Problems on generalizing planar graphs and thrackles

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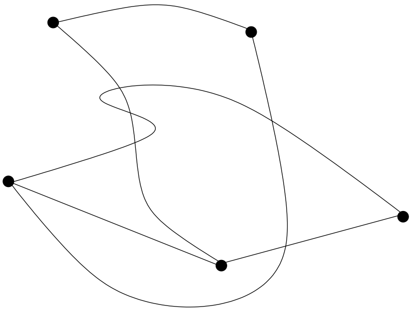
Definition

A *topological graph* is a graph drawn in the plane with vertices represented by points and edges represented by curves connecting the corresponding points. A topological graph is *simple* if every pair of its edges intersect at most once.



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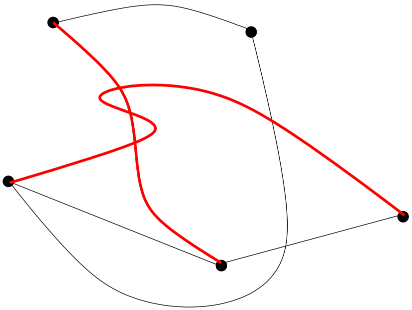
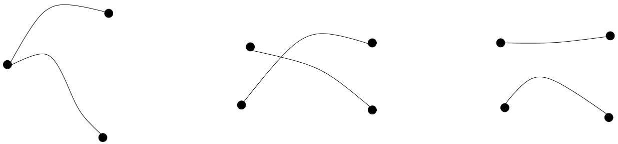


Special Case 2

G is a **simple** topological graph



Classic results

Planar graphs:

Theorem (Euler)

Every n-vertex simple topological graph with no two **crossing** *edges has at most $3n - 6$ edges.*

Dual

Thrackles:

Conjecture (Conway)

Every n-vertex simple topological graph with no two **disjoint** *edges has at most n edges.*

Classic results

Planar graphs:

Theorem (Euler)

Every n-vertex simple topological graph with no two **crossing** *edges has at most $3n - 6$ edges.*

Dual

Thrackles:

Theorem (Lovász, Pach, Szegedy, 1997)

Every n-vertex simple topological graph with no two disjoint edges, has at most $2n$ edges.

Best known $1.43n$ by Fulek and Pach, 2010.

Relaxing planarity/thrackle condition

k-quasi-planar graphs

Conjecture

Every n-vertex simple topological graph with no k pairwise crossing edges has at most $O(n)$ edges.

Dual

k-quasi-thrackles

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Relaxing planarity/thrackle condition

k-quasi-planar graphs

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Every n-vertex simple topological graph with no k pairwise crossing edges has at most $O(n)$ edges.

Proven: for $k = 3, 4$. **Open**: for $k \geq 5$ Dual

k-quasi-thrackles

Conjecture (Pach and Tóth, 2005)

Every n-vertex simple topological graph with no k pairwise disjoint edges, has at most $O(n)$ edges.

Open: for $k \geq 3$

Best known bounds

k-quasi-planar graphs

Conjecture

Every n-vertex simple topological graph with no k pairwise crossing edges has at most $O(n)$ edges.

Best bound: $O(n \log n)$ for $k \geq 5$ (S. and Walczak 2012)

Dual

k-quasi-thrackles

Conjecture (Pach and Tóth, 2005)

Every n-vertex simple topological graph with no k pairwise disjoint edges, has at most $O(n)$ edges.

Best bound: $O(n \log^{4k-8} n)$ for $k \geq 3$ (Pach and Tóth, 2005)

A coloring problem

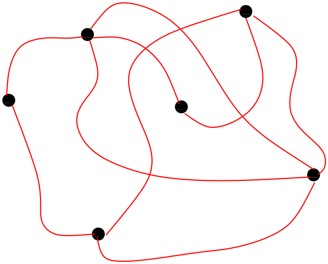
k-quasi-planar graphs

Conjecture

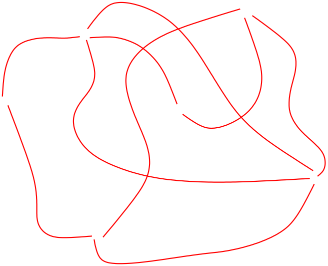
Every n-vertex simple topological graph with no k pairwise crossing edges has at most $O(n)$ edges.

Best bound: $O(n \log n)$ for $k \geq 5$ (S. and Walczak 2012)

$G = (V, E)$ k-quasi-planar graph.



