

Immersions and Albertson's conjecture

Andrew Suk (UC San Diego)

June 21, 2025

Jacob Fox and János Pach



Conjecture (Hadwiger)

If G is a graph with $\chi(G) = k$, then G contains a K_k -minor.

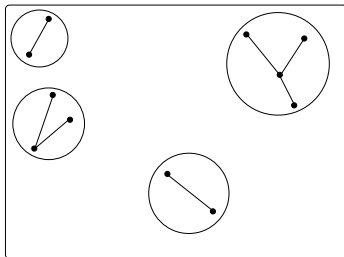
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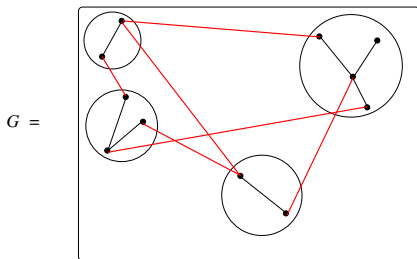
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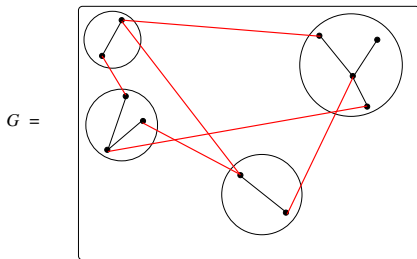
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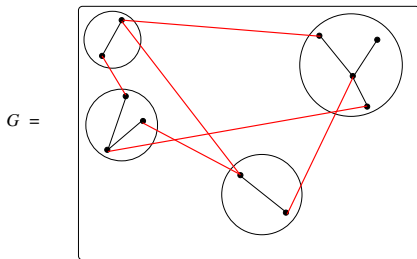
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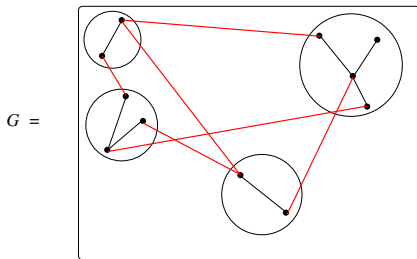
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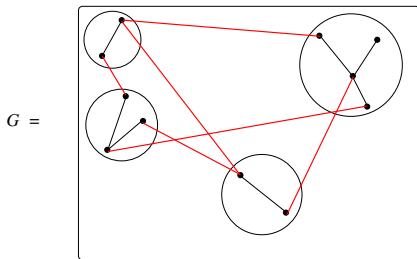
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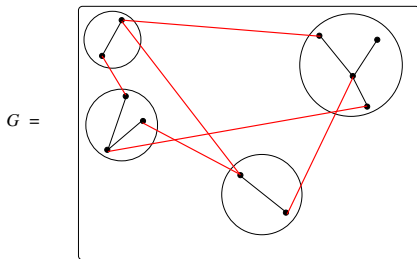
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- **Delcourt-Postle 2021:** $k \geq 7$, K_t -minor with $t = \Omega(\frac{k}{\log \log k})$.

A stronger conjecture: topological minor

Conjecture (Hajós, 1961)

If G is a graph with $\chi(G) = k$, then G contains a subdivision of K_k

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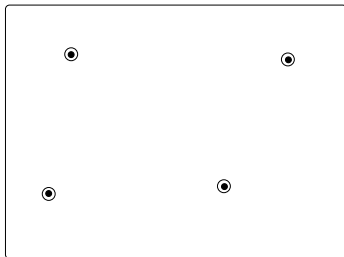


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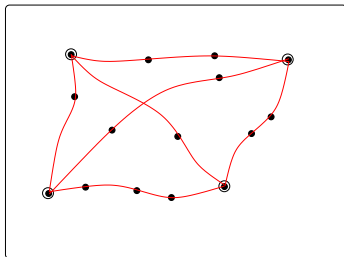


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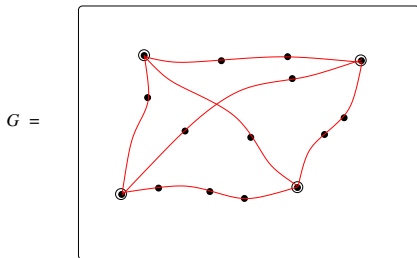
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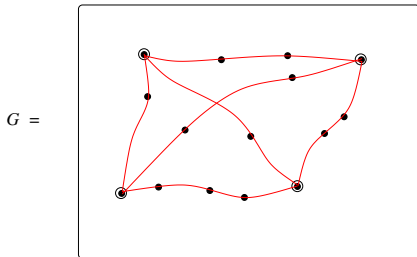


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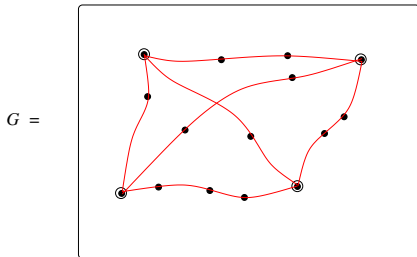


