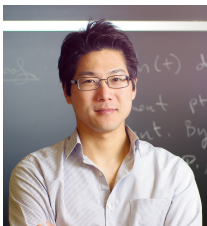


# A note on the no- $(d + 2)$ -on-a-sphere problem

Andrew Suk (UC San Diego)

June 21, 2025

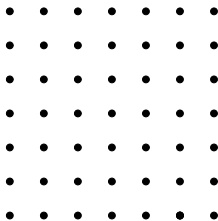
Andrew Suk and Ethan White



# The no-three-in-line problem

## Problem (Dudeney 1917)

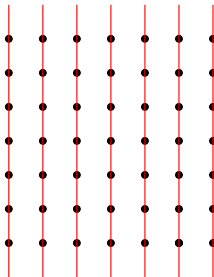
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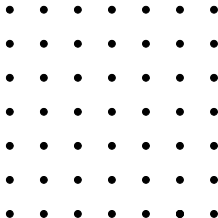
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Problem (Erdős and Purdy 1981)

*What is the largest number of points that can be selected from an  $n \times n$  lattice square with no four on a circle?*

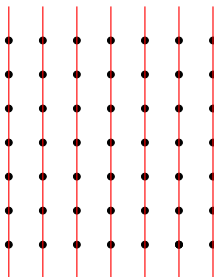


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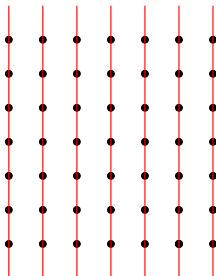
line = degenerate circle. **Trivial upper bound:**  $3n$



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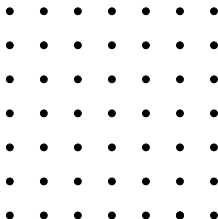
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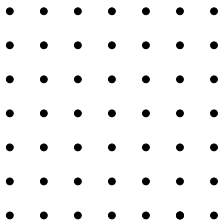
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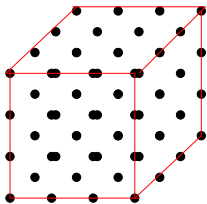
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# Higher dimensions

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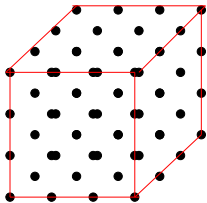
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**Answer.**  $\Theta(n)$ . At most  $dn$  and at least  $n$ .

**Moment modular curve:** For  $n$  prime and  $t = 1, 2, \dots, n$

$$(t, t^2, \dots, t^d) \pmod n$$

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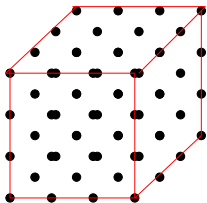
**No  $k + 2$  on a  $k$ -flat.**

- 1 A. Pór, D. Wood: No-three-in-line-in-3D, GD 2004.
- 2 H. Lefmann, Extensions of the No-Three-In-Line problem, 2008.
- 3 B. Sudakov, I. Tomon, Evasive sets, covering by subspaces, 2023.
- 4 A. Suk, J. Zeng, On higher dimensional point sets in general position, SoCG 2023.

# No- $(d + 2)$ -on-a-sphere

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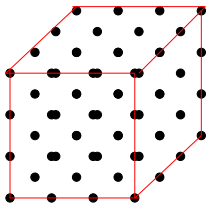
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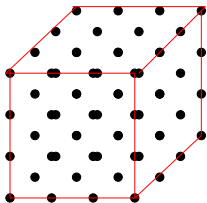


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At least ???

# More than $d + 2$ on a sphere

$$(x_1 - a_1)^2 + \cdots + (x_d - a_d)^2 = b$$

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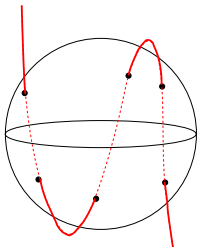
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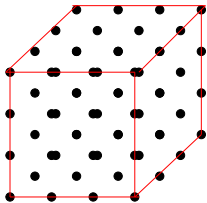


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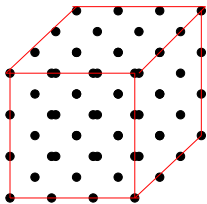
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## Theorem (Thiele 1995)

