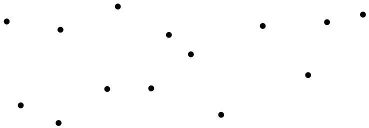
Erdős-Szekeres-type theorems for monotone paths and convex bodies

Jacob Fox, János Pach, Benny Sudakov, Andrew Suk

November 24, 2010

History



Theorem

(Erdős-Szekeres 1935) For any positive integer n, there exists an integer ES(n), such that any set of at least ES(n) points in the plane such that no three are collinear contains n members in convex position. Moreover

$$2^{n-2} + 1 \leq ES(n) \leq \binom{2n-4}{n-2} + 1 = O(4^n/\sqrt{n}).$$

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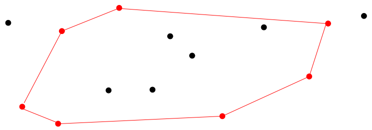

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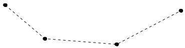

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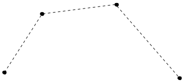


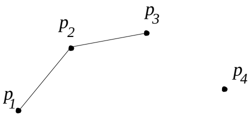
Theorem

(Erdős-Szekeres 1935) For any positive integers k and l, there exists an integer $f(k,l)$, such that any set of at least $f(k,l)$ points in the plane such that no three are collinear contains either a k-cup or an l-cap. Moreover

$$f(k,l) = \binom{k+l-4}{k-2} + 1$$

Proof is very combinatorial. The only geometric fact used was the following: Order the points from left to right $\{p_1, ..., p_N\}$

transitive property: If (p_1, p_2, p_3) is a cap (cup), and (p_2, p_3, p_4) is a cap (cup), then p_1, p_2, p_3, p_4 is a 4-cap (4-cup).



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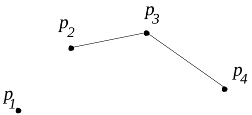

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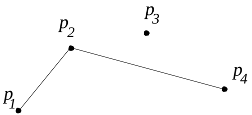

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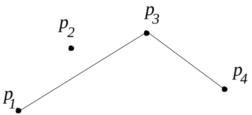

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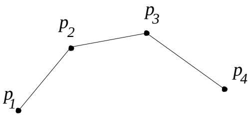

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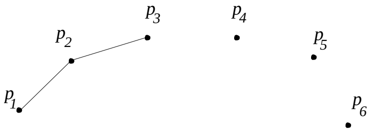

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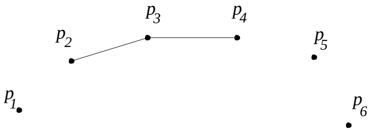

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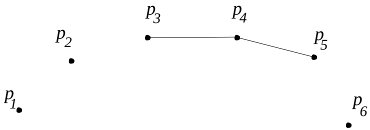

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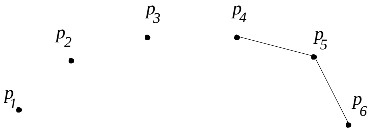

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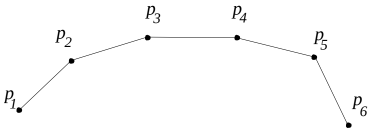
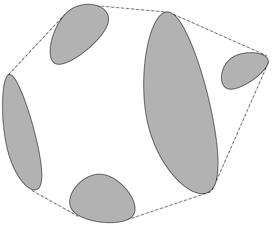
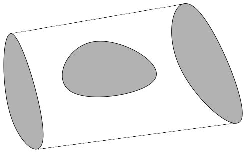


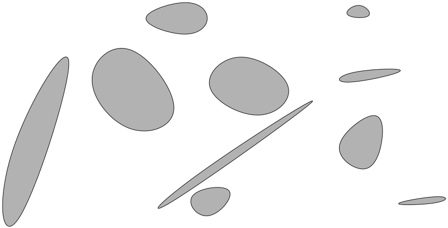
Generalizing to convex bodies

Definition

A family $\mathcal{C}$ of convex bodies (compact convex sets) in the plane is said to be in *convex position* if none of its members is contained in the convex hull of the union of the others. We say that $\mathcal{C}$ is in *general position* if every three members are in convex position.





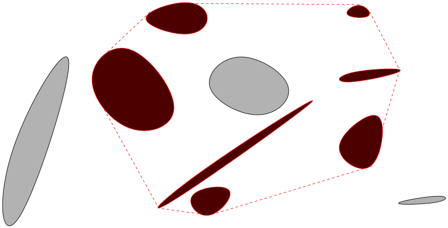


Theorem

(Bisztriczky and Fejes Tóth 1989) For any positive integer n, there exists an integer $D(n)$, such that every family of at least $D(n)$ disjoint convex bodies in the plane in general position contains n members in convex position. Moreover

$$2^{n-2}+1 \leq D(n) \leq 2^{2^{2^n}}.$$

They conjectured $D(n) = ES(n)$.



