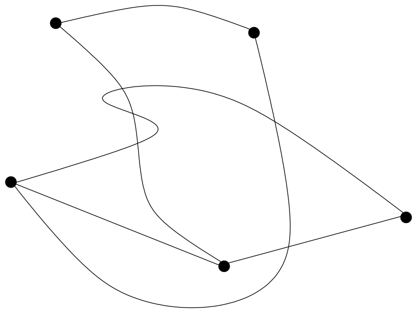
New bounds on the maximum number of edges in k-quasi-planar graphs

Andrew Suk and Bartosz Walczak

September 21, 2013

Definition

A *topological graph* is a graph drawn in the plane with vertices represented by points and edges represented by curves connecting the corresponding points. A topological graph is *simple* if every pair of its edges intersect at most once.



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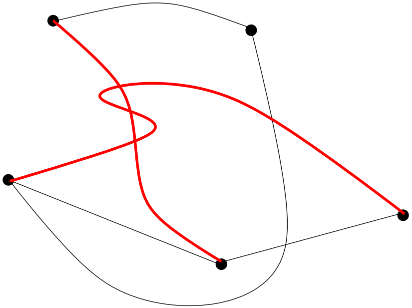
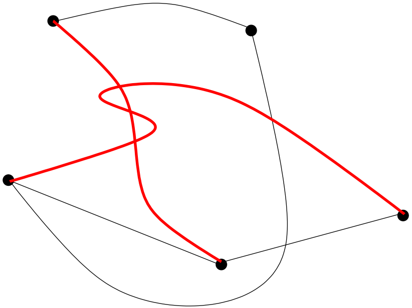


Crossing edges

Two edges in a topological graph **cross** if they have a common interior point.

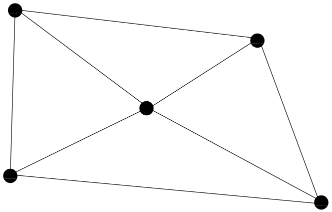


Planar graphs

Application of Euler's Polyhedral formula:

Theorem

Every n-vertex topological graph with no crossing edges contains at most $3n - 6 = O(n)$ *edges.*

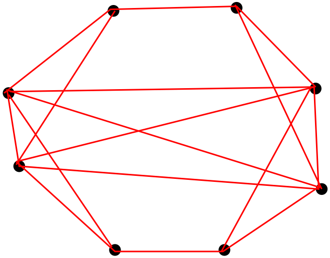


Relaxation of planarity.

Conjecture

Every n-vertex topological graph with no k pairwise crossing edges contains at most $O(n)$ edges.

All such graphs are called *k-quasi-planar*.



Conjecture

Every n-vertex k-quasi-planar graph has at most $O(n)$ edges.

Generated a lot of research, 1990's - present, different variations.

Conjecture has been proven for

1. $k = 3$ by Pach, Radoičić, Tóth 2003, Ackerman and Tardos 2007.
2. $k = 4$ by Ackerman 2008.

Open for $k \geq 5$.

Best known bound for $k \geq 5$

Theorem (Pach, Radoičić, Tóth 2003)

Every n-vertex k-quasi-planar graph has at most $n(\log n)^{4k-12}$ edges.

As an application of a separator Theorem by Matoušek 2013:

Theorem (Fox and Pach 2013)

Every n-vertex k-quasi-planar graph has at most $n(\log n)^{O(\log k)}$ edges.

Best known bound for $k \geq 5$

Theorem (Fox and Pach 2013)

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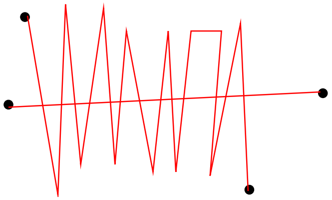

Two edges may cross n^n times.

Best known bound for $k \geq 5$

Theorem (Fox and Pach 2013)

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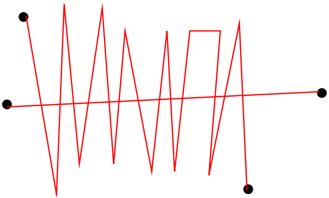


Contribution: We improve this bound in two special cases.

Special Case 1

- G is an n-vertex k-quasi planar graph,
- **extra condition:** every pair of edges have at most t (say 1000) points in common.
- $|E(G)| \leq n(\log n)^{O(\log k)}$, Fox and Pach 2008

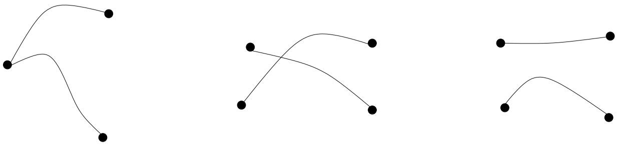
Theorem (Main Result, Suk and Walczak 2013)

Every n-vertex k-quasi-planar graph with no two edges having more than t points in common, has at most $c_{k,t}n(\log n)^{1+\epsilon}$ edges.