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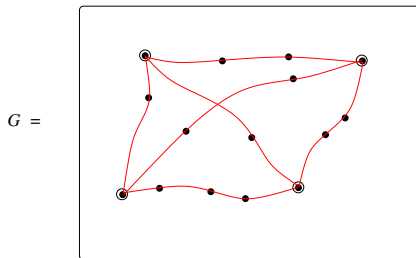


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- $cr(G) \geq c \frac{|E|^3}{|V|^2}$
- $cr(K_k) \leq \frac{1}{4} \left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{k-1}{2} \right\rfloor \left\lfloor \frac{k-2}{2} \right\rfloor \left\lfloor \frac{k-3}{2} \right\rfloor$

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If G is a graph on n vertices and is k -critical, then $cr(G) \geq cr(K_k)$.

- **Barát-Tóth 2010:** $n \leq k + 4$.
- **Ackerman:** $n \geq 3.03k$.

New result for large k

Theorem (Fox-Pach-S. 2025+)

For k sufficiently large, if G is a graph on $n \leq 1.4k$ vertices and is k -critical, then $cr(G) \geq cr(K_k)$.

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Key ingredients in the proof.

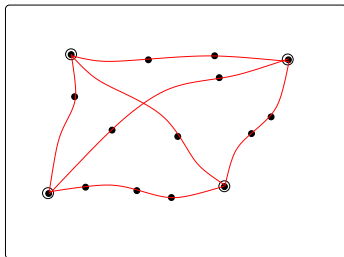
- 1 Crossing lemma: Büngener and Kaufmann (2024).
- 2 **(Weak) immersions**

Weak immersions

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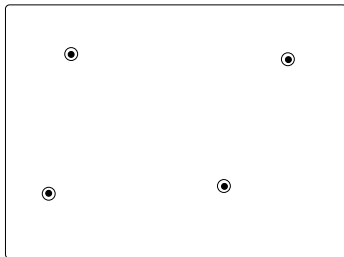
Definition. G contains K_k as a **weak immersion** if there are k branch vertices $v_1, \dots, v_k$ in G , and $\binom{k}{2}$ **edge**-disjoint paths between them.

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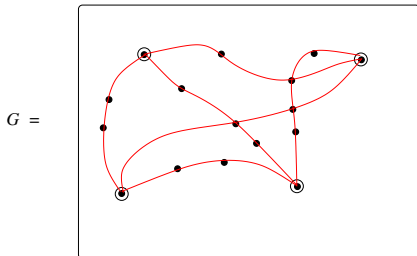
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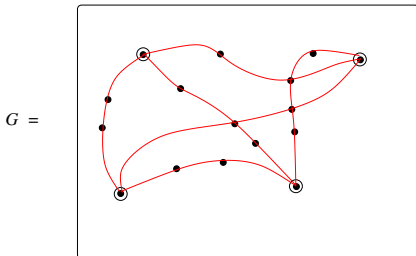
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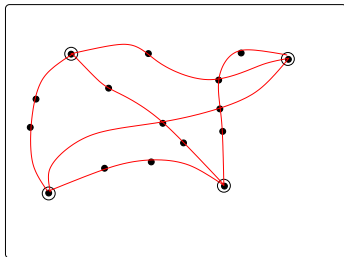
Note: The paths may go through the branch vertices.

Weak immersions

Conjecture (Lescure-Meyniel, 1989)

If G is a graph with $\chi(G) = k$, then G contains a weak immersion of K_k .

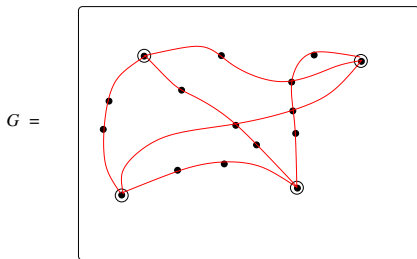
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- 1 **Lescure-Meyniel:** $k = 3$.
- 2 **DeVos-Kawarabayashi-Mohar-Okamura:** $k = 4, 5, 6$.
- 3 **Open** for $k \geq 7$. Can find weak immersion of K_t , $t = k/3.54$.

New result for immersions

We verify the Lescure-Meyniel conjecture for “small” graphs.

Theorem (Fox-Pach-S. 2025+)

If G is a graph with $\chi(G) = k$ and at most $1.4k$ vertices, then G contains K_k as a weak immersion.

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Theorem (Fox-Pach-S. 2025+)

If G is a graph on $n \leq 1.4k$ vertices and is k -critical, then $cr(G) \geq cr(K_k) - k^3/3$.

Proof. Draw G in the plane with $cr(G)$ crossings.