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$D(n)$ was later improved:

Theorem

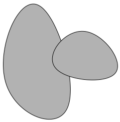
(Pach and Tóth 1998) $D(n) \leq \binom{2n-4}{n-2}^2 + 1 = O(16^n)$

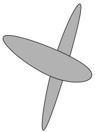
Theorem

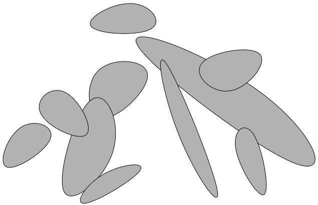
(Hubard, Montejano, Mora, S. 2010)
$D(n) \leq (\binom{2n-5}{n-2} + 1)\binom{2n-4}{n-2} + 1$

Definition

We say that a family of convex bodies in the plane is *noncrossing* if any two members share at most two boundary points.



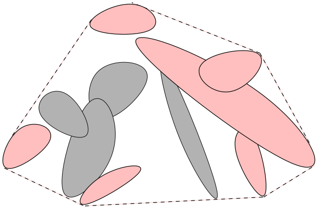




Theorem

(Pach and Tóth 2000) For any positive integer n, there exists an integer $N(n)$, such that any family of at least $N(n)$ noncrossing convex bodies in the plane in general position contains n members in convex position. Moreover

$$2^{n-2}+1 \leq N(n) \leq 2^{2^{2^n}}.$$



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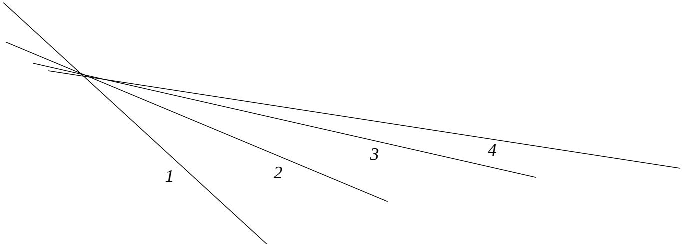
Note: We cannot drop the noncrossing assumption. Pach and Tóth gave a construction of n pairwise crossing rectangles that which are in general position, but no four of them are in convex position



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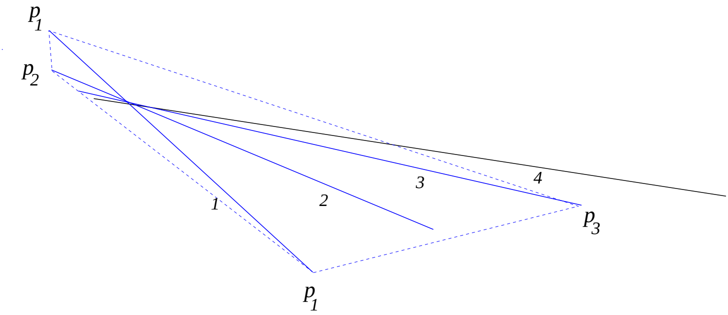

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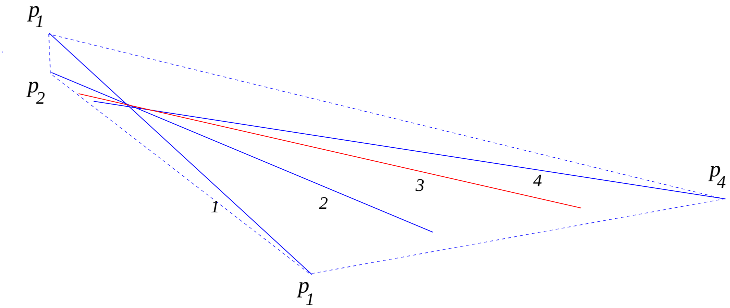

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Theorem

(Hubard, Montejano, Mora, S. 2010)

$$N(n) \leq 2^{2^n}$$

Proof introduces order types for convex bodies. Our result:

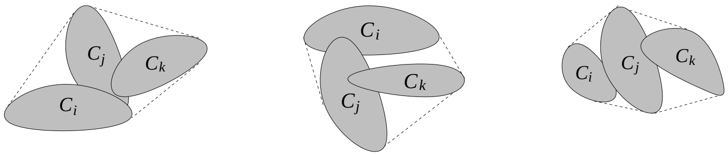
Theorem

(Fox, Pach, Sudakov, S.)

$$2^{n-2} + 1 \leq N(n) \leq n^{n^2} = 2^{cn^2 \log n}.$$

Order types of convex bodies

Given an ordering on $\mathcal{C}$, (C_i, C_j, C_k) $(i < j < k)$ has a *clockwise (counterclockwise) orientation* if there exist distinct points $p_i \in C_i, p_j \in C_j, p_k \in C_k$ such that they lie on the boundary of $conv(C_i \cup C_j \cup C_k)$ and appear there in clockwise (counterclockwise) order. For $i < j < k$



