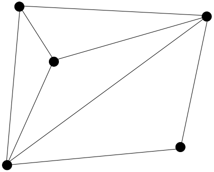
Intersection patterns of pseudo-segments

Andrew Suk (UC San Diego)

February 20, 2024

Planar graphs

$G =$ 

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Every n-vertex planar graph has at most $3n - 6$ edges.

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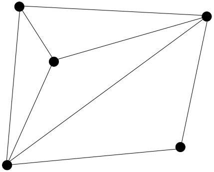
$G =$ 

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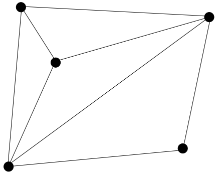
$G =$ 

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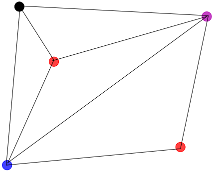
$G =$ 

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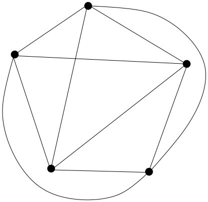
K_5 = 

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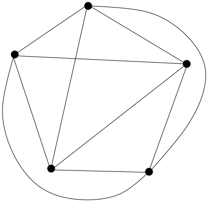

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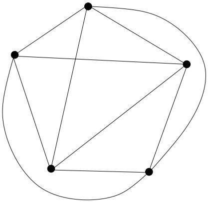

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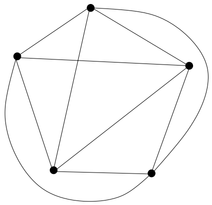

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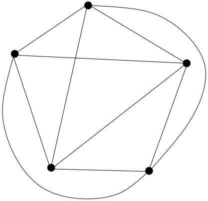

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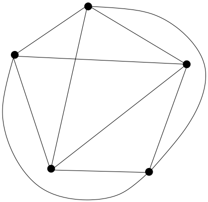

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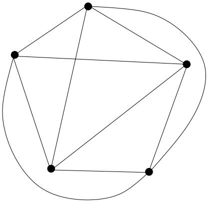

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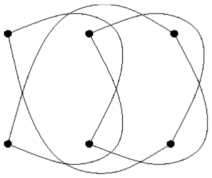

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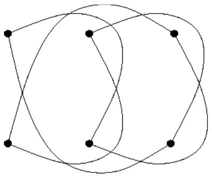

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Every n-vertex thrackle has at most n edges.

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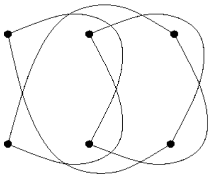

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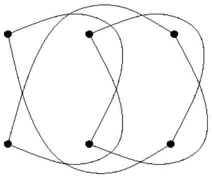

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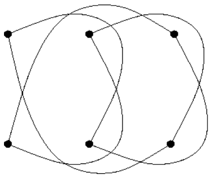

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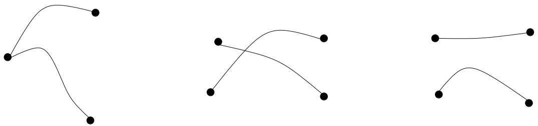
If G is an n-vertex graph with a simple drawing in the plane with no k-pairwise disjoint edges, then $|E(G)| = O_k(n)$.

Simple Drawing:



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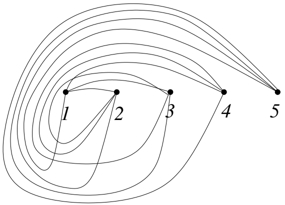
Drawing of K_n with every pair of edges crossing once or twice.

$T_5 =$



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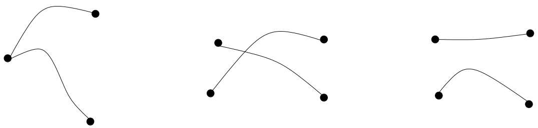

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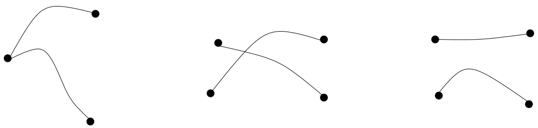