*One can select  $\Omega(n^{\frac{1}{d-1}})$  points from  $[n]^d$  with no  $d + 2$  on a sphere or hyperplane.*

## Theorem (S.-White 2025)

*One can select  $n^{\frac{3}{d+1}-o(1)}$  points from  $[n]^d$  with no  $d+2$  on a sphere or a hyperplane.*

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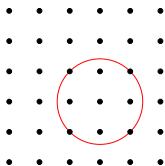
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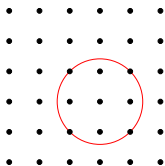
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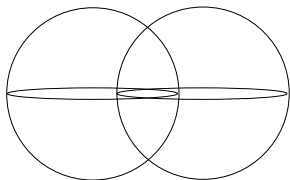
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**Jacobi's two-square theorem:** Number of ways  $n$  can be written as the sum of two squares is  $n^{c/\log \log n}$ .

# Grid points on a sphere

## Observation

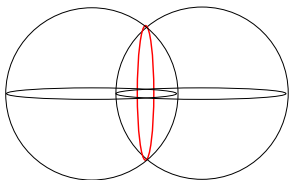
*For  $d \geq 2$ , any sphere in  $\mathbb{R}^d$  contains at most  $n^{d-2+o(1)}$  points from  $[n]^d$ .*



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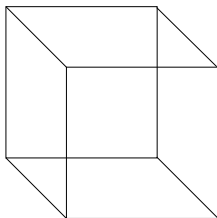
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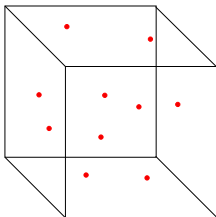
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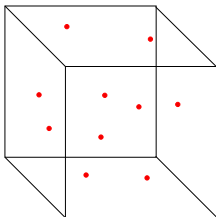


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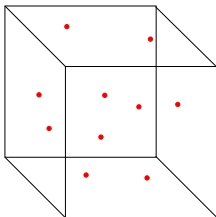
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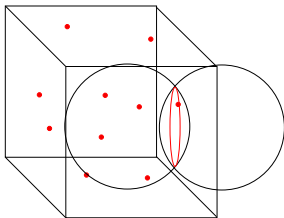
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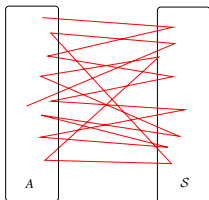


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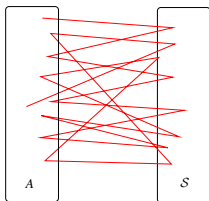


❶ No  $K_{n^{o(1)}, 2}$ .

$$|I(A, \mathcal{S})| \leq n^{o(1)} |\mathcal{S}| |A|^{1/2} + n^{o(1)} |A|.$$

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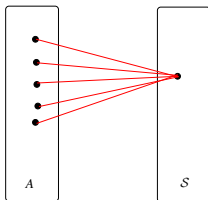


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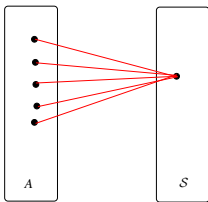


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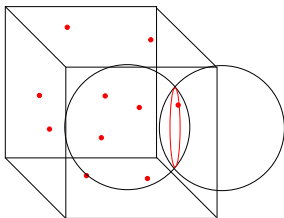
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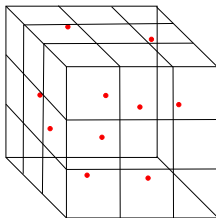
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# Simple "cutting lemma"

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**Partition:**  $D^d$  parts, each part has  $(n/D)^d$  points from  $[n]^d$

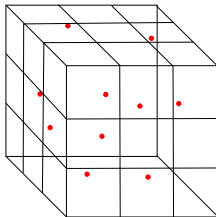


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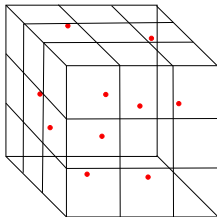
①  $|A_i| = \Theta(n^3/D^d)$ .