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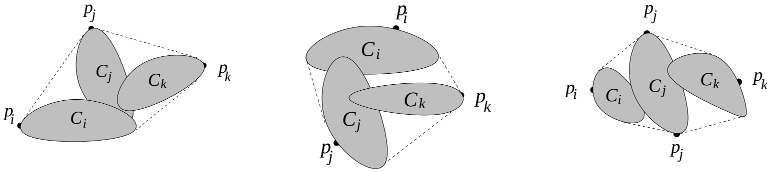
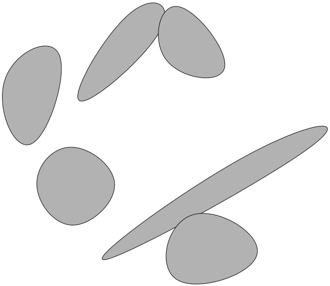
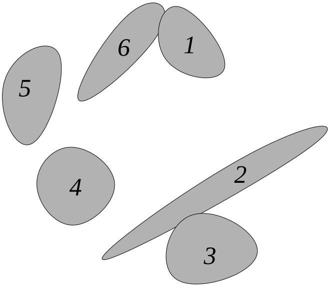




Theorem

(Hubard, Montejano, Mora, S. 2010) A family $\mathcal{C}$ of noncrossing convex sets is in covex position if and only if there exists an ordering on the member of $\mathcal{C}$ such that every triple has a clockwise orientation.

Which implies $N(n) \leq 2^{2^{cn}}$ by Ramsey Theory.



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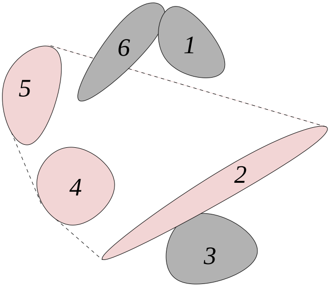

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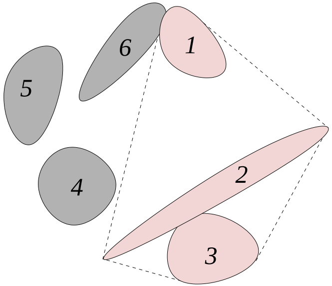

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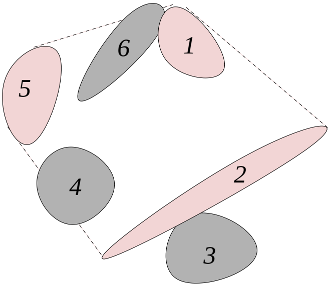

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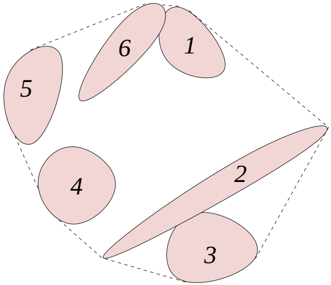

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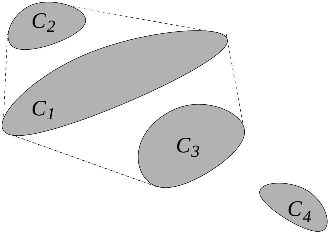

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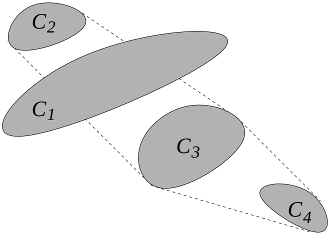

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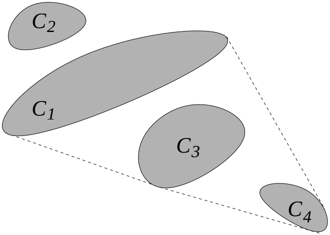
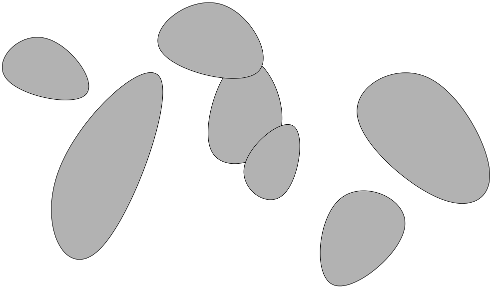


Strong orientations

Order the member of $\mathcal{C}$ from "left to right" according to their "left endpoint".



