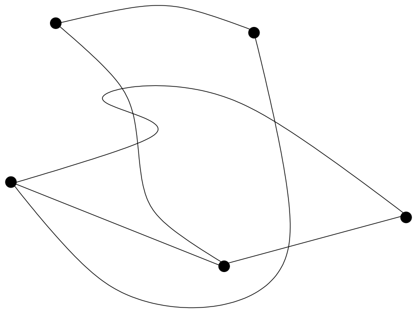
We will consider the case when $\mathcal{C}$ is the collection of edges in a *simple topological graph*.

Definition

A *topological graph* is a graph drawn in the plane with vertices represented by points and edges represented by curves connecting the corresponding points. A topological graph is *simple* if every pair of its edges intersect at most once.



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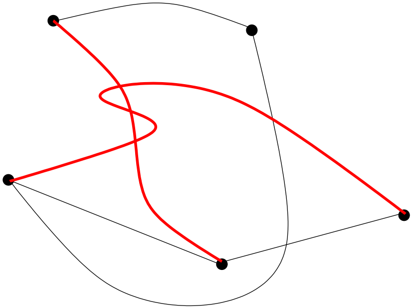
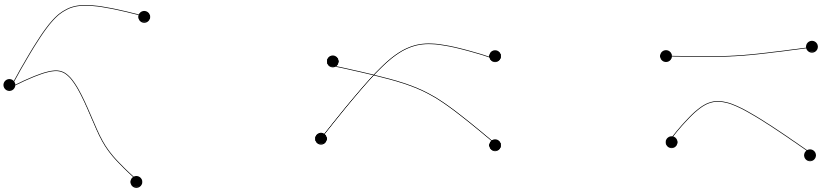


We will only consider *simple* topological graphs.



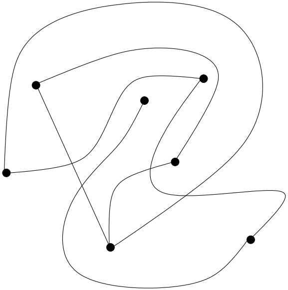
Three problems in topological graph theory.

Problem 1: Thrackle conjecture.

Problem (Conway)

Does every n-vertex simple topological graph with $|E(G)| > n$ have two disjoint edges?

Fulek and Pach 2010: $|E(G)| \geq 1.43n$.
Best known $1.43n$ by Fulek and Pach, 2010.



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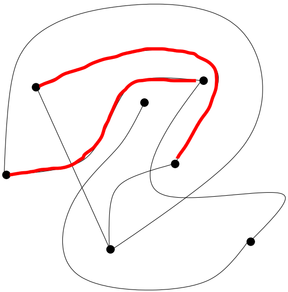


Generalization.

Theorem (Pach and Tóth, 2005)

Every n-vertex simple topological graph with no k pairwise disjoint edges, has at most $C_k n \log^{5k-10} n$ *edges.*

Conjecture to be at most $O(n)$ (for fixed k). By solving for k in $C_k n \log^{5k-10} n = \binom{n}{2}$.

Corollary (Pach and Tóth, 2005)

Every complete n-vertex simple topological graph has at least $\Omega(\log n / \log \log n)$ *pairwise disjoint edges.*

Conjecture (Pach and Tóth)

There exists a constant δ, such that every complete n-vertex simple topological graph has at least $\Omega(n^{\delta})$ pairwise disjoint edges.

History

Pairwise disjoint edges in complete n-vertex simple topological graphs:

1. $\Omega(\log^{1/6} n)$, Pach, Solymosi, Tóth, 2001.
2. $\Omega(\log n / \log\log n)$, Pach and Tóth, 2005.
3. $\Omega(\log^{1+\epsilon} n)$, Fox and Sudakov, 2008.

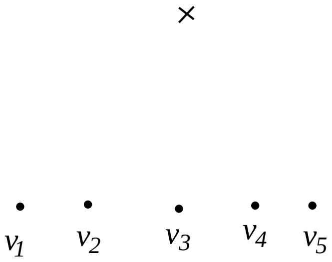
Note $\epsilon \approx 1/50$. All results are slightly stronger statements.

Main result

Theorem (Suk, 2011)

Every complete n-vertex simple topological graph has at least $\Omega(n^{1/3})$ pairwise disjoint edges.

