Turán-type problems for point-line incidences

Andrew Suk (UC San Diego)

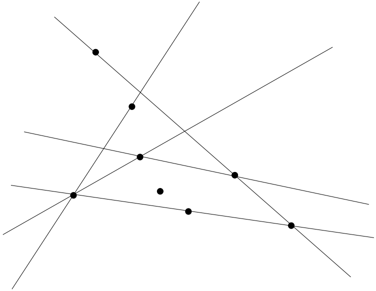
June 8, 2021

The Szemerédi-Trotter Theorem

$P = n$ points in the plane.

$L = m$ lines in the plane.

$$I(P, L) = \{(p, \ell) \in P \times L : p \in \ell\}.$$



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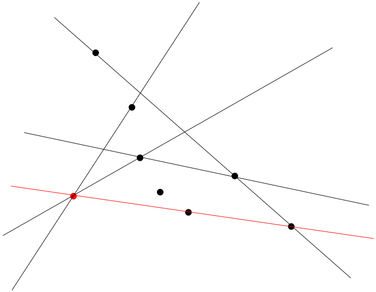

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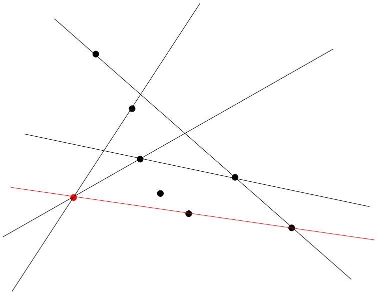
Theorem (Szemerédi-Trotter 1983)

$$|I(P, L)| \leq O(m^{2/3}n^{2/3} + m + n).$$



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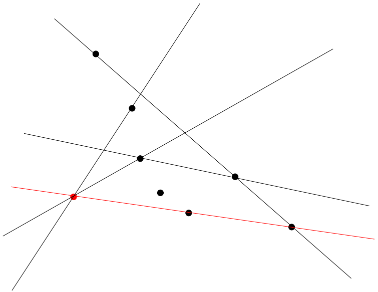
Theorem (Szemerédi-Trotter 1983)

$$|I(P, L)| \leq O(n^{4/3}).$$



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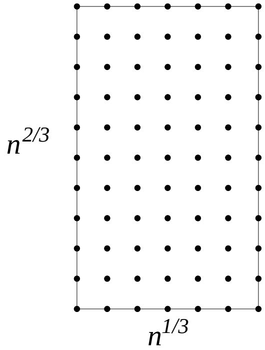

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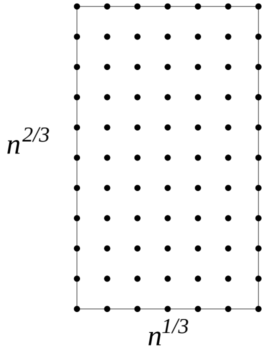
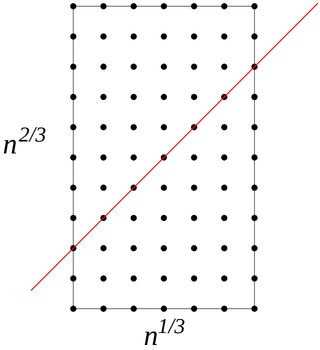


The Szemerédi-Trotter Theorem

$P = n^{1/3} \times n^{2/3}$ grid.

$L : y = mx + b$, $m \in \{1, ..., \lfloor \frac{n^{1/3}}{2} \rfloor\}$ and $b \in \{1, ..., \lfloor \frac{n^{2/3}}{2} \rfloor\}$



$$|I(P, L)| = \Theta(n^{4/3})$$

The Szemerédi-Trotter Theorem

$P = n$ points in the plane $L = n$ lines in the plane.

Theorem (Szemerédi-Trotter 1983)

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Variations:

1. Points and Curves in $\mathbb{R}^2$
2. Points and lines in $\mathbb{R}^d$
3. Points and hyperplanes in $\mathbb{R}^d$

Applicitions:

1. Sums versus Products
2. Distinct Distances

Old Question of Erdős

$P = n$ points in the plane $\qquad$ $L = n$ lines in the plane.

$$|I(P, L)| \geq \Omega(n^{4/3})$$

Does (P, L) show any kind of structure?

Old Question of Erdős

$P = n$ points in the plane $\qquad L = n$ lines in the plane.

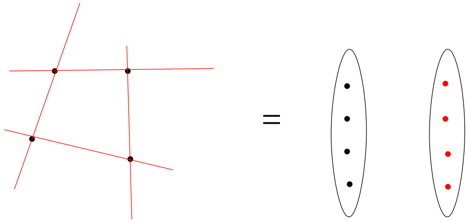
$$|I(P, L)| \geq \Omega(n^{4/3})$$

Does (P, L) show any kind of structure?