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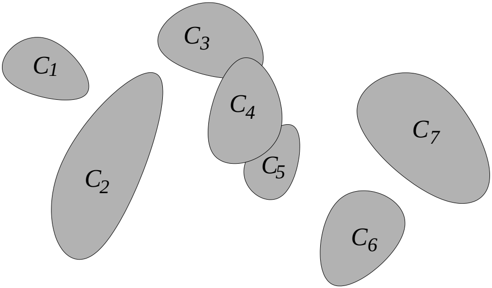
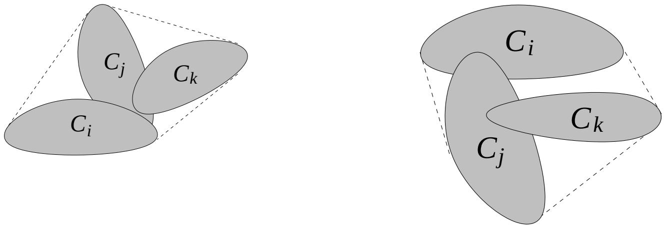


Given the ordering as above, for $i < j < k$, (C_i, C_j, C_k) is said to have a *strong-clockwise (strong-counterclockwise) orientation* if there exist points $p_j \in C_j, p_k \in C_k$ such that, starting at the left endpoint p_i^* of C_i, the triple (p_i^*, p_j, p_k) appears in clockwise (counterclockwise) order along the boundary of $conv(C_i \cup C_j \cup C_k)$.



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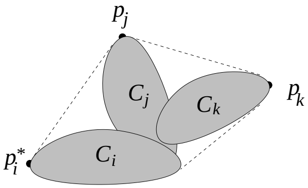
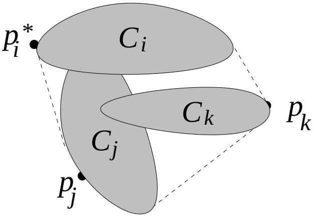
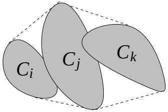




(C_i, C_j, C_k) has *both strong orientations* if it has both a strong-clockwise and a strong-counterclockwise orientation.



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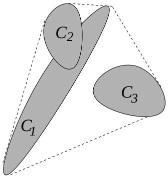

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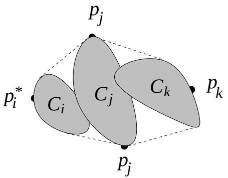

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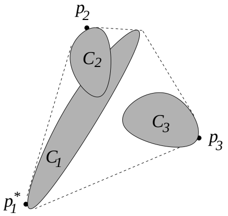
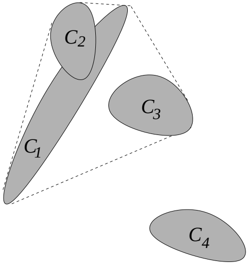


Theorem

(Fox, Pach, Sudakov, S.) **transitive property:** *If* (C_1, C_2, C_3) *and* (C_2, C_3, C_4) *have only a strong clockwise (strongcounter clockwise) orientation, then* (C_1, C_3, C_4) *and* (C_1, C_2, C_4) *must also only have strong clockwise (counter clockwise) orientations.*



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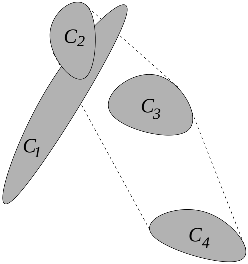

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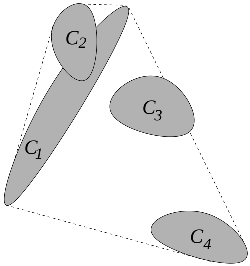

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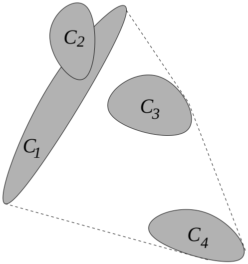

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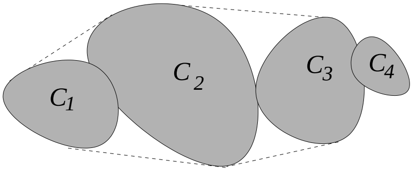

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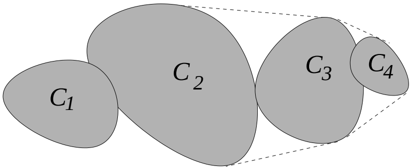

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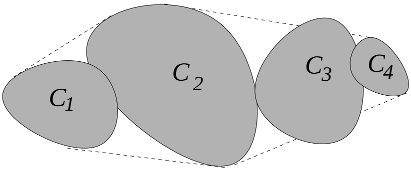

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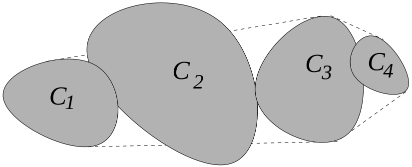

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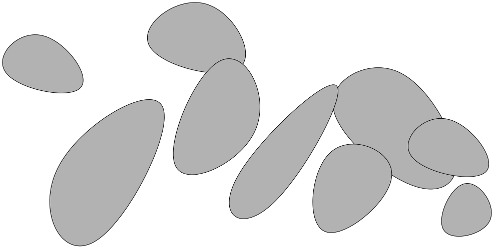