Clearly the simple condition is required.



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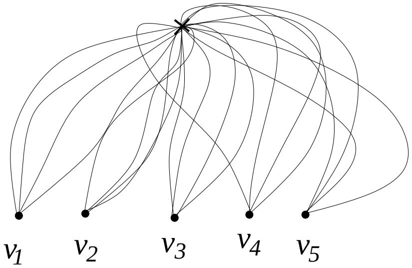

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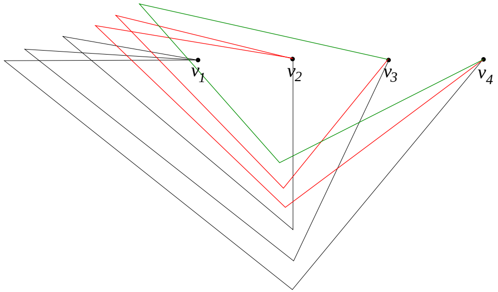

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combinatorial tools

Let $\mathcal{F}$ be a set system with ground set X.

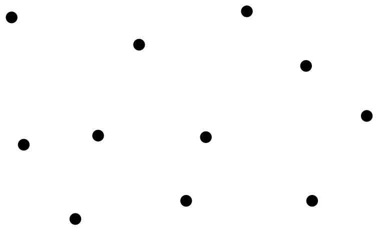
Definition (Dual shatter function)

The dual shatter function $\pi^*_{\mathcal{F}}(m)$, is defined to be the maximum number of equivalence classes on X, defined by an m-element subfamily of $\mathcal{F}$.

For m sets $S_1, S_2, ..., S_m$, $x \sim y$ if BOTH x, y are in exactly the same sets among $S_1, ..., S_m$ (i.e. no set S_i contains x and not y or vise versa).

I.e. $\pi^*_{\mathcal{F}}(m)$ is the number of nonempty cells in the Venn diagram of m sets of $\mathcal{F}$.

Example: X is a set of n points in the plane, $\mathcal{F}$ is the set of all halfplanes.



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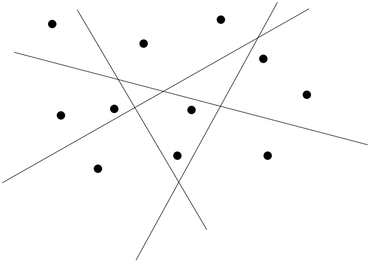

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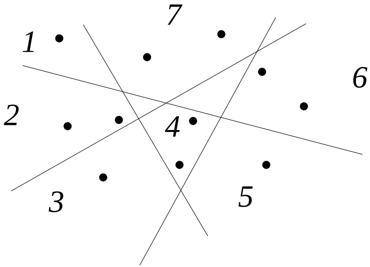

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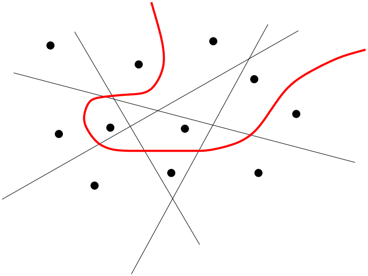

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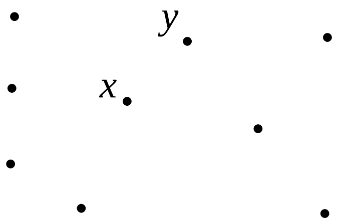
definition

A set $S \in \mathcal{F}$ *stabs* the pair (of vertices) x, y if $|S \cap \{x, y\}| = 1$.



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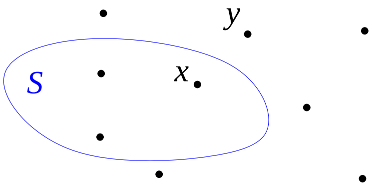

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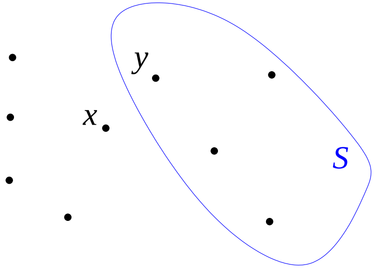

