Daisy-free hypergraphs and arithmetic Hilbert cubes

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Abstract

For integers $r \geq 3$ and $t \geq 2$, an r-uniform *t-daisy* $\mathcal{D}_r^t$ is a family of $\binom{2t}{t}$ r-element sets of the form

$$\{S \cup T \ : T \subset U, \ |T| = t\}$$

for some sets S, U with $|S| = r - t$, $|U| = 2t$ and $S \cap U = \emptyset$. It was conjectured by Bollobás, Leader and Malvenuto (and independently by Bukh) that the Turán densities of t-daisies satisfy $\lim_{r \to \infty} \pi(\mathcal{D}_r^t) = 0$ for all $t \geq 2$. This problem has become well known and is still open. Bollobás, Leader and Malvenuto observed that the complete r-partite hypergraph is daisy-free; we improve these lower bounds for the Turán densities. To do so, we introduce the following natural problem in additive combinatorics: for integers $m \geq 2t \geq 4$, what is the maximum cardinality $g(m, t)$ of a subset R of $\mathbb{Z}/m\mathbb{Z}$ such that for any $x \in \mathbb{Z}/m\mathbb{Z}$ and any $2t$-element subset X of $\mathbb{Z}/m\mathbb{Z}$, there are t distinct elements of X whose sum is not in the translate $x + R$? This question is a slice-analogue of an extremal Hilbert cube problem considered by Cilleruelo and Tesoro and separately by Gunderson and Rödl. This talk is based upon joint work with David Ellis.