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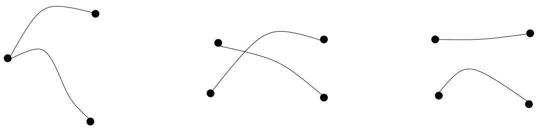

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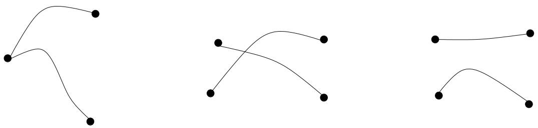

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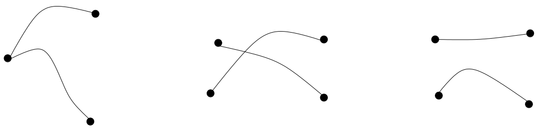

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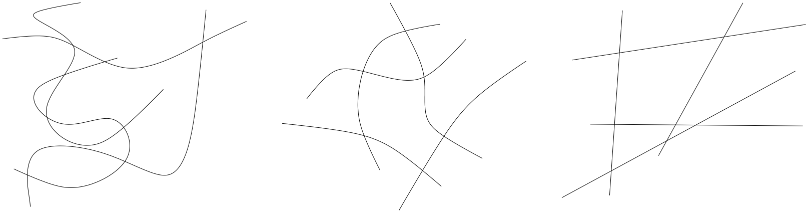
General curves vs. Pseudo-segments vs. Segments



Crossing patterns of curves

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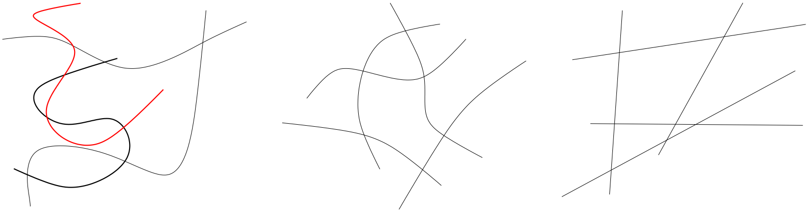

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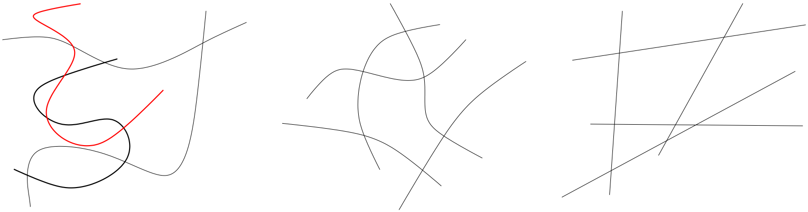

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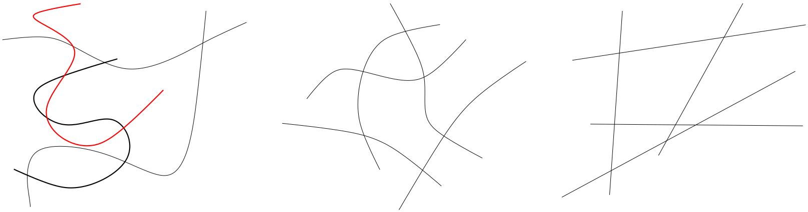

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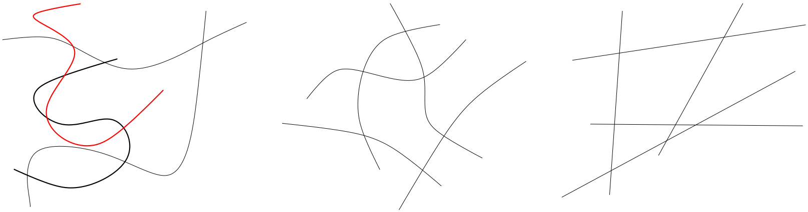

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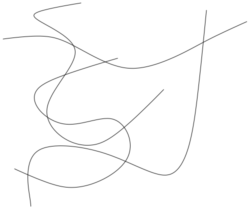
General curves (string graphs)



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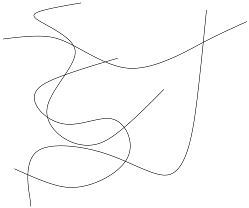

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String graphs have the Erdős-Hajnal property.