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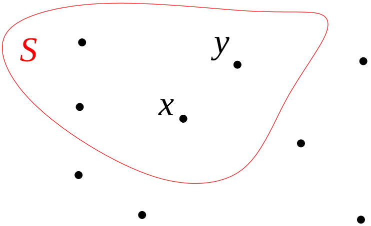
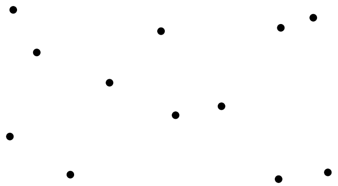


Main tool

Theorem (Matching theorem, Chazelle and Welzl, 1989)

Let $\mathcal{F}$ be a set system on an n element point set X (n is even), such that $\pi^*_{\mathcal{F}}(m) \leq O(m^d)$. Then there exists a perfect matching M on X such that each set in $\mathcal{F}$ stabs at most $O(n^{1-1/d})$ members in M.



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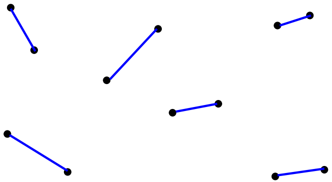

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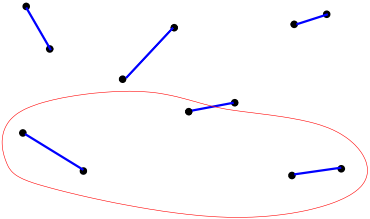

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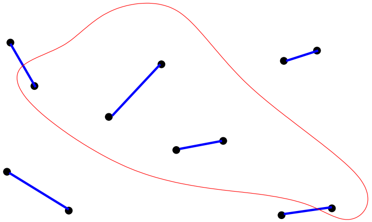

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Sketch of proof

Theorem (Suk, 2011)

Every complete n-vertex simple topological graph has at least $\Omega(n^{1/3})$ pairwise disjoint edges.

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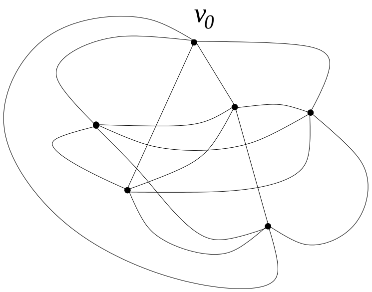

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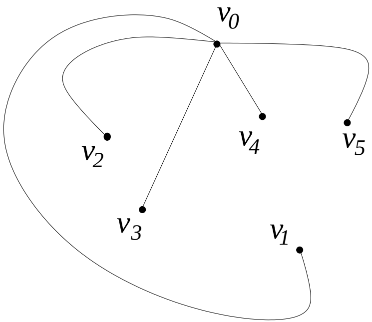
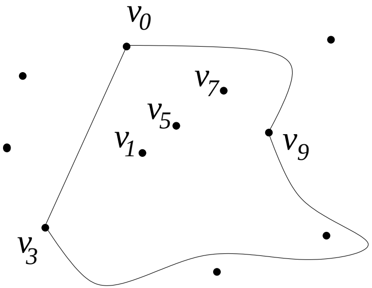


Define $\mathcal{F}_1 = \bigcup_{1 \leq i < j \leq n} S_{i,j}$, where $S_{i,j}$ is the set of vertices inside triangle v_0, v_i, v_j.



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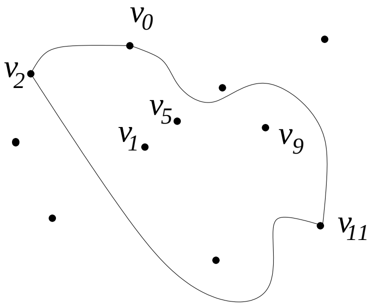

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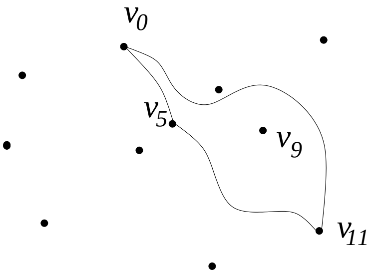

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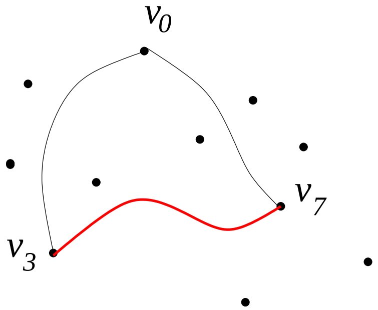
$\mathcal{F}_1$ is not "complicated".