Does (P, L) show any kind of **grid** structure?

Turán-type problem

Let (P_0, L_0) be a fixed point-line configuration.



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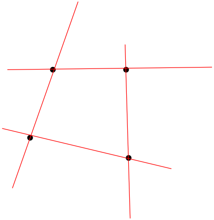
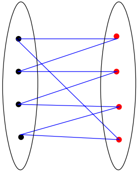


$=$



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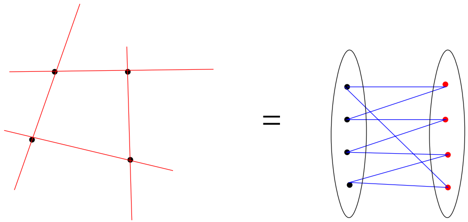


$P = n$ points in the plane. $L = n$ lines in the plane.

How large can $|I(P, L)|$ be if (P, L) does not contain (P_0, L_0) as subconfiguration?

Research problems in Discrete Geometry, Brass, Moser, Pach

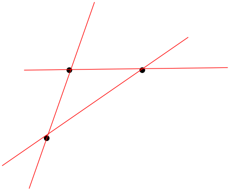
Let (P_0, L_0) be a fixed point-line configuration

Conjecture (Solymosi)

Any set of n points and n lines in the plane that does not contain (P_0, L_0) as a subconfiguration, determines at most $o(n^{4/3})$ incidences.

$C_6 = $ 1-subdivision of K_3

$(P_0, L_0) =$



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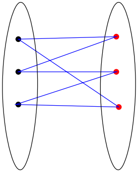

Theorem (Solymosi 2006)

Any set of n points and n lines in the plane does not contain (P_0, L_0) as a subconfiguration, determines $o(n^{4/3})$ incidences.

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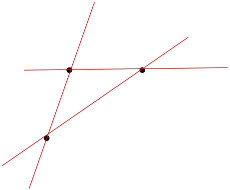

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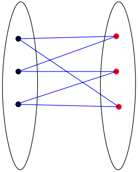
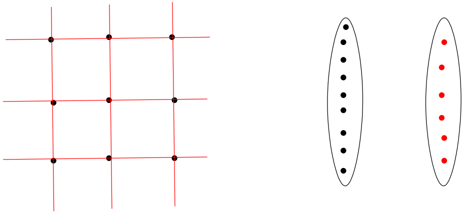


Theorem (Solymosi 2006)

Any set of n points and n lines in the plane that does not contain a 1-subdivision of K_k in its incidence graph, determines $o(n^{4/3})$ incidences.

Forbidding $k \times k$-grid

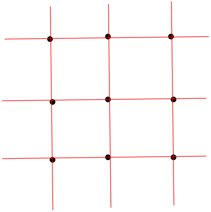
$(P_0, L_0) =$

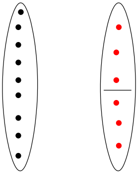


Theorem (Mirzaei-S. 2020)

Any set of n points and n lines in the plane that does not contain (P_0, L_0) as a subconfiguration, determines $O(n^{\frac{4}{3} - \frac{1}{9k-6}})$ incidences.

1-subdivision of $K_{k,k}$

$(P_0, L_0) =$ 



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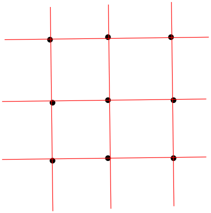

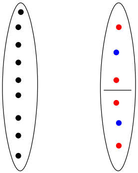

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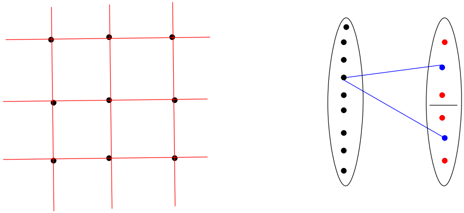

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Bounds

Theorem (Mirzaei-S. 2020)

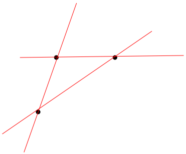
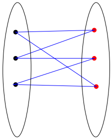
Any set of n points and n lines in the plane that does not contain $k \times k$-grid subconfiguration, determines $O(n^{\frac{4}{3} - \frac{1}{9k-6}})$ incidences.

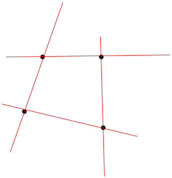
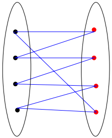
In the other direction

Theorem (S.-Tomon 2021)

For $k \geq 3$, there is a set of n points and n lines in the plane that does not contain a $k \times k$ grid, and determines at least $\Omega(n^{\frac{4}{3} - \frac{4}{3(k-2)}})$ incidences.