Theorem (Tomon, 2023)

Every n-vertex string graph contains a clique or independent set of size n^{ε}, where ε is an absolute constant.

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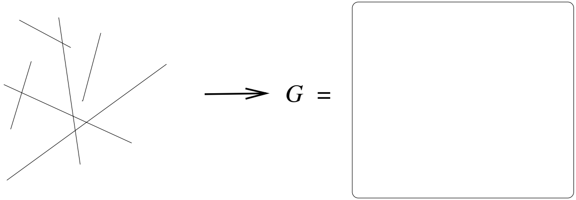
Let $G = (V, E)$ be an n-vertex intersection graph of a collection of segments in the plane. Then there are subsets $A, B \subset V$ of size $\Omega(n)$, such that either every segment in A crosses every segment in B, or every segment in A is disjoint to every segment in B.



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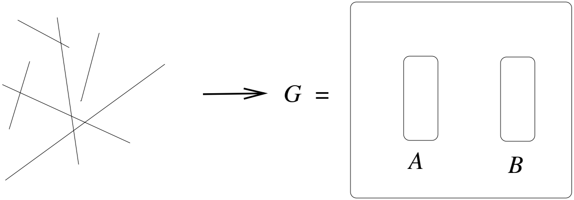

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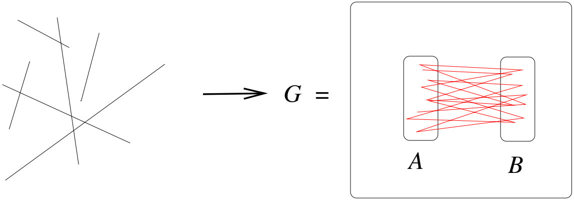

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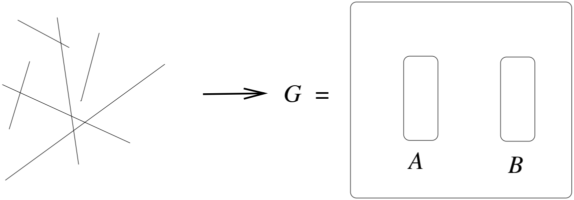

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$E = \{(u,v) : \Phi(f_1(u,v) \geq 0, \ldots, f_t(u,v) \geq 0)\}$, where each f_i is a polynomial of bounded degree.

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Erdős-Hajnal property

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Mighty Erdős-Hajnal property Erdős-Hajnal property

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Applications: Need the **Mighty** Erdős-Hajnal property.

The mighty Erdős-Hajnal property

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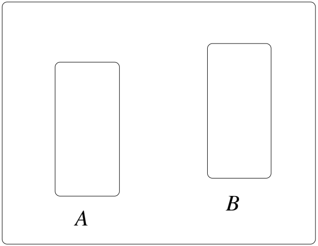
$G =$ 

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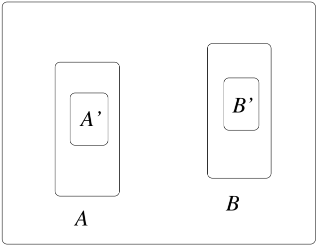
$G =$ 

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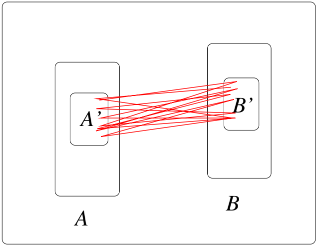
$\mathcal{F}$ has the mighty Erdős-Hajnal property if there is a constant $\varepsilon > 0$ such that for every graph $G \in \mathcal{F}$ and every pair of disjoint subsets $A, B \subset V(G)$ there are subsets $A' \subset A$ and $B' \subset B$ with $|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$ such that the bipartite graph between A and B in G is complete or empty.

$G =$ 

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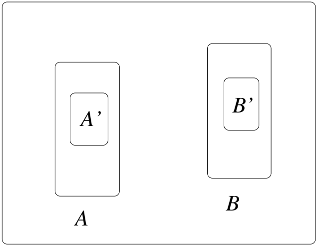
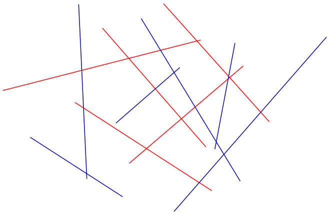


Segments

$\mathcal{S}_n$ has the **mighty** Erdős-Hajnal property.