Lemma

$\pi^*_{\mathcal{F}_1}(m) \leq O(m^2)$.

Proof: Basically m "triangles" divides the plane into at most $O(m^2)$ regions. Proof is by induction on m.

Define set system $\mathcal{F}_2 = \bigcup_{1 \le i < j \le n} S'_{i,j}$, where $v_k \in S'_{i,j}$ if topological edges $v_0 v_k$ and $v_i v_j$ cross.



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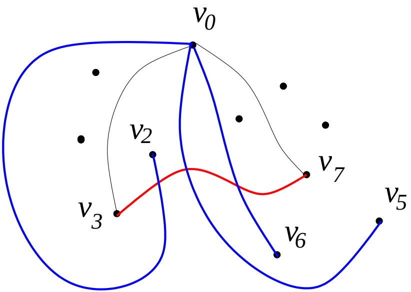

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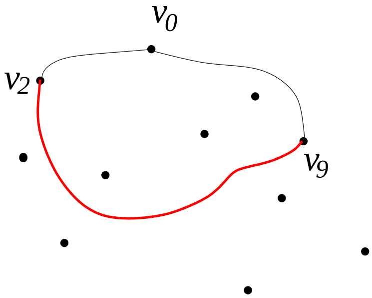

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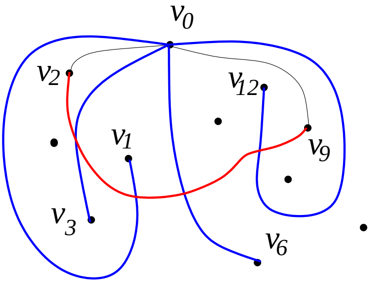

$S'_{3,7} = \{v_2, v_6, v_5\}, S'_{2,9} = \{v_1, v_3, v_6, v_{12}\}$.

Again, $\mathcal{F}_2$ is not "complicated". Set $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. One can show

Lemma

$\pi^*_{\mathcal{F}}(m) = O(m^3)$.

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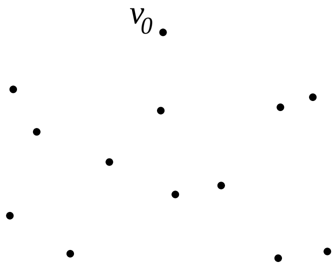
By the **matching lemma** (Chazelle and Welzl), there is a perfect matching M such that each set in $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ stabs at most $O(n^{2/3})$ members in M. Recall $|M| = n/2$.



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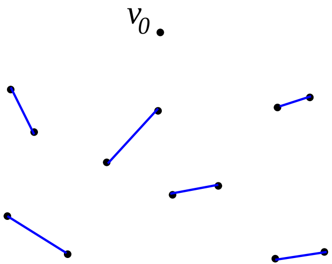

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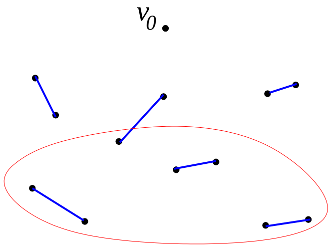
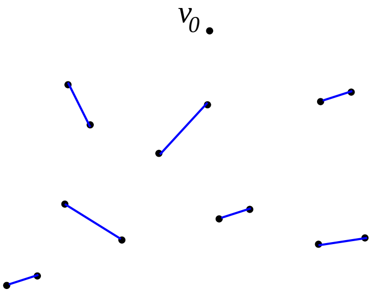


Auxiliary graph G, where $V(G) = M$ and $v_i v_j \rightarrow v_k v_l$ if $S_{i,j}$ or $S'_{i,j}$ stabs $v_k v_l$.