Forbidding Even Cycles

$(P_0, L_0) =$  $=$ 

 $=$ 

Old results

Let $ex(n, C_{2k})$ denote the maximum number of edges in an n-vertex graph that is C_{2k}-free.

Theorem (Erdős, Bondy-Simonovitz 1938,1974)

For fixed $k \geq 2$, $ex(n, C_{2k}) = O(n^{1+\frac{1}{k}})$.

Tight for C_4, C_6, C_{10} (Benson 1966).

Theorem (Lazebnik, Ustimenko and Woldar 1995)

For even k, $ex(n, C_{2k}) = \Omega(n^{1+\frac{2}{3k-2}})$.

For odd k, $ex(n, C_{2k}) = \Omega(n^{1+\frac{2}{3k-3}})$.

Point-line configurations without cycles

Theorem (Erdős, Bondy-Simonovitz 1938,1974)

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Conjecture (Mirzaei-S.-Verstraëte 2020)

Any set of n points and n lines in the plane, whose incidence graph is C_{2k}*-free, determines* $o(n^{1+\frac{1}{k}})$ *incidences.*

Point-line configurations without cycles

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Theorem (Solymosi 2006)

Any set of n *points and* n *lines in the plane, whose incidence graph is* C_6*-free, determines* $o(n^{4/3})$ *incidences.*

Point-line configurations without cycles

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Theorem (S.-Tomon 2021)

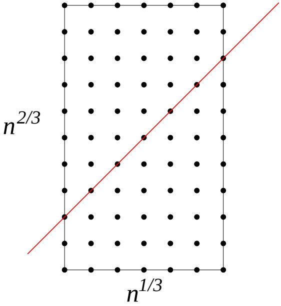
For every positive integer $k \geq 2$, there exists a set of n points and n lines in the plane whose incidence graph is C_{2k}-free, and determines at least $n^{1+\frac{1}{2k}-o(1)}$ incidences.

Construction

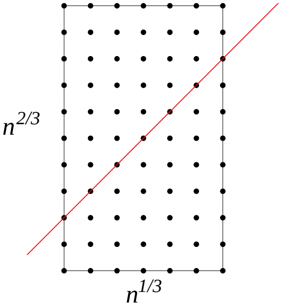
Theorem (S.-Tomon 2021)

For every positive integer $k \geq 2$, there exists a set of n points and n lines in the plane whose incidence graph is C_{2k}-free, and determines at least $n^{1+\frac{1}{2k}-o(1)}$ incidences.

Proof. Standard Construction.



Proof. Standard Construction.
$P = n^{1/3} \times n^{2/3}$ grid.
$L : y = mx + b,\ m \in [\frac{n^{1/3}}{2}]$ and $b \in [\frac{n^{2/3}}{2}]$



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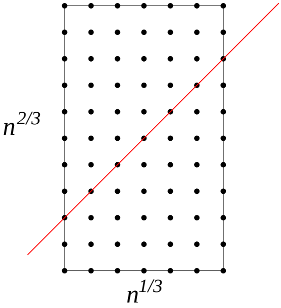

$$|I(P, L)| = \Theta(n^{4/3}) \qquad \#C_{2k} < n^{2k}$$

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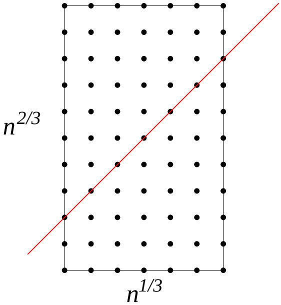

$$|I(P, L)| = \Theta(n^{4/3}) \qquad \#C_{2k} < n^{k+o(1)}$$

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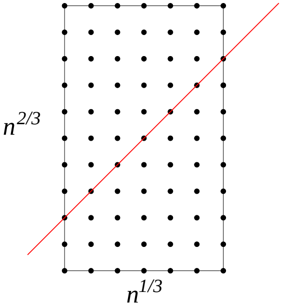
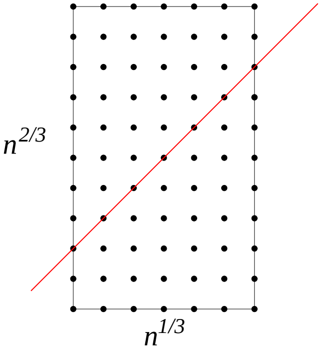


$$|I(P, L)| = \Theta(n^{4/3}) \qquad \#C_{2k} < n^{2k/3+o(1)}$$

Key Lemma

Lemma

Let $p, p' \in P$ be distinct vertices. The number of "common neighbors" of p and p' is at most $n^{\frac{1}{3}+o(1)}$.