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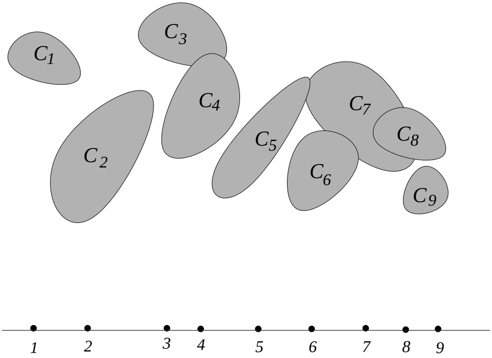

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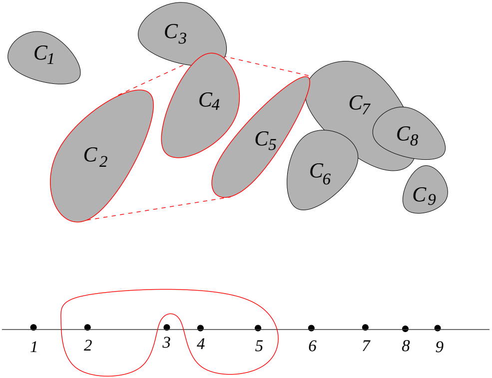

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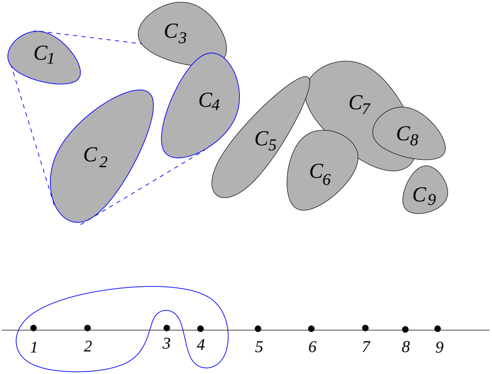

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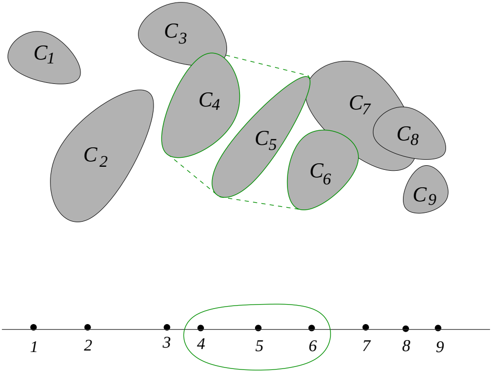
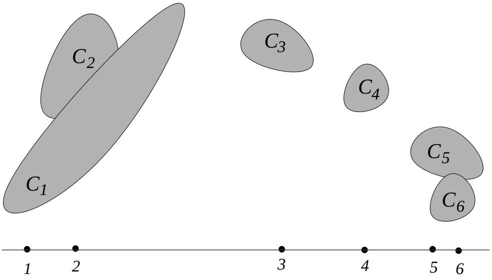


Find bodies in convex position by looking for a path and applying the transitive property.



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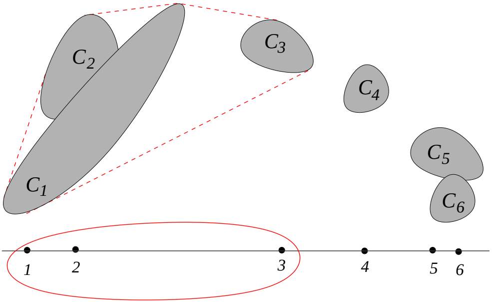

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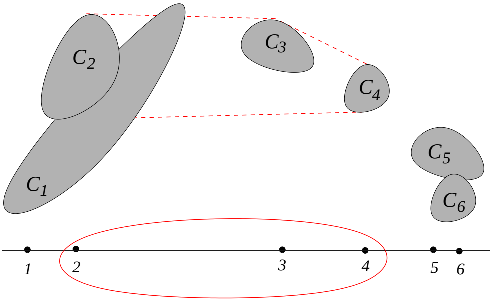

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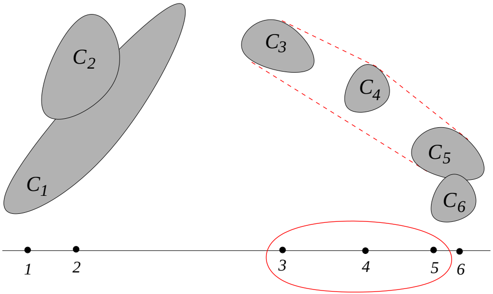

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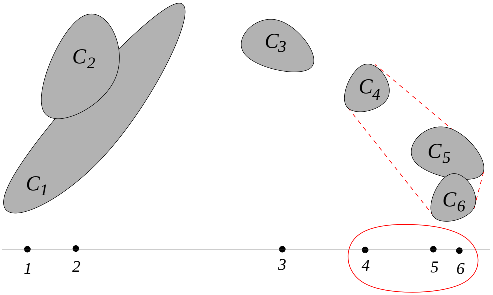