$G =$



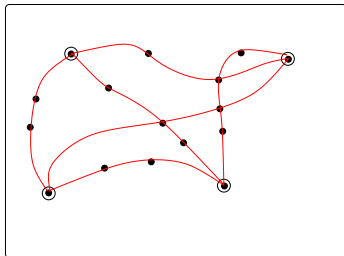
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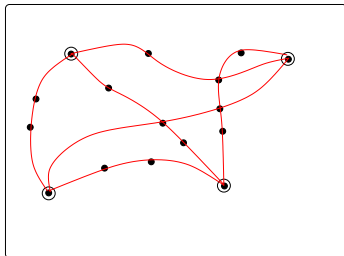
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Proof. Draw K_k by following the paths in the immersion.

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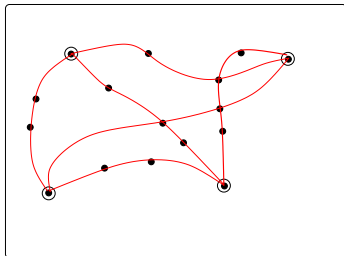
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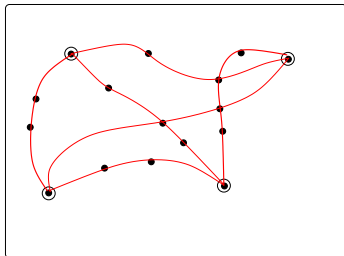
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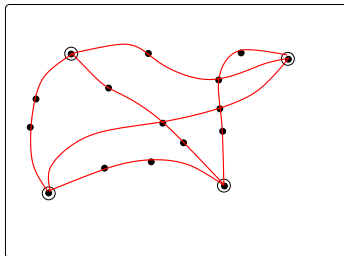
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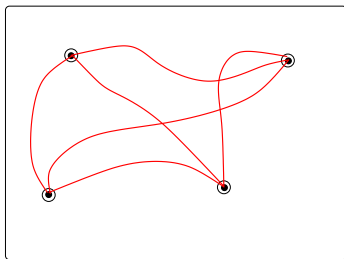
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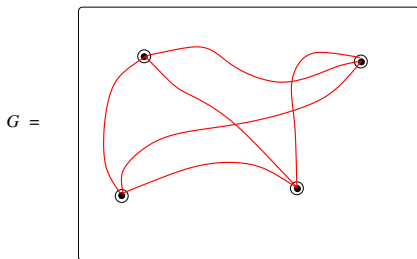


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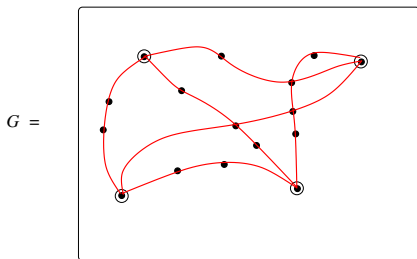
$$cr(K_k) \leq cr(G) + \text{crossings at a vertex}$$

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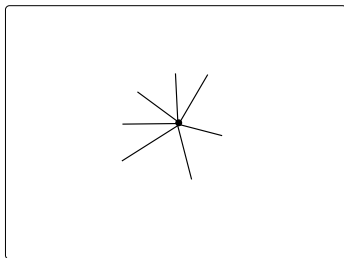


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Crossings at a non-branch vertex

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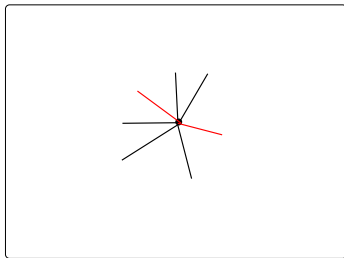
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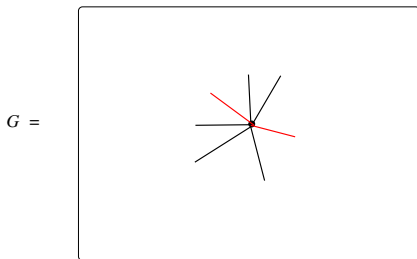
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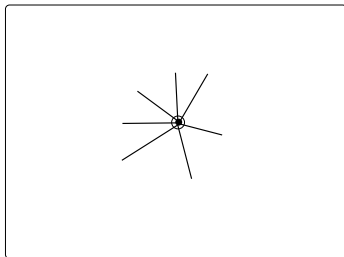


$$cr(K_k) \leq cr(G) + (n-k)n^2/8 + \text{crossings at a branch vertex}$$

Redrawing and crossings at a branch vertex

$$k \frac{n-k}{2} \frac{n}{2}$$

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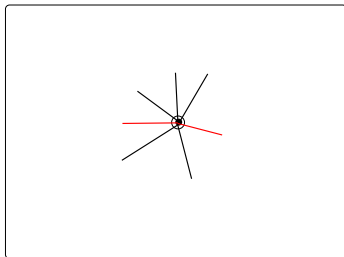


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