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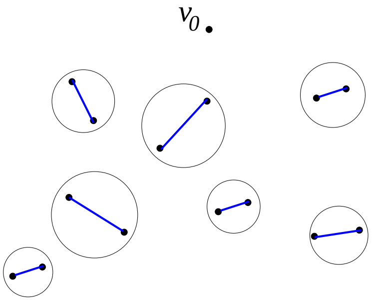

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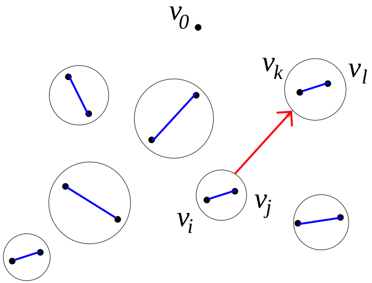

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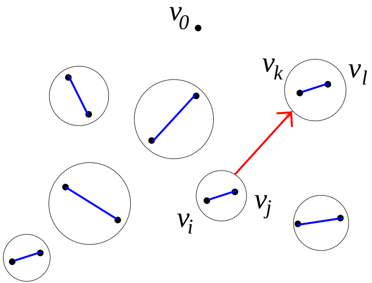
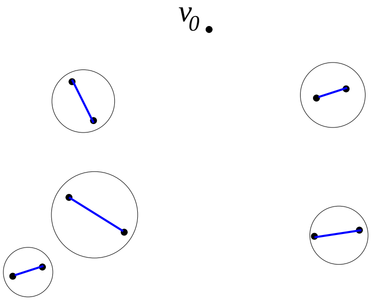


$S_{i,j}$ and $S'_{i,j}$ stabs (in total) at most $O(n^{2/3})$ members in $M = V(G)$. $|E(G)| \leq O(n^{5/3})$.

$|E(G)| \leq O(n^{5/3})$, by Turán, G contains an independent set of size $\Omega(n^{1/3})$.



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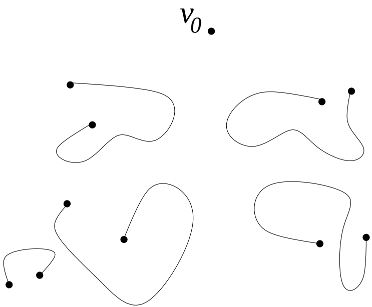

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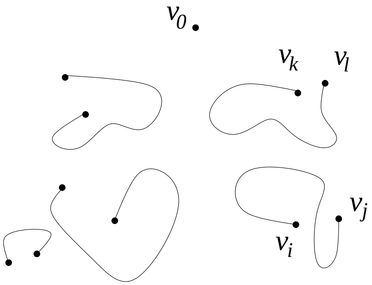
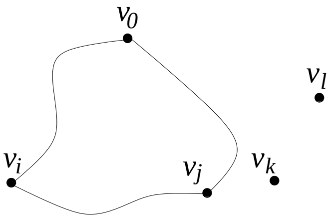
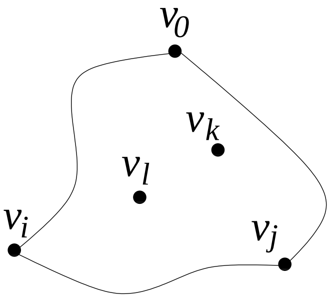
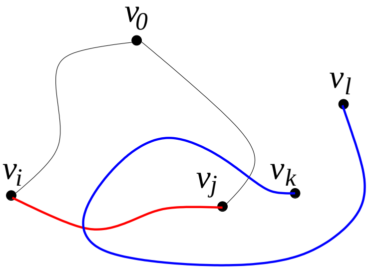


Claim!

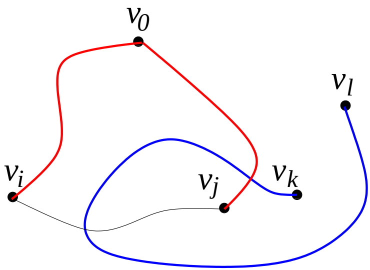
Since $S_{i,j}$ does NOT stab $v_k v_l$



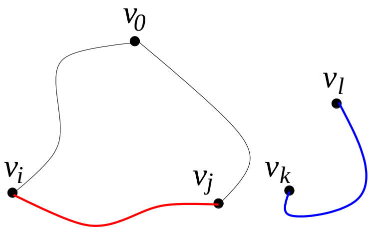




Assume edges cross.