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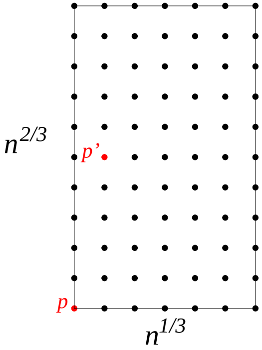

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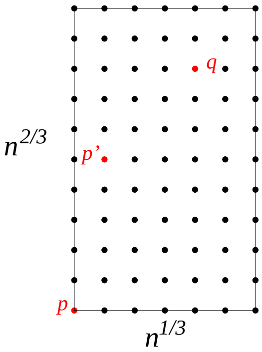

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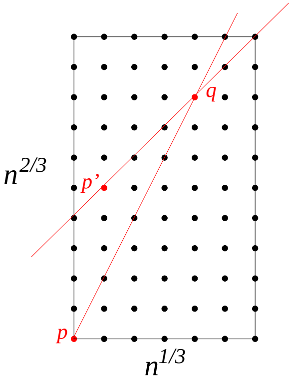

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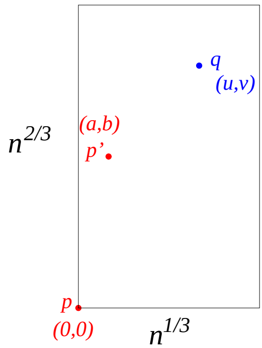
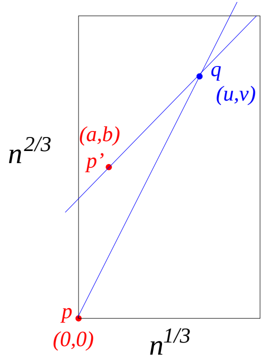


Fix $p = (0, 0)$ and $p' = (a, b)$.

How many choices do we have for $q = (u, v)$?

Key Lemma

Proof.



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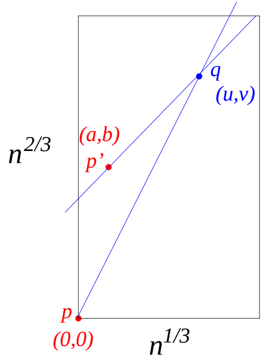
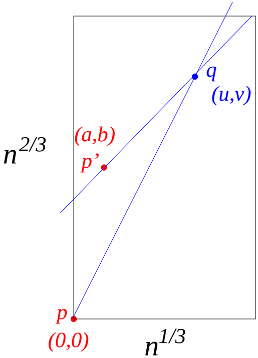


Fix $p = (0,0)$ and $p' = (a,b)$.

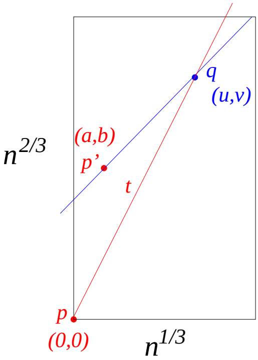
How many choices do we have for $q = (u,v)$? **Claim.** $n^{1/3+o(1)}$.

Proof.



We will determine q by slope $t = \frac{u}{v}$ and x-coordinate u.

Proof.

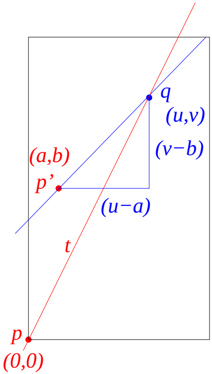


Fix slope $t \in [\frac{n^{1/3}}{2}]$. How many choices for u? **Claim.** $n^{o(1)}$.

Key Lemma

$L : y = mx + b,\ m \in [\frac{n^{1/3}}{2}]$ and $b \in [\frac{n^{2/3}}{2}]$

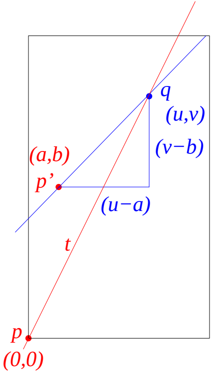
Fix a, b, t.



(# of choices for u) $=$ (# of choices for $(u - a)$).

