A multipartite analogue of Dilworth's Theorem

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Abstract

Dilworth's Theorem says that any partially ordered set on n elements contains a chain or antichain of size $\sqrt{n}$; in other words, the comparability graph of any partially ordered set of size n contains a clique or an independent set of size $\sqrt{n}$. Fox (2006) showed that any such partially ordered set contains two subsets A, B of size at least $\Omega(n/\log n)$ such that either $a < b$ for all $a \in A$ and $b \in B$, or a and b are incomparable for all $a \in A$ and $b \in B$. Thus, one can guarantee significantly larger complete or anticomplete bipartite subgraphs in comparability graphs. Fox asked if a similar result holds for the largest complete or anticomplete k-partite subgraph in comparability graphs for $k \geq 3$.

Answering Fox's question, we show that for any $k \geq 3$ and any partially ordered set on n elements with n sufficiently large, there exist subsets $A_1, \ldots, A_k$ of size at least $\Omega(n/(k^2 \log n))$ such that either $a_1 < a_2 < \cdots < a_k$ for all $a_i \in A_i$; or a_i and a_j are incomparable for any $a_i \in A_i$, $a_j \in A_j$ and $i \neq j$. Both the dependence on k and n are best possible up to constant factors. As a corollary, we show that for any h partial orders $<_1, \ldots, <_h$, there exist k subsets $A_1, \ldots, A_k$ each of size at least $n/(k \log n)^{O_h(1)}$ such that for each partial order $<_\ell$, either $a_1 <_\ell a_2 <_\ell \cdots <_\ell a_k$ for any $a_i \in A_i$; or $a_1 >_\ell a_2 >_\ell \cdots >_\ell a_k$ for any $a_i \in A_i$; or a_i is incomparable with a_j for any $a_i \in A_i$, $a_j \in A_j$ and $i \neq j$. This improves an earlier result of Fox and Pach (2009).

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