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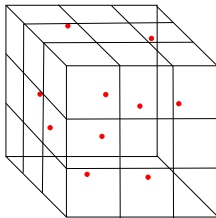
**Partition:**  $D^d$  parts, each part has  $(n/D)^d$  points from  $[n]^d$



- 1  $|A_i| = \Theta(n^3/D^d)$ .
- 2 Any sphere  $S \in \mathcal{S}$  will enter at most  $D^{d-1}$  cubes.

# Simple "cutting lemma"

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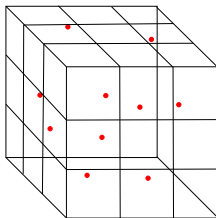


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$$|I(A, \mathcal{S})| \leq n^{o(1)} \sum_{i=1}^{D^d} |A_i| |\mathcal{S}_i|^{\frac{d}{d+1}} + |\mathcal{S}_i|$$

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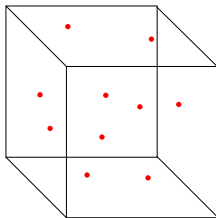
- ①  $|A_i| = \Theta(n^3/D^d)$ .
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$$|I(A, \mathcal{S})| \leq n^{o(1)} \sum_{i=1}^{D^d} |A_i| |\mathcal{S}_i|^{\frac{d}{d+1}} + |\mathcal{S}| \leq n^{\frac{3(d^2-1)}{d^2+d-1} + o(1)} |\mathcal{S}|^{\frac{d^2}{d^2+d-1}}$$

# Spheres with at least $r$ points from $A$

$A \subset [n]^d$ ,  $|A| = \Theta(n^3)$ , such that

$\mathcal{S}_r = r$ -rich spheres in  $\mathbb{R}^d$

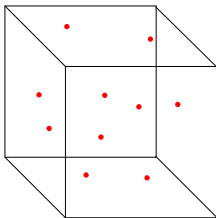


$$r|\mathcal{S}_r| \leq |I(A, \mathcal{S}_r)| \leq n^{\frac{3(d^2-1)}{d^2+d-1} + o(1)} |\mathcal{S}_r|^{\frac{d^2}{d^2+d-1}}$$

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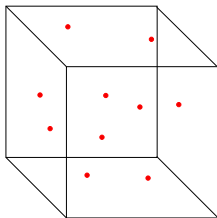
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$$|\mathcal{S}_r| \leq \frac{n^{3(d+1)+o(1)}}{r^{\frac{d^2+d-1}{d-1}}}$$

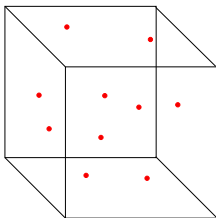
# $(d + 2)$ -tuples of $A$ on a sphere



$$\sum_{i=1}^{\log |A|} |\mathcal{S}_{2^i, 2^{i+1}}| \binom{2^{i+1}}{d+2} \leq n^{3(d+1)+o(1)}$$

- 1  $|A| = \Theta(n^3)$
- 2 At most  $O(n^{2d+5})$   $(d + 2)$ -tuples of  $A$  lie on a hyperplane.

# Deletion method



Random subset  $A' \subset A$

- 1  $|A'| = n^{\frac{3}{d+1}} + o(1)$
- 2 No  $d + 2$  on a hyperplane or a sphere.

# Open problems

## Theorem (S.-White 2025)

*One can select  $n^{\frac{3}{d+1}-o(1)}$  points from  $[n]^d$  with no  $d+2$  on a sphere or a hyperplane.*

## Conjecture (S.-White 2025)

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## Problem

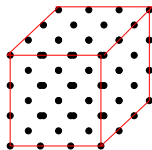
*Can we improve the trivial upper bound  $(d+1)n$ .*

# Open problems

No  $(d + 2)$  points on a sphere, but many may lie on a hyperplane.

**Upper bound:**  $d(d + 1)n^2$ .

$$x_1^2 + \cdots + x_d^2 = r^2$$

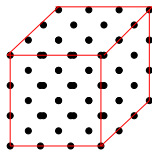


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No  $(d + 2)$  points on a sphere, but many may lie on a hyperplane.

**Upper bound:**  $d(d + 1)n^2$ .

$$x_1^2 + \cdots + x_d^2 = r^2$$



**Lower bound:** At least  $n^{2 - \frac{4}{d+1} - o(1)}$  from  $[n]^2 \subset [n]^d$ .

**Thank you!**