Key Lemma

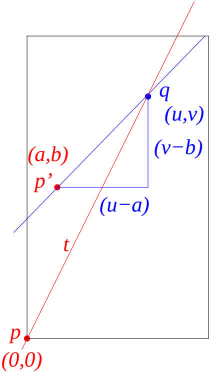
Fix a, b, t. $t = \frac{v}{u}$



$$v - b = t(u - a) + (ta - b).$$

Key Lemma

Fix a, b, t. $t = \frac{v}{u}$

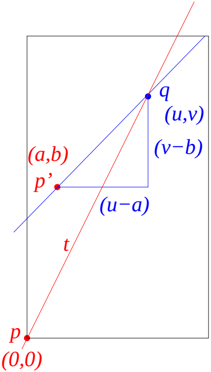


$$v - b = t(u - a) + (ta - b).$$

$$\frac{v - b}{u - a} = t + \frac{ta - b}{u - a}.$$

Key Lemma

Fix a, b, t. $t = \frac{v}{u}$



$$\frac{v-b}{u-a} = t + \frac{ta-b}{u-a}.$$

$(\#$ of choices for $(u-a)) \leq (\#$ of divisors of $(ta-b))$

Key Lemma

(# of choices for $(u-a)$) $\leq$ (# of divisors of $(ta-b)$)

Lemma

For $N > 1$, the number of distinct divisors of N is $N^{\Theta(\frac{1}{\log\log N})} = N^{o(1)}$.

$ta - b \leq n^{2/3}$

(# of divisors of $(ta-b)$) $\leq n^{o(1)}$.

$\square$

Standard Construction

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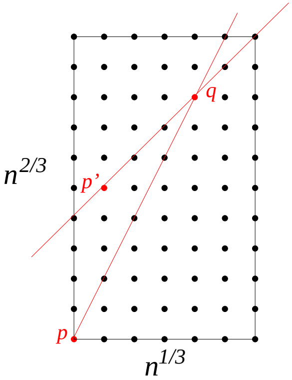
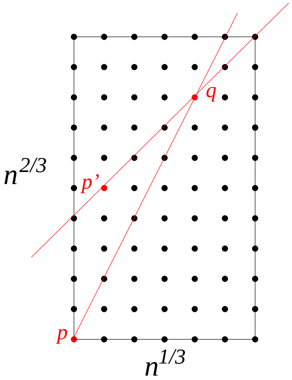
$L : y = mx + b,\ m \in [\frac{n^{1/3}}{2}]$ and $b \in [\frac{n^{2/3}}{2}]$



Standard Construction

Lemma

Let $p, p' \in P$ be distinct vertices. The number of "common neighbors" of p and p' is at most $n^{\frac{1}{3}+o(1)}$.



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$$|I(P, L)| = \Theta(n^{4/3})$$

Lemma

There are at most $n^{\frac{2k}{3}+o(1)}$ copies of C_{2k} in the incidence graph.

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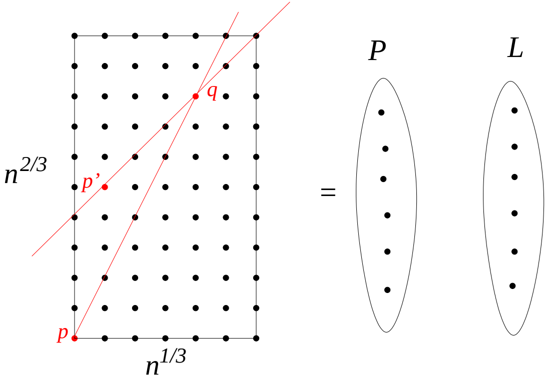

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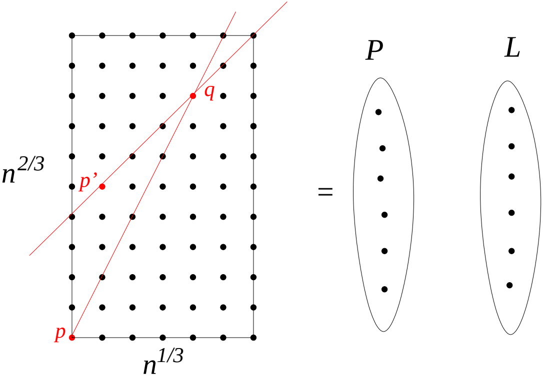

$\# \ C_{2k} \leq$

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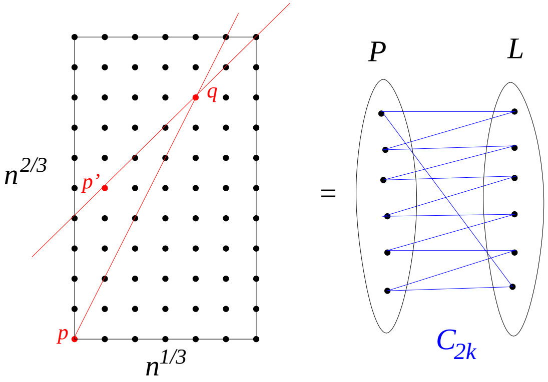