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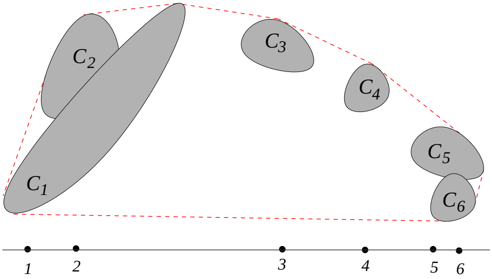
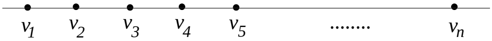


By the transitive property, every triple has a strong clockwise orientation. Hence by the previous theorem, $C_1, \ldots, C_6$ is in convex position.

Finding monochromatic paths in ordered hypergraphs

For an ordered 3-uniform hypergraph $H = ([N], E)$, a monotone 3-path of length n are edges $(v_1, v_2, v_3), (v_2, v_3, v_4), (v_3, v_4, v_5), \ldots, (v_{n-2}, v_{n-1}, v_n)$.



In general, for an ordered k-uniform hypergraph $H = ([N], E)$, a monotone k-path of length n are edges $(v_1, v_2, \ldots, v_k), (v_2, v_3, \ldots, v_{k+1}) \ldots, (v_{n-k+1}, \ldots, v_n)$.

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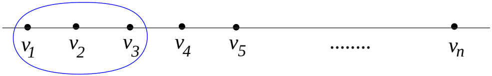

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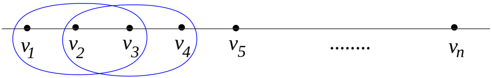

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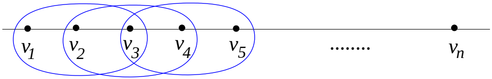

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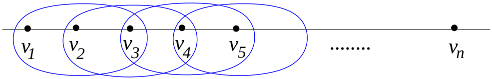

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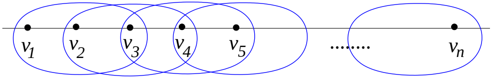

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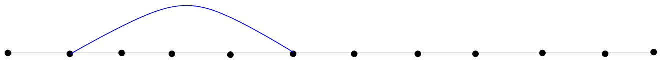
Ordered hypergraphs

Definition

Let $N_k(q, n)$ denote the smallest integer N such that every q coloring on the k-tuples of $[N]$ contains a monochromatic path of length n.

$N_2(q, n) = (n-1)^q + 1$ by Dilworth's theorem.



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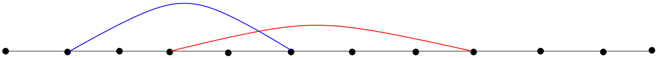

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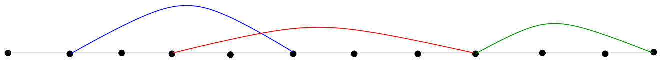

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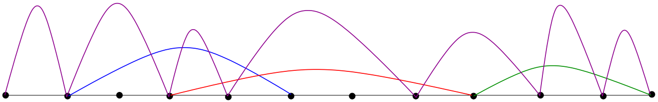

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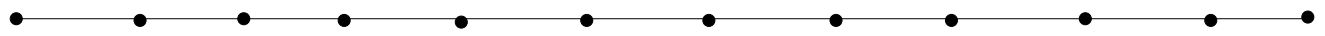

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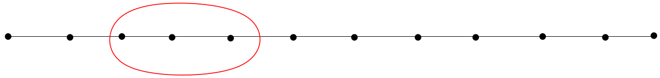

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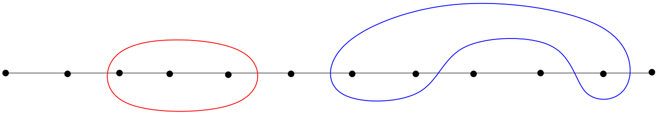

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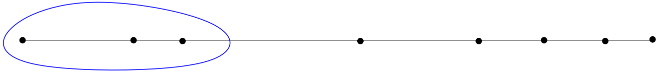

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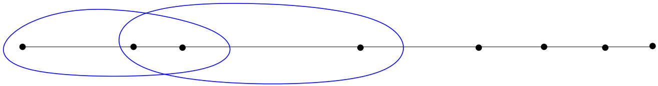

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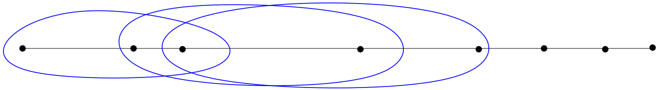

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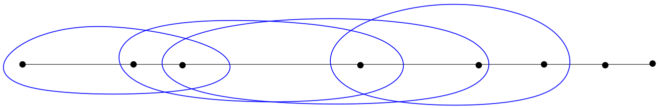

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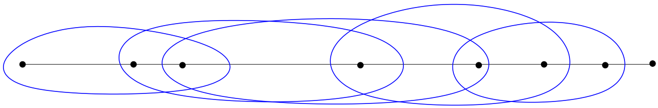

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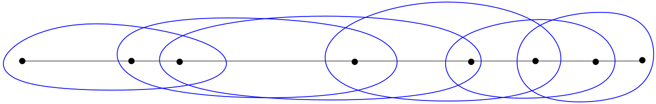


For more colors.