For any $\epsilon > 0$.

Special Case 2

G is a **simple** k-quasi-planar graph:



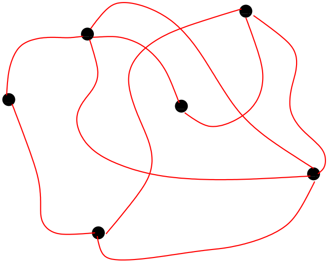
1. $|E(G)| \leq n(\log n)^{O(k)}$, Pach, Shahrokhi, Szegedy 1996.
2. $|E(G)| \leq n(\log n)^{O(\log k)}$, Fox and Pach 2008.
3. $|E(G)| \leq c_k n(\log n)^{1+\epsilon}$, Fox, Pach, Suk 2012.

$|E(G)| \leq c_k n(\log n)^{1+\epsilon}$, Fox, Pach, Suk 2012.
Using new/different methods:

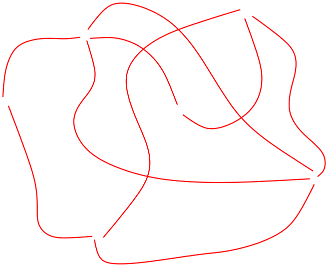
Theorem (Main Result, Suk and Walczak 2013)

Every n-vertex **simple** *k-quasi-planar graph has at most $O(n \log n)$ edges.*

$G = (V, E)$ k-quasi-planar graph.



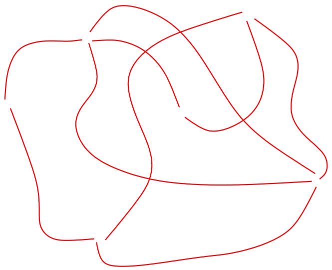
E is a family of $|E(G)|$ curves in the plane, no k pairwise intersecting.



Conjecture (Fox and Pach)

Let F be a family of curves in the plane such that no k members pairwise intersect. Then $\chi(F) \leq c_k$.

Color the curves such that each color class consists of pairwise disjoint curves.



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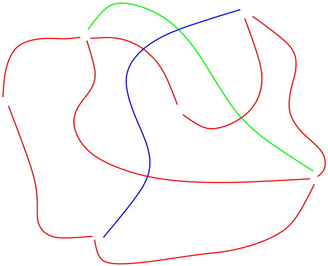

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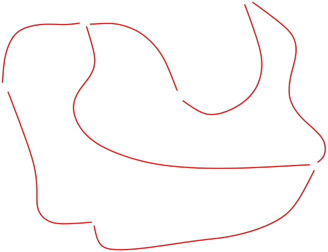

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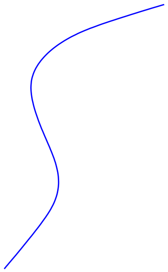

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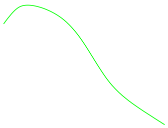

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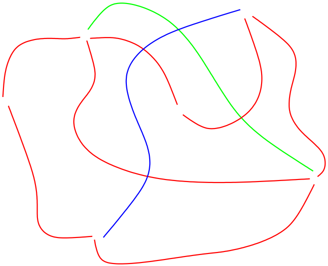
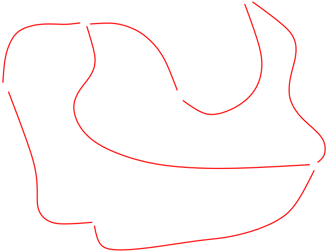


Conjecture (Fox and Pach)

Let F be a family of curves in the plane such that no k members pairwise intersect. Then $\chi(F) \leq c_k$.

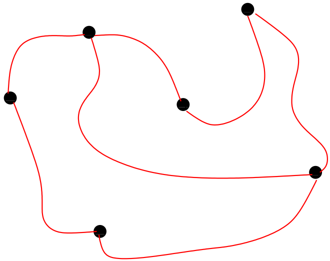
One of the color classes has at least $|E(G)|/c_k$ curves (edges).



Conjecture (Fox and Pach)

Let F be a family of curves in the plane such that no k members pairwise intersect. Then $\chi(F) \leq c_k$.

$$\frac{|E(G)|}{c_k} \leq 3n - 6$$



Conjecture (Fox and Pach, False)

Let F be a family of curves in the plane such that no k members pairwise intersect. Then $\chi(F) \leq c_k$.

Conjecture is False!