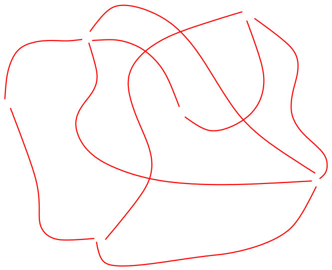
E is a family of $|E(G)|$ curves in the plane, no k pairwise intersecting.



Conjecture (Coloring conjecture)

Let F be a family of curves in the plane such that no k members pairwise intersect. Then $\chi(F) \leq c_k$.

Color the curves such that each color class consists of pairwise disjoint curves.



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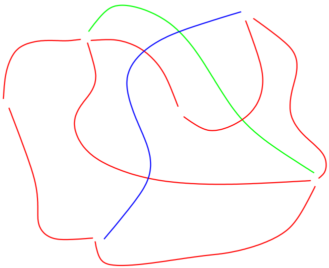

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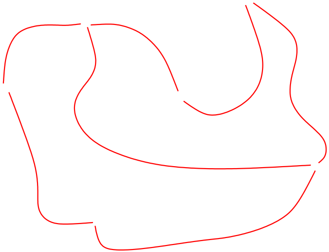

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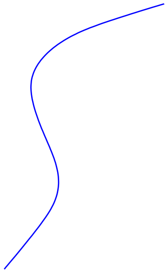

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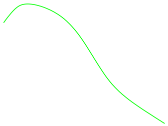

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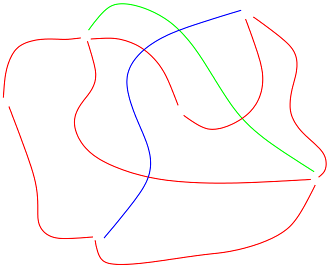
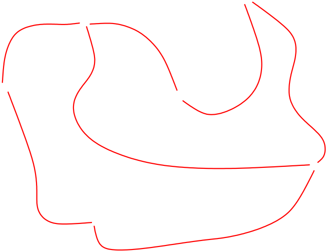


Conjecture (Coloring conjecture)

Let F be a family of curves in the plane such that no k members pairwise intersect. Then $\chi(F) \leq c_k$.

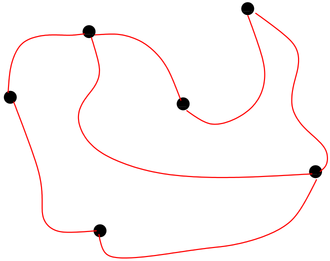
One of the color classes has at least $|E(G)|/c_k$ curves (edges).



Conjecture (Coloring conjecture)

Let F be a family of curves in the plane such that no k members pairwise intersect. Then $\chi(F) \leq c_k$.

$$\frac{|E(G)|}{c_k} \leq 3n - 6$$



Conjecture (Coloring conjecture, FALSE)

Let F be a family of curves in the plane such that no k members pairwise intersect. Then $\chi(F) \leq c_k$.

Conjecture is False!

Theorem (Pawlik, Kozik, Krawczyk, Lason, Micek, Trotter, Walczak, 2012)

For infinite values n, there exists a family F of n segments in the plane, no three members pairwise cross, and $\chi(F) > \Omega(\log \log n)$.

Conjecture (Coloring conjecture, FALSE)

Let F be a family of curves in the plane such that no k members pairwise intersect. Then $\chi(F) \leq c_k$.

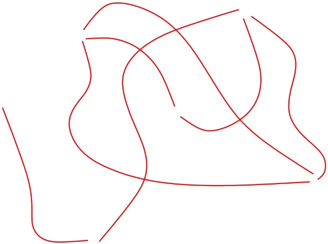
Conjecture true under extra conditions?

Theorem (Suk and Walczak, 2013)

Let F be a family of curves in the plane such that no k members pairwise intersect. Furthermore, suppose

1. *F is* **simple**,
2. *there is a curve β that intersects every member in F exactly once,*

then $\chi(F) \leq c_k$.



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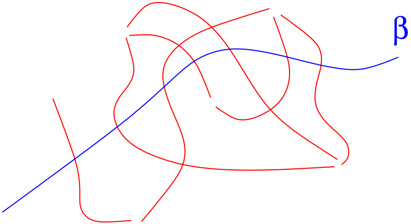

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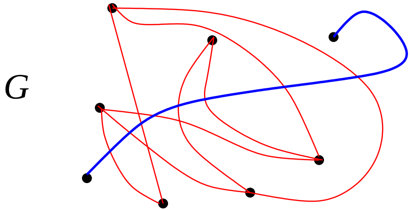
1. Coloring intersection graphs of arcwise connected sets in the plane, Lason, Micek, Pawlik and Walczak 2013.
2. Coloring intersection graphs of x-monotone curves in the plane, Suk 2012.
3. On bounding the chromatic number of L-graphs, McGuinness 1996.