Theorem

(Fox, Pach, Sudakov, S.) For $q \geq 3$, we have

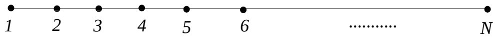
$$2^{(n/q)^{q-1}} \leq N_3(q,n) \leq n^{n^{q-1}},$$

Therefore for the noncrossing convex bodies problem:

$$N(n) \leq N_3(3,n) \leq n^{n^2} = 2^{cn^2 \log n}$$

Proof of $N_3(q,n) \leq N_2(n^{q-1},n) \leq n^{n^{q-1}}$:

1. Set $N = N_2(n^{q-1},n)$
2. $\chi : \binom{[N]}{3} \to [q]$ be q-coloring on the triples of $[N]$.
3. Then define $\phi : \binom{[N]}{2} \to [n]^{q-1}$ as follows. We color $(i,j) \in \binom{[N]}{2}$ with color $(a_1, a_2, ..., a_{q-1})$ where a_t denotes the length of the longest t-colored 3-path ending with vertices i,j.



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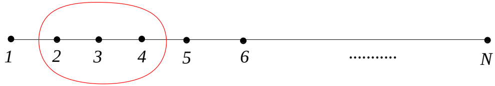

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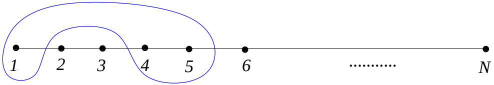

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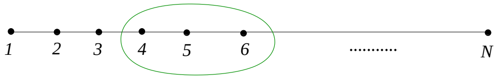

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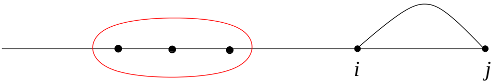

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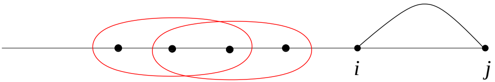

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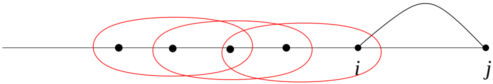

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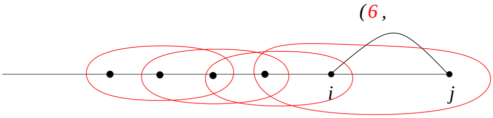

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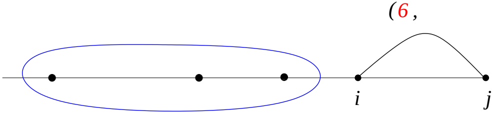

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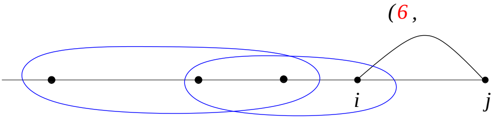

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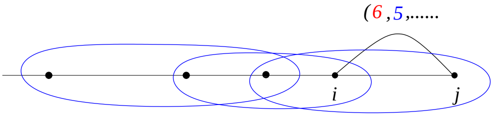
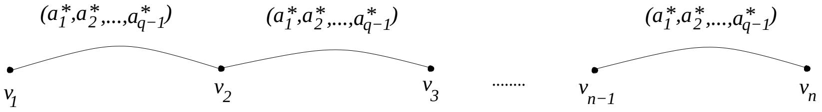
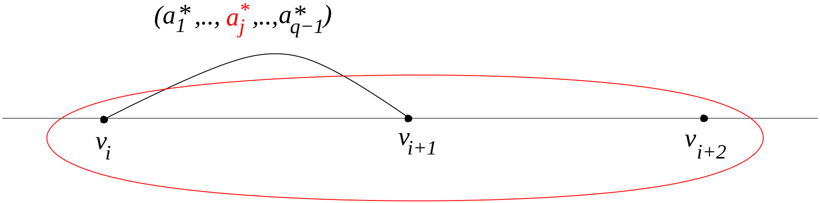


By definition of $N_2(n^{q-1}, n)$, there is monochromatic 2-path on vertices $v_1 < v_2 < ... < v_n$ with color $(a_1^*, ..., a_{q-1}^*)$.



Claim: $(v_1, \ldots, v_n)$ is a monochromatic 3-path (with color q)!
Indeed, Assume (v_i, v_{i+1}, v_{i+2}) has color $j \neq q$.