Theorem (Pach-Solymosi, 2001)

Let $\mathcal{R}$ be a set of red segments in the plane, and $\mathcal{B}$ be a set of blue segments in the plane. Then there are subsets $\mathcal{R}' \subset \mathcal{R}$ and $\mathcal{B}' \subset \mathcal{B}$, where $|\mathcal{R}'| \geq |\mathcal{R}|/330$ and $|\mathcal{B}'| \geq |\mathcal{B}|/330$, such that either red segment in $\mathcal{R}'$ crosses every blue segment in $\mathcal{B}'$, or every red segment in $\mathcal{R}'$ is disjoint to every blue segment in $\mathcal{B}'$.



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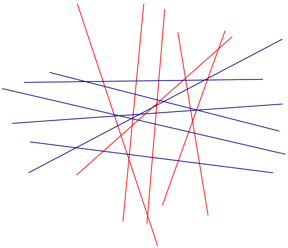

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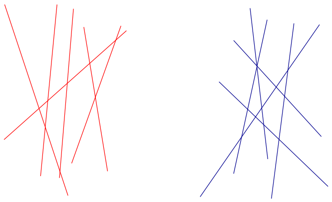

Segments vs. Pseudo-Segments vs. General curves

$$\mathcal{S}_n \subset \mathcal{P}_n \subset \mathcal{G}_n$$

$|\mathcal{S}_n| = 2^{\Theta(n \log n)}$ $|\mathcal{G}_n| = 2^{\Theta(n^2)}$

Mighty Erdős-Hajnal property Erdős-Hajnal property

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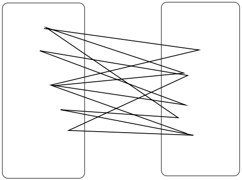

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Theorem (Fox-Pach-Tóth, 2010)

Intersection graphs of convex sets have the strong Erdős-Hajnal property, but not the **mighty** *Erdős-Hajnal property.*

Pseudo-Segments: Old results

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$$2^{\Omega(n \log n)} < |\mathcal{P}_n| < 2^{O(n^{3/2} \log n)}$$

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Point-line incidences

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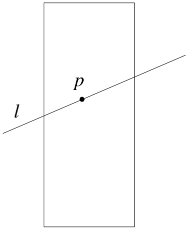

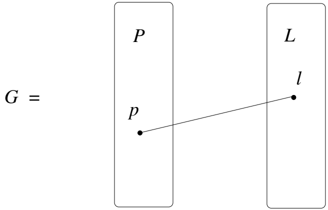

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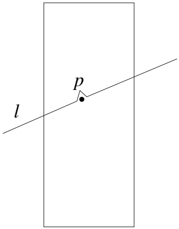

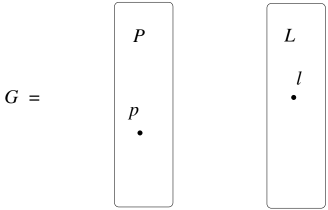

Pseudo-Segments: New results

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Mighty Erdős-Hajnal property

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Mighty Erdős-Hajnal property | Erdős-Hajnal property

Theorem (Fox-Pach-Tóth, 2010)

$\mathcal{P}_n$ has the strong Erdős-Hajnal property.

New result

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*$\mathcal{P}_n$ has the **mighty** Erdős-Hajnal property.*

New result

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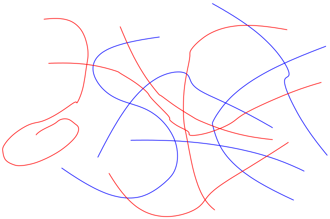
$\mathcal{R} = n$ red curves, $\mathcal{B} = n$ blue curves, $\mathcal{R} \cup \mathcal{B}$ pseudo-segments.