Application of coloring result.

Corollary (Suk and Walczak, 2013)

For fixed $k > 1$, let G be a **simple** *n-vertex k-quasi planar graph. If G contains an edge that crosses every other edge, then $|E(G)| \leq O(n)$.*

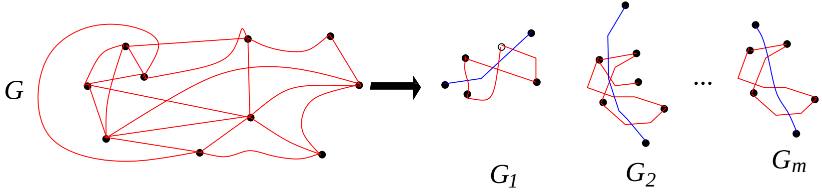


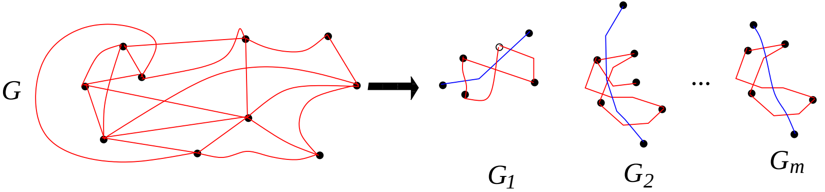
Lemma (Fox, Pach, Suk, 2012)

Let G be a simple topological graph on n vertices. Then there are subgraphs $G_1, G_2, \ldots, G_m \subset G$ such that

$$\frac{|E(G)|}{c\log n} \leq \sum_{i=1}^{m} |E(G_i)|,$$

every edge in G_i is disjoint to every edge in G_j. G_i has an edge that crosses every other edge in G_i.





Let $n_i = |V(G_i)|$.

- $|E(G_i)| \leq c_k n_i$, Suk and Walczak 2013.

$$\frac{|E(G)|}{c \log n} \leq \sum_{i=1}^{m} |E(G_i)| \leq \sum_{i=1}^{m} c_k n_i = c_k(n_1 + n_2 + \cdots + n_m) = c_k n.$$

□

Disjoint edges

k-quasi-thrackles

Conjecture (Pach and Tóth, 2005)

Every n-vertex simple topological graph with no k pairwise disjoint edges, has at most $O(n)$ edges.

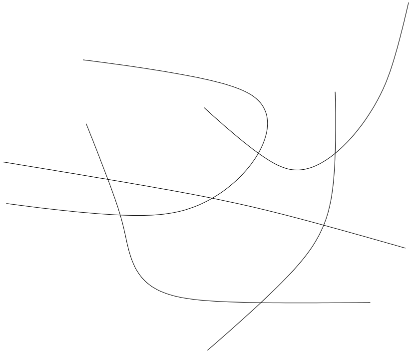
Conjecture

Let F be a family of curves in the plane such that no k pairwise are disjoint. Then $\chi(\overline{F}) \leq c_k$.

OPEN

Disjoint edges

F = family of curves no k pairwise crossing.



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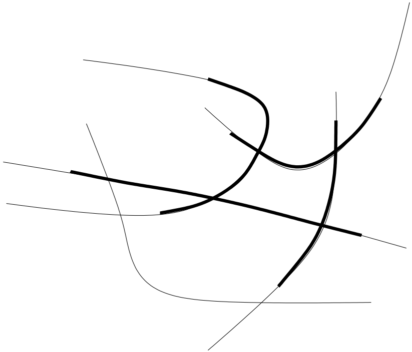
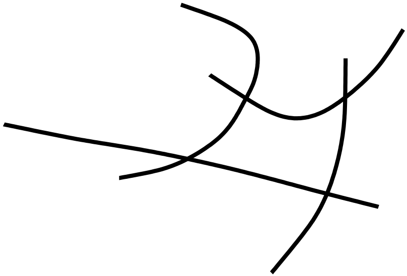


Disjoint edges

F = family of curves no k pairwise crossing. Still no k pairwise crossing (did not introduce crossing pairs).



Not true if F had no k pairwise **disjoint** members (new disjoint pairs can be introduced).