1. Longest jth-colored 3-path ending with vertices (v_i, v_j) must be shorter than the longest jth-colored 3-path ending with vertices (v_{j+1}, v_{j+2}).
2. Contradicts $\phi(v_i, v_{i+1}) = \phi(v_{i+1}, v_{i+2})$.
3. Hence (v_i, v_{i+1}, v_{i+2}) must have color q for all i.



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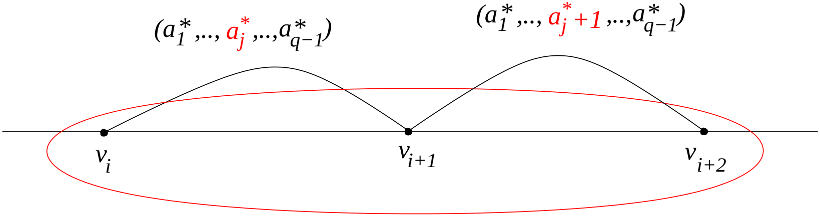


The upper bound proof can easily be generalized to show

$$N_k(q,n) \le N_{k-1}((n-k+1)^{q-1}, n)$$

Using the stepping-up approach we have

Theorem

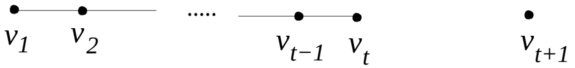
(Fox, Pach, Sudakov, S.) Define $t_1(x) = x$ *and* $t_{i+1}(x) = 2^{t_i(x)}$. *Then for* $k \ge 4$ *we have*

$$t_{k-1}(cn^{q-1}) \le N_k(q,n) \le t_{k-1}(c' n^{q-1} \log n).$$

Another upper bound on $N_3(q, n)$

Consider the following game played by two players, **Builder** and **Painter**.

1. vertex v_{t+1} is revealed.
2. Builder decides whether to draw the edge (v_i, v_{t+1}) for $i \leq t$. item If Builder draws an edge, Painter must immediately color it one of q colors.



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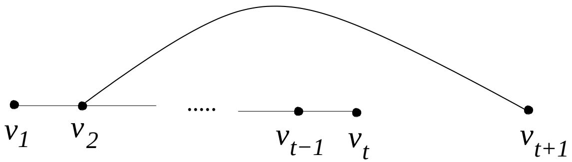

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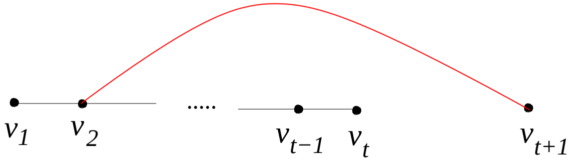

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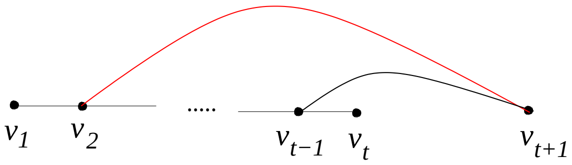

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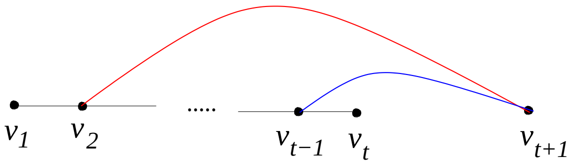

The vertex online Ramsey number $V_2(q,n)$ is the minimum number of edges builder has to draw to guarantee a monochromatic path of length n. Clearly $V_2(q,n) \leq \binom{(n-1)^q+1}{2}$.

Theorem

(Fox, Pach, Sudakov, S.) We have

$$V_2(q,n) \leq q^2 n^q \log n$$

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$$N_3(q,n) \leq q^{V_2(q,n)} + 1 = q^{q^2 n^q \log n}$$

For $q = 3$, the formula above implies $N_3(3,n) \leq 2^{cn^3 \log n}$ (Not as strong as the previous bound $2^{c' n^2 \log n}$). A weaker upper bound, but gives us an algorithm of finding a monochromatic 3-path of length n.

Summary

1. Points (Tóth and Valtr 2005):

$$2^{n-1}+1 \leq ES(n) \leq \binom{2n-5}{n-2}+1 = O(4^n/\sqrt{n}).$$

2. Disjoint convex bodies (Hubard, Montejano, Mora, S. 2010):

$$2^{n-1}+1 \leq D(n) \leq (\binom{2n-5}{n-2}+1)\binom{2n-4}{n-2}+1 = O(16^n).$$

3. Noncrossing convex bodies (Fox, Pach, Sudakov, S.):

$$2^{n-1}+1 \leq N(n) \leq n^{n^2} = 2^{O(n^2 \log n)}.$$

Problem

$ES(n) = D(n) = N(n)$?

Thank you!