$\mathcal{R}' \subset \mathcal{R}$, $\mathcal{B}' \subset \mathcal{B}$ of size $\Omega(n)$



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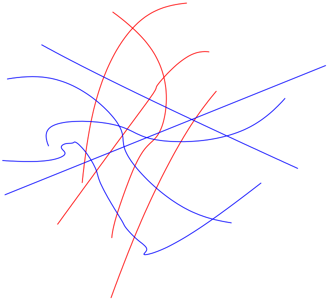

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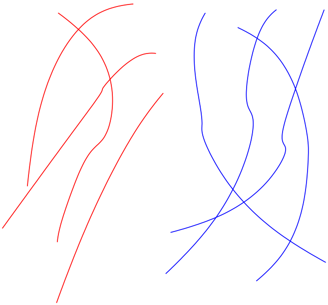

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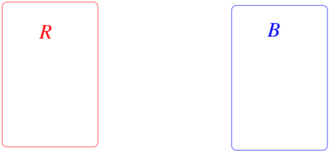

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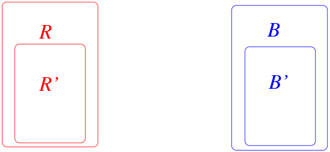

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Intersection graphs $G(\mathcal{R}')$ and $G(\mathcal{B}')$ has edge density less than ε or greater than $1 - \varepsilon$.

New result

Case 1. Both $G(\mathcal{R}')$ and $G(\mathcal{B}')$ have edge density less than ε.

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Case 3. Both $G(\mathcal{R}')$ and $G(\mathcal{B}')$ have edge density at least $1-\varepsilon$. Repeat the arguments in Case 2.

Applications

Theorem (Fox-Pach-S., 2024+)

$\mathcal{P}_n$ has the **mighty** *Erdős-Hajnal property.*

homogeneous density property

Theorem (Fox-Pach-S., 2024+)

There is an absolute constant $c > 0$ such that the following holds. Let $\mathcal{R}$ be a collection of n red curves, and $\mathcal{B}$ be a collection of n blue curves in the plane such that $\mathcal{R} \cup \mathcal{B}$ is a collection of pseudo-segments.

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1. *If there are at least δn^2 disjoint pairs in $\mathcal{R} \times \mathcal{B}$, then there are subsets $\mathcal{R}'$ and $\mathcal{B}'$, each of size $\delta^c n$, such that every red curve in $\mathcal{R}'$ is disjoint to every blue curve in $\mathcal{B}'$.*

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Mighty EH property $\Rightarrow$ Density theorems

$\mathcal{R} = n$ red curves.

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$G =$ disjointness graph between $\mathcal{R}$ and $\mathcal{B}$. $|E(G)| \geq \delta n^2$.

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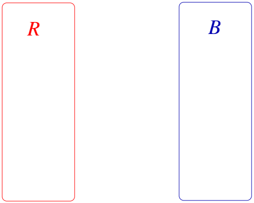

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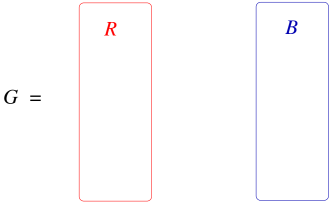

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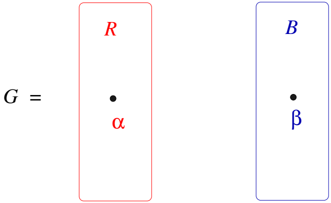

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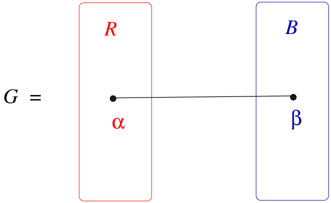

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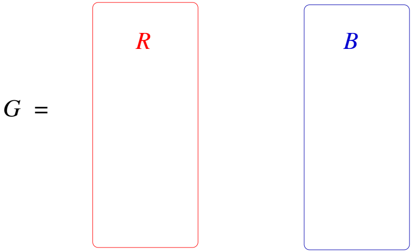
Set ϵ to be the constant from the Mighty EH property.

Theorem (Szemerédi 1978, Komlós 1996)

Let $G = (\mathcal{R} \cup \mathcal{B}, E)$ be a bipartite graph with at least δn^2 edges. Then for any $\varepsilon > 0$, there are subsets $\mathcal{R}'$ and $\mathcal{B}'$, each of size $\delta^{1/\varepsilon^2} n$, such that every subset $X \subset \mathcal{R}'$ and $Y \subset \mathcal{B}'$ of size at least $\varepsilon|\mathcal{R}'|$ and $\varepsilon|\mathcal{B}'|$ respectively contains an edge.