Disjoint edges

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OPEN

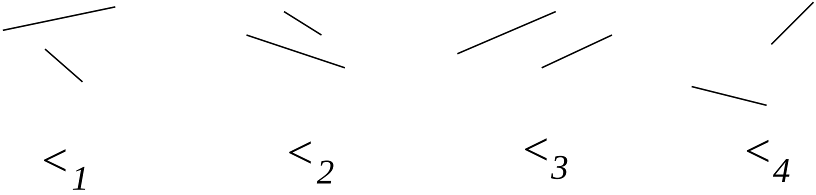
Coloring theorem for disjoint curves

Theorem (Pach and Töröcsik, 1994)

Let F be family of segments in the plane such that no k members are pairwise disjoint. Then $\chi(\overline{F}) \leq ck^4$.

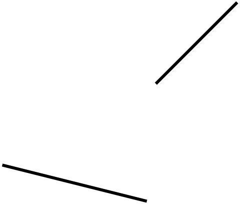
Partial orders

Disjoint edges can be compared by one of four partial orders:



Partial orders

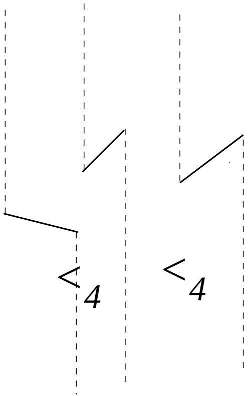
Example



$$<_4$$

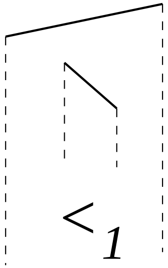
Partial orders

Example



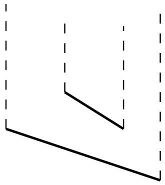
Partial orders

Example



Partial orders

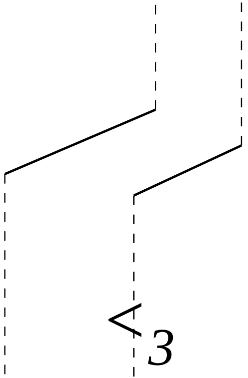
Example



$<_2$

Partial orders

Example



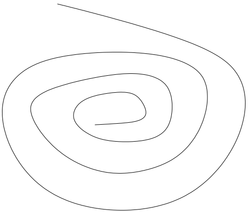
Generalizes

Theorem (Pach and Törőcsik, 1994)

Let F be family of x-monotone curves in the plane such that no k members are pairwise disjoint. Then $\chi(\overline{F}) \leq ck^4$.

General curves

No (clear) partial ordering for general curves



Back to the classic results

Planar graphs:

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Every n-vertex simple topological graph with no two **crossing** *edges has at most $3n - 6$ edges.*

Dual

Thrackles:

Theorem (Lovász, Pach, Szegedy, 1997)

Every n-vertex simple topological graph with no two disjoint edges, has at most $2n$ edges.

Best known $1.43n$ by Fulek and Pach, 2010.

Another generalization

Planar graphs:

Theorem (Fox and Pach)

Let G be an n-vertex simple topological graph with k edges crossing another set of k edges. Then G has at most $O(n)$ edges.

Dual

Thrackles:

Theorem (Ruiz-Vargas, S., Tóth, 2014)

Let G be an n-vertex simple topological graph with k edges disjoint to another set of k edges. Then G has at most $O(n)$ edges.

A stronger result

Theorem (Fox and Pach)

Let F be a simple family of n curves in the plane. If the intersection graph of F is $K_{k,k}$-free, then the number of edges in the intersection graph is at most $O(n)$.

Dual

Problem

Let F be a simple family of n curves in the plane. If the non-intersection graph of F is $K_{k,k}$-free, then the number of edges in the non-intersection graph is at most $O(n)$.

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Dense graphs

Theorem (Fox and Pach, 2008)

Every n-vertex simple topological graph with $\Omega(n^2)$ edges, has n^ϵ pairwise crossing edges.

Dual

Problem

Every n-vertex simple topological graph $\Omega(n^2)$ edges, has n^ϵ pairwise disjoint edges.

OPEN. Best known result: $\log^{1+\delta} n$ (Fox and Sudakov 2006).

Dense graphs

Theorem (Fox and Pach, 2008)

Every n-vertex simple topological graph with $\Omega(n^2)$ edges, has n^ϵ pairwise crossing edges.

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Problem

Every n-vertex simple topological graph $\Omega(n^2)$ edges, has n^ϵ pairwise disjoint edges.

OPEN. True for *complete* graphs: $\Omega(n^{1/3})$ (S. 2011, Fulek and Ruiz-Vargas 2013).

Thank you!