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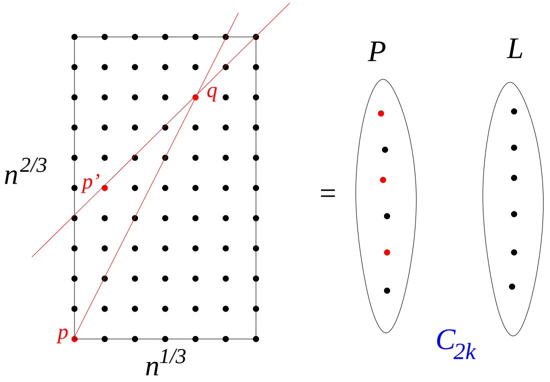

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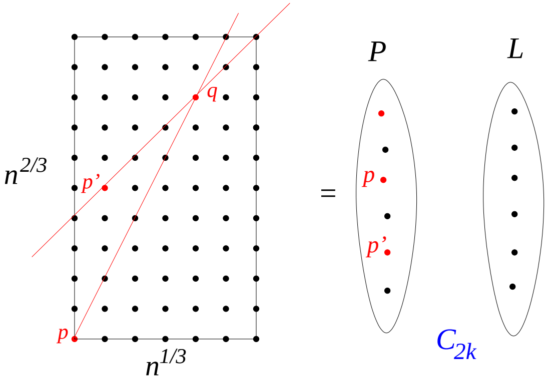

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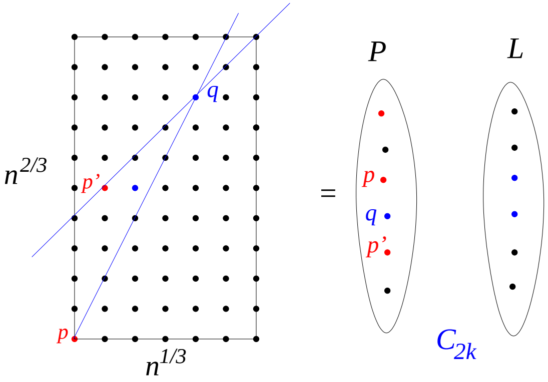

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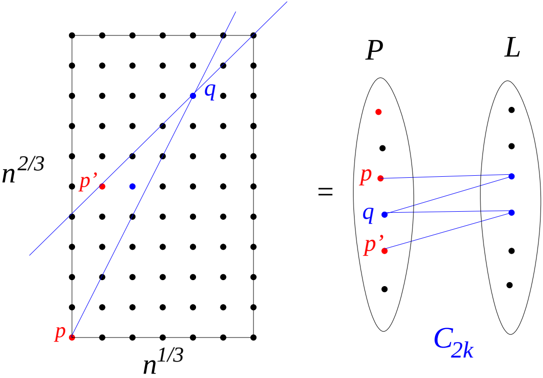
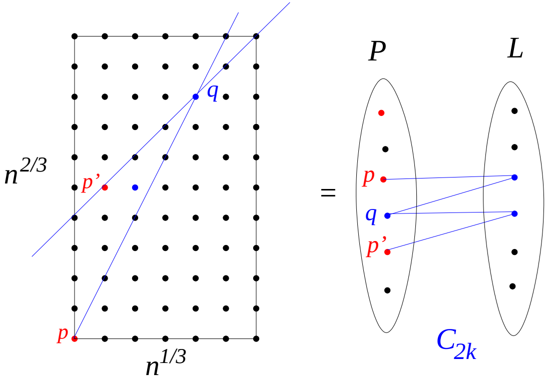


$$\# \; C_{2k} \leq \left(n^{k/2}\right)\left(n^{1/3+o(1)}\right)$$



$$\# \; C_{2k} \leq \left(n^{k/2}\right)\left(n^{1/3+o(1)}\right)^{k/2} = n^{2k/3+o(1)}.$$

Probabilistic argument

$|P| = n$, $|L| = n$, $|I(P,L)| = \Theta(n^{4/3})$, $\#\ C_{2k} \leq n^{2k/3+o(1)}$.

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$|P| = n$, $|L| = n$, $|I(P,L)| = \Theta(n^{4/3})$, $\#\ C_{2k} \leq n^{2k/3+o(1)}$.

Pick each point and line with probability $q = n^{\frac{-2k+3}{6k-3} - o(1)}$

$$\mathbb{E}[\#\text{ points/lines}] = nq = N$$

$$\mathbb{E}[\#\ C_{2k}] = n^{2k/3+o(1)}q^{2k} \leq N/8.$$

Delete 1 point from each cycle.

$$\mathbb{E}[\#\text{ incidences}] = \left(\frac{qn^{1/3}}{2}\right)\left(\frac{qn}{2}\right) = \geq N^{1+\frac{1}{2k}-o(1)}$$

Lower bounds

Theorem (S.-Tomon 2021)

For every positive integer $k \geq 2$, there exists a set of n points and n lines in the plane whose incidence graph is C_{2k}-free, and determines at least $n^{1+\frac{1}{2k}-o(1)}$ incidences.