Theorem (Pawlik, Kozik, Krawczyk, Lason, Micek, Trotter, Walczak, 2012)

For infinite values n, there exists a family F of n segments in the plane, no three members pairwise cross, and $\chi(F) > \Omega(\log \log n)$.

Conjecture (Fox and Pach, False)

Let F be a family of curves in the plane such that no k members pairwise intersect. Then $\chi(F) \leq c_k$.

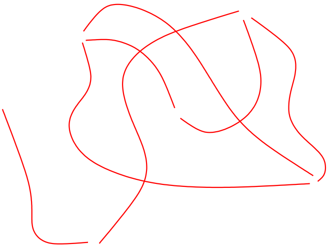
Conjecture true under extra conditions?

Theorem (Suk and Walczak, 2013)

Let F be a family of curves in the plane such that no k members pairwise intersect. Furthermore, suppose

1. *F is* **simple**,
2. *there is a curve β that intersects every member in F exactly once,*

then $\chi(F) \leq c_k$.



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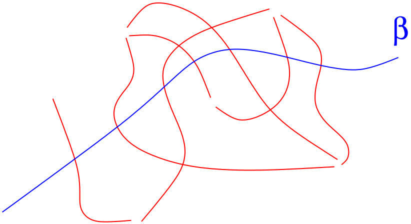

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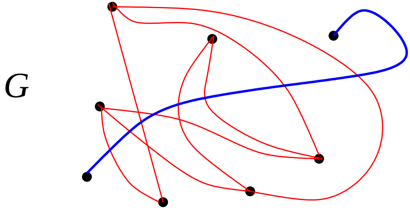
1. Coloring intersection graphs of arcwise connected sets in the plane, Lason, Micek, Pawlik and Walczak 2013.
2. Coloring intersection graphs of x-monotone curves in the plane, Suk 2012.
3. On bounding the chromatic number of L-graphs, McGuinness 1996.

Application of coloring result.

Corollary (Suk and Walczak, 2013)

For fixed $k > 1$, let G be a **simple** *n-vertex k-quasi planar graph. If G contains an edge that crosses every other edge, then $|E(G)| \leq O(n)$.*

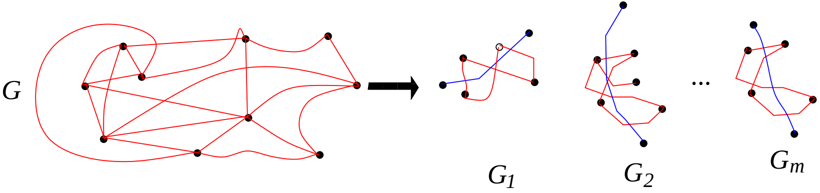


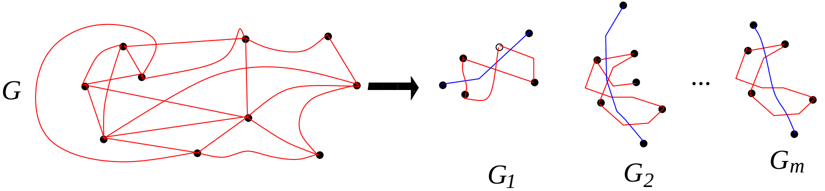
Lemma (Fox, Pach, Suk, 2012)

Let G be a simple topological graph on n vertices. Then there are subgraphs $G_1, G_2, \ldots, G_m \subset G$ such that

$$\frac{|E(G)|}{c \log n} \leq \sum_{i=1}^{m} |E(G_i)|,$$

every edge in G_i is disjoint to every edge in G_j. G_i has an edge that crosses every other edge in G_i.





Let $n_i = |V(G_i)|$.

- $|E(G_i)| \leq c_k n_i$, Suk and Walczak 2013 (main result).

$$\frac{|E(G)|}{c \log n} \leq \sum_{i=1}^{m} |E(G_i)| \leq \sum_{i=1}^{m} c_k n_i = c_k(n_1 + n_2 + \cdots + n_m) = c_k n.$$

□

Theorem (Main Result, Suk and Walczak 2013)

Every n-vertex k-quasi-planar graph with no two edges having more than t points in common, has at most $c_{k,t}n(\log n)^{1+\epsilon}$ edges.

Theorem (Main Result, Suk and Walczak 2013)

Every n-vertex **simple** *k-quasi-planar graph has at most $O(n \log n)$ edges.*

Goal: $|E(G)| \leq O(n)$.

Other problems

Conjecture (Fox and Pach)

Let F be a family of curves in the plane such that no k members pairwise intersect. Then $\chi(F) \leq c_k$.

False.

Conjecture

Let F be a family of curves in the plane such that no k members pairwise intersect. Then F contains $\frac{|F|}{c_k}$ pairwise disjoint members.

Open!

Thank you!