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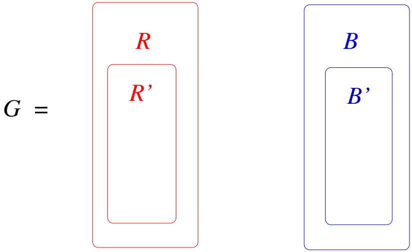

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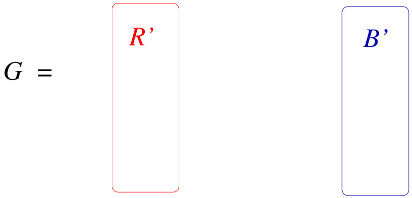

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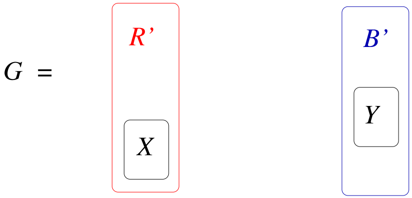

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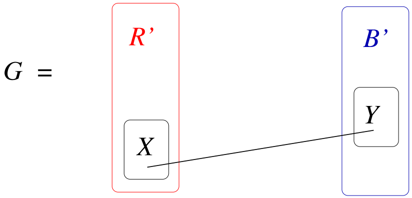

Mighty EH property $\Rightarrow$ Density theorems

Set ϵ to be the constant from the Mighty EH property.

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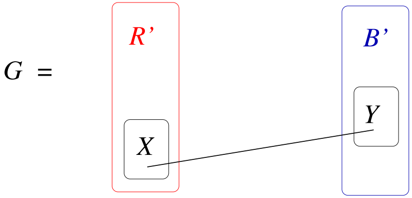

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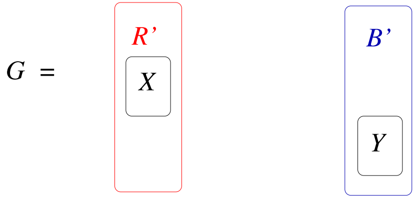

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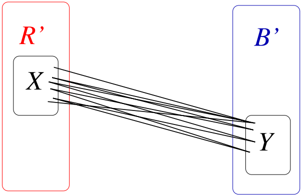
$G =$ 

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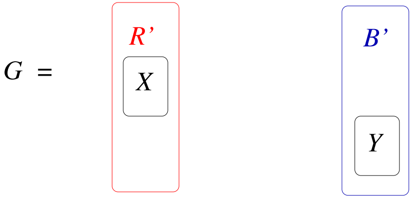

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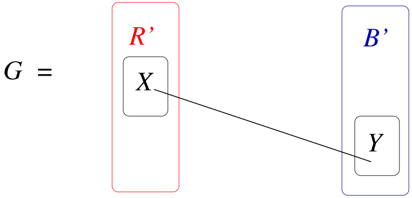

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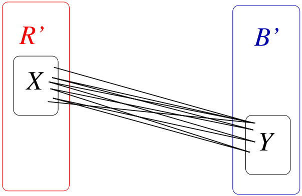
$G =$ 

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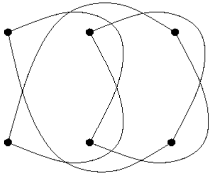
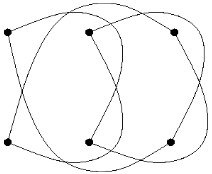
Application of the density theorems



Conjecture (Pach-Tóth, 2005)

If G is an n-vertex graph with a simple drawing in the plane with no k pairwise disjoint edges, then $|E(G)| = O_k(n)$.

Application of the density theorems



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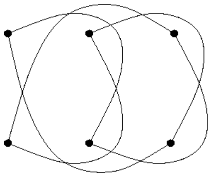

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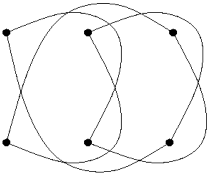

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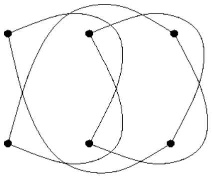

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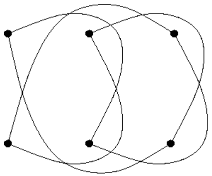

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If G is an n-vertex graph with $n^{1+\varepsilon}$ edges, then any simple drawing of G in the plane contains $n^{\frac{\varepsilon}{10 \log \log n}}$ pairwise disjoint edges.