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Theorem (S.-Tomon 2021)

For $k \geq 3$, there is a set of n points and n lines in the plane that does not contain a $k \times k$ grid, and determines at least $\Omega(n^{\frac{4}{3}-\frac{4}{3(k-2)}})$ incidences.

Application: Hasse diagrams

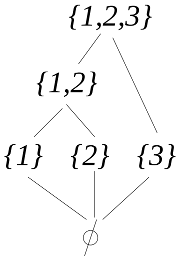
Poset $(P, \prec)$

Definition. Hasse Diagram $G = (V, E)$

$V = P$

$E = (x, y)$ such that $x \prec y$ and $\nexists z \in P$ such that $x \prec z \prec y$.

Example: $P = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,2,3\}\}$, $\prec = \subset$.



Coloring Hasse diagrams

Problem

Given a Hasse diagram G on n vertices, how large can $\chi(G)$ be?

Fact. Hasse diagrams are K_3-free

Theorem (Ajtai-Komlos-Szemeredi 1980)

Every n-vertex Hasse diagram G satisfies $\chi(G) \leq c\sqrt{\frac{n}{\log n}}$

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Theorem (Erdős-Hajnal 1964)

There are Hasse diagrams G on n vertices with $\chi(G) \geq \Omega(\log n)$.

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There are Hasse diagrams G on n vertices with girth k and $\chi(G) \geq \Omega(\frac{\log n}{\log \log n})$.

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Every n-vertex Hasse diagram G satisfies $\chi(G) \leq \log^c n$

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Every n-vertex Hasse diagram G satisfies $\chi(G) \leq n^{o(1)}$

New results

Theorem (S.-Tomon 2021)

There are Hasse diagrams G on n vertices with $\chi(G) \geq \Omega(n^{1/4})$.

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There are Hasse diagrams G on n vertices with girth k and $\chi(G) \geq n^{\frac{1}{2k-4}-o(1)}$.

Ordered graphs

$G =$

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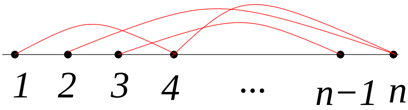

Definition. Ordered cycle of length k, C_k^{ord}

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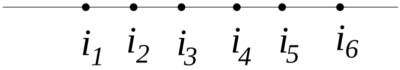

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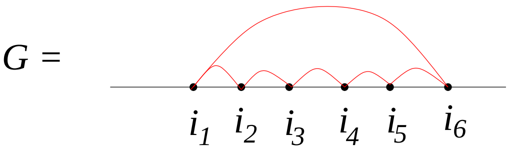

Lemma

G is a Hasse diagram if and only if it can be represented as an ordered graph with no ordered cycle.

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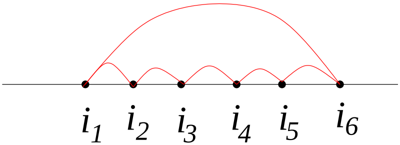

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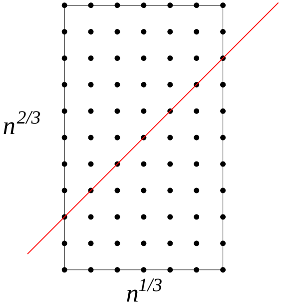
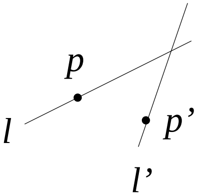
Proof. Standard point-line construction, $|P| = n$, $|L| = n$,

$$|I(P, L)| = \Omega(n^{4/3}) \qquad \text{Rotate.}$$



Define $G = (V, E)$, where $V = I(P, L)$, $|V| = cn^{4/3}$.
$(p, \ell) < (p', \ell')$ if

1. $x(p) < x(p')$
2. or $p = p'$, $s(\ell) < s(\ell')$



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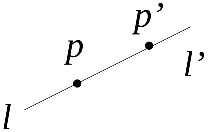

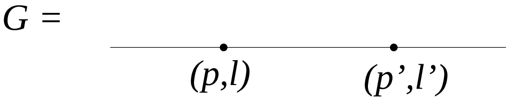

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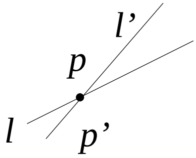
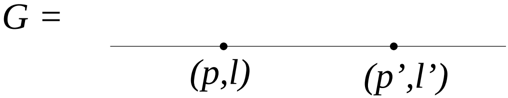
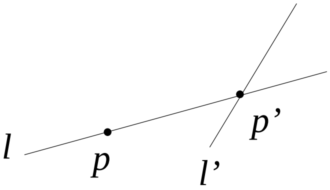




