Ramsey Theory Workshop

University of California, San Diego, February 18-20 2023

Organizers D. Mubayi, A. Suk, J. Verstraete

OPEN PROBLEM SESSION

OPEN PROBLEMS

Problem 1.

D. Mubayi

Given a graph F and a real $x \in [0,1]$, let $I(F,x)$ be the maximum proportion of induced copies of F in a (large) graph with edge density x.

Problems: For each of the following, is there a graph F such that

1. There is a real x such that $I(F,x) = rand(F,x)$, where $rand(F,x)$ is the expected number of induced copies of F in the random graph with density x?
2. $I(F,x)$ has at least two global maxima ?
3. There is a nontrivial interval J such that $I(F,x) = sup I(F,x)$ for all $x \in J$?
4. $I(F,x)$ has a nontrivial local maximum (minimum)?

Problem 2.

J. Tidor

This conjecture comes out of some joint work with MingYang Deng and Yufei Zhao.

Let $s(k;r)$ be the largest N such that there exists an r-coloring $\phi\colon \mathbb{Z}/N\mathbb{Z} \to [r]$ with no symmetrically-colored $2k$-APs. A $2k$-AP $a, a+d, a+2d, \ldots, a+(2k-1)d$ with $d \neq 0$ is symmetrically-colored if $\phi(a+id) = \phi(a+(2k-i-1)d)$ for all $0 \le i < k$.

For each $k \geq 2$, prove that there exists some r such that $s(k;r) > r^k$. (This is known for $k = 2,3,4$ and open in general.) A harder question is to prove for each fixed $k \geq 2$, that $s(k;r)$ grows superpolynomially in r, i.e., $s(k;r) \geq r^{\omega_{k;r\to\infty}(1)}$.

Problem 3.

C. Toth

A *lattice polytope* P is the convex hull of a finite set $S \subset \mathbb{Z}^d$ for $d \in \mathbb{N}$. Andrews (1963) proved that for every $d \in \mathbb{N}$, every lattice polytope of volume V in d-space has at most $O(V^{\frac{d-1}{d+1}})$ faces, and this bound is the best possible; see also Barany and Pach (1992) and Barany and Vershik (1992).

A facet F of a lattice polytope P is *t-rich* if F contains more than t lattice points, for $t \in \mathbb{N}$, that is, $|F \cap \mathbb{Z}^d| > t$. For $d \geq 2$ let $f(d) > 0$ denote the largest real such that for all $t \geq 1$ every d-dimensional lattice polytope of volume V has

$$O\left(\left(\frac{V}{t^{f(d)}}\right)^{\frac{d-1}{d+1}}\right)$$

t-rich facets. It is not difficult to see that $f(2) = 2$. Determine $f(d)$ for $d \geq 3$.

Problem 4.

S. Spiro

Given two graphs K, H, define $r_K(H)$ to be the minimum number of copies of K that a graph G can have such that any 2-coloring of G contains a monochromatic copy of H. For example, when $K = K_1$ this is the usual Ramsey number, and when $K = K_2$ this is the size Ramsey number. The general question is to try and generalize bounds for size Ramsey numbers to other K, e.g. when K is a clique or complete bipartite graph. In particular, is it true that $r_{K_s}(K_t) = \binom{r(K_t)}{s}$ for all $s \leq t$ (which is known to hold when $s = 2$)?

Problem 5.

J. Verstraete

Let G be a graph with n vertices and independence number $o(n)$ as $n \to \infty$. Does G contain a 3-regular subgraph? If G has independence number $o(n/\log^* n)$ then indeed G has a 3-regular subgraph. This is a consequence of the following result of Pyber, Rödl and Szemerédi (1991) (more recently, Janzer and Sudakov), who proved the following:

Theorem 1 *Let $d \geq 1$ and let G be a graph of average degree at least $2^{10}d$ and maximum degree at most 2^{10d}. Then G has a 3-regular subgraph.*

Indeed, if G is an n-vertex graph of average degree d then G contains at most $n/2$ vertices of degree at least $4d$. Therefore G has an induced subgraph with at least $n/2$ vertices and maximum degree at most $4d$. By the theorem, this has a 3-regular subgraph unless the average degree is $O(\log d)$, and it suffices to repeat the argument if d is roughly $\log^* n$ to get a large independent set. This is a Ramsey version of the Turán problem asking for the maximum number of edges in an n-vertex graph with no 3-regular subgraph. The latter was solved by Janzer and Sudakov recently, who showed the answer is $\Theta(n \log\log n)$. If F is the family of all 3-regular graphs, then our problem is asking whether $r(K_n, F) = O(n)$, and the above argument gives $r(K_n, F) = O(n \log^* n)$.

Problem 6. J. Fox

Does there exists a constant $c > 0$ such that in any red-blue-edge-coloring of K_n, either there is a red triangle or a blue $K_{s,t}$ where $s \geq n^c$ and $t \geq cn$? It is know that there is a blue $K_{s,t}$ with $s \geq (\log n)/\log\log n$ and t linear in n, but even $s = \log n$ is not known. In other words, is there a constant $c > 0$ such that

$$r(K_3, K_{n^c,n}) = O(n)?$$

This is a toy version of a more general problem which is related to the Erdős-Hajnal conjecture.

Problem 7. D. Conlon

An interesting problem of Horn, Milans and Rödl [2] asks whether for every d there exists D such that, for every graph H with maximum degree at most d, there is a graph G with maximum degree at most D with the property that every two-colouring of the edges of G contains a monochromatic copy of H. Such a result is known for some families of graphs, like blowups of trees [2], but seems likely to be false in general. However, just as for size Ramsey numbers [1], the question of whether such a result holds for grid graphs is an interesting test case. This was raised in [2], but it would already be interesting to prove that we can take G to have maximum degree $n^{o(1)}$ when H is the $n \times n$ grid.

[1] D. Conlon, R. Nenadov and M. Trujićc, On the size-Ramsey number of grids, to appear in Combin. Probab. Comput.

[2] P. Horn, K. G. Milans and V. Rödl, Degree Ramsey numbers of closed blowups of trees, Electron. J. Combin. 21 (2014), Paper 2.5, 6 pp.