A new regularity lemma for pseudo-segments

Mighty EH property $\Leftrightarrow$ density theorems $\Leftrightarrow$ strong regularity lemma

Theorem (Fox-Pach-S., 2024+)

For every ε, there is a $K = K(\varepsilon)$, such that every intersection graph of pseudo-segments in the plane has an equipartition on its vertex set into K parts, $V_1, \ldots, V_K$, such that for all but an ε fraction of pairs of parts (V_i, V_j) are complete or empty in G.

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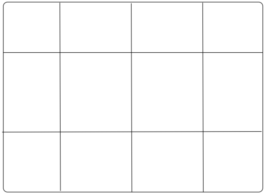

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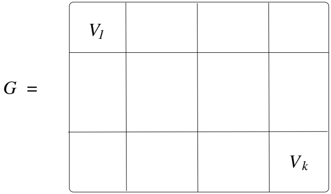

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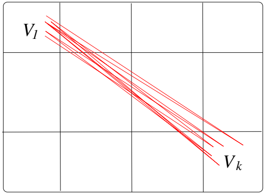
$G =$ 

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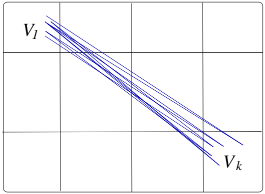

Open problems: Polynomial strong regularity lemma

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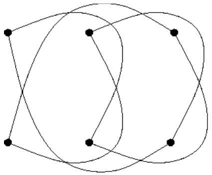
Conjecture (Fox-Pach-S., 2024+)

$K = (1/\varepsilon)^c$

Fox-Pach-S.: K is a tower of 2's of height $(1/\varepsilon)^c$

Open problems: k-quasi-thrackle conjecture



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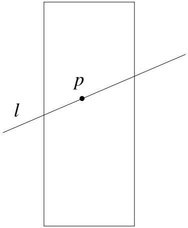
Open problems: Incidences between points and 2-intersecting curves

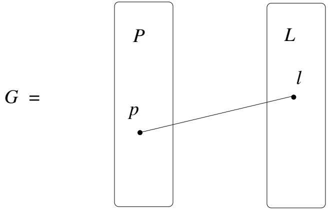
Theorem (Fox-Pach-S., 2024+)

$$2^{\Omega(n^{4/3})} < |\mathcal{P}_n^{mono}| \leq |\mathcal{P}_n| \leq 2^{O(n^{3/2}\log n)}.$$

$P = n^{1/3} \times n^{2/3}$ grid $\qquad L = n$ lines

$|I(P, L)| = \Theta(n^{4/3})$





Open problems: Incidences between points and 2-intersecting curves

Problem

What is the maximum number of incidences between n points and n 2-intersecting curves in the plane?

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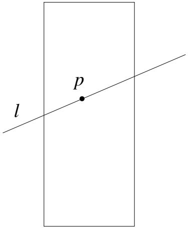

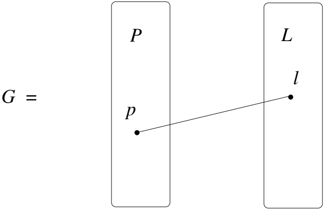

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$P = n$ points $\qquad L = n$ 2-intersecting curves

Pach-Sharir, 1998

$\Omega(n^{4/3}) \leq |I(P, L)| = O(n^{7/5})$

Open problems: Incidences between points and k-intersecting curves

Problem

What is the maximum number of incidences between n points and n k-intersecting curves in the plane?

$P = n$ points $\qquad$ $L = n$ k-intersecting curves

Pach-Sharir, 1998

$\Omega(n^{4/3}) \leq |I(P, L)| = O(n^{\frac{3k-2}{2k-1}})$

Open problems: Incidences between points and k-intersecting curves

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What is the maximum number of incidences between n points and n k-intersecting curves in the plane?

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Application:

$$2^{\Omega(n^{4/3})} < |\mathcal{P}_n^{(k)}| < 2^{O(n^{2-\varepsilon})}.$$

Thank you!