$(p,\ell)(p',\ell')$ is an edge if

$$x(p) < x(p') \qquad \text{and} \qquad s(\ell) < s(\ell')\text{and} \qquad p' \in \ell$$



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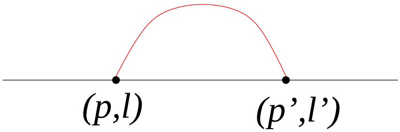
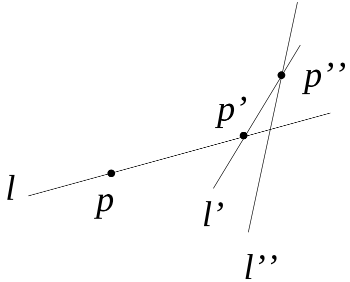
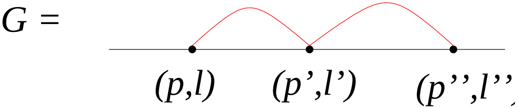


G is an ordered graph with no ordered cycle.





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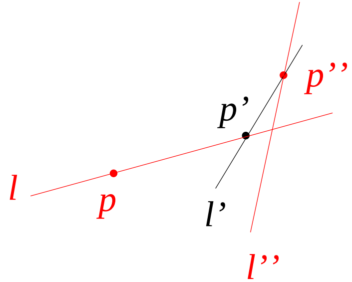

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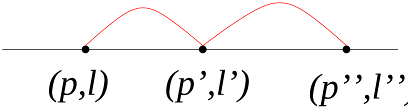
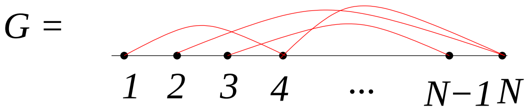


G is an ordered graph with no ordered cycle. $|V(G)| = cn^{4/3} = N$

Claim. $\alpha(G) \leq O(n) = O(N^{3/4})$.



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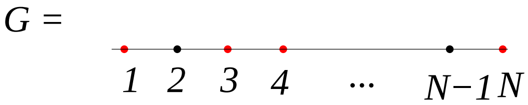

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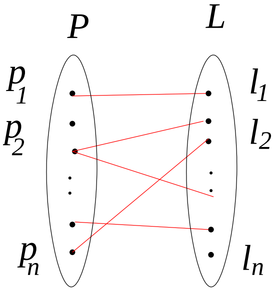

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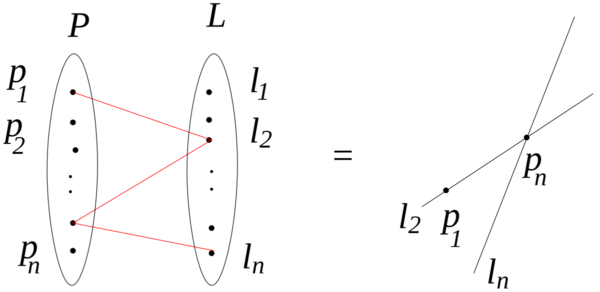

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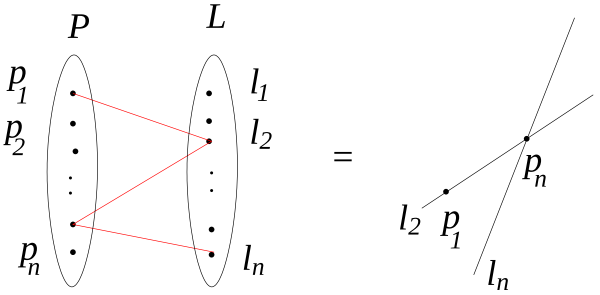

$\square$

Theorem (S.-Tomon 2021)

There are Hasse diagrams G on n vertices with $\chi(G) \geq \Omega(n^{\frac{1}{4}})$.

High girth and high chromatic number

Theorem (S.-Tomon)

For every positive integer $k \geq 2$, there exists a set of n points and n lines in the plane whose incidence graph is C_{2k}-free, and determines at least $n^{1+\frac{1}{2k}-o(1)}$ incidences.

Same argument

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Theorem (S.-Tomon 2021)

There is a set of n curves in the plane, whose disjointness graph G has girth k, $\chi(G) \geq n^{\frac{1}{2k-4}-o(1)}$

Conclusion

Theorem (Erdős, Bondy-Simonovitz)

For fixed $k \geq 2$, $ex(n, C_{2k}) = O(n^{1+\frac{1}{k}})$.

Theorem (Solymosi)

Any set of n lines and n points that does not contain C_6 *in its incidence graph, determines* $o(n^{4/3})$ *incidences.*

Questions

1. Polynomial improvement?
2. Maximum number of incidences C_8-free incidence graphs?
3. Other configurations?

Thank you!