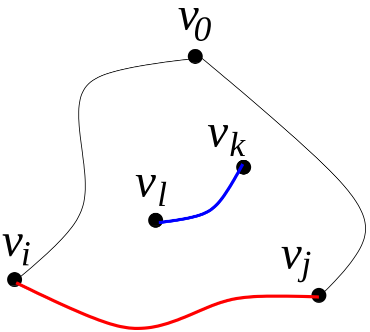


$S'_{k,l}$ stabs $v_i v_j$, which is a contradiction.

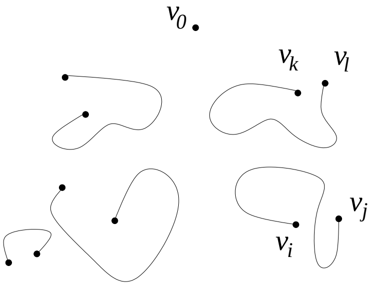


Two edges must be disjoint.

Same argument shows



$\Omega(n^{1/3})$ pairwise disjoint edges in K_{n+1}.



Open Problems.

1. Can the $\Omega(n^{1/3})$ bound be improved? Perhaps to $\Omega(n^{1/2})$?
2. Just need to show $\pi^*_{\mathcal{F}}(m) = O(m^2)$.
3. Best known upper bound construction: $O(n)$ pairwise disjoint edges.

Pairwise crossing edges in K_n?

Theorem (Fox and Pach, 2008)

Every complete n-vertex simple topological graph has at least $\Omega(n^\delta)$ pairwise crossing edges.

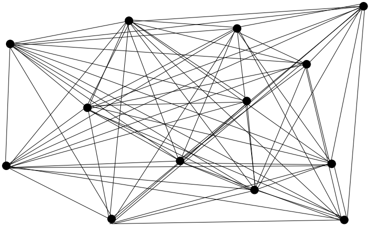
$\delta \approx 1/50$.

Problem: Can one improve this bound?

This problem is interesting for complete geometric graphs (edges are straight line segments)!

Theorem (Aronov, Erdős, Goddard, Kleitman, Klugerman, Pach, Schulman, 1997)

Every complete n-vertex geometric graph has at least $\Omega(n^{1/2})$ *pairwise crossing edges.*



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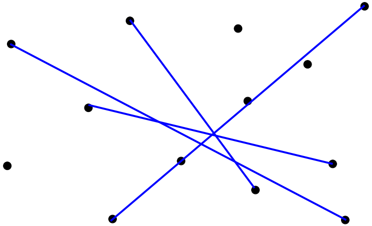

Conjecture: $\Omega(n)$.

This problem is interesting for complete geometric graphs!

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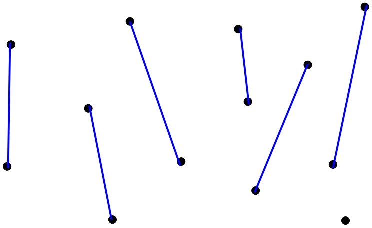

Similar dual problems.

Theorem

Every n-vertex topological graph with no crossing edges contains at most $3n - 6 = O(n)$ edges.

Relaxation of planarity.

Conjecture

Every n-vertex topological graph with no k pairwise crossing edges contains at most $O(n)$ edges.

Special Cases

1. Conjecture is true for $k = 3, 4$ ($k = 3$ by Agarwal et. al. '97, Pach, Radoičić, Tóth '03, Ackerman and Tardos '08, $k = 4$ by Ackerman '09)

Best known bound for general k:

1. Every n-vertex (not simple) topological graph with no k pairwise crossing edges has at most $n(\log n)^{O(\log k)}$ edges. (Fox and Pach, 2008)
2. Every n-vertex simple topological graph with no k pairwise crossing edges has at most $(n \log n) \cdot 2^{\alpha^{c_k}(n)}$ edges. (Fox, Pach, Suk, 2011)

Problem (Erdős)

Let $\mathcal{S}$ be a family of n segments in the plane, such that no three members pairwise cross. Can you color the members in $\mathcal{S}$ with at most c colors, such that each color class consists of pairwise disjoint segments.

Best known $O(\log n)$, McGuinness 1997.

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Is there a subcollection of $\Omega(n)$ pairwise disjoint segments? Best known $\Omega(n/\log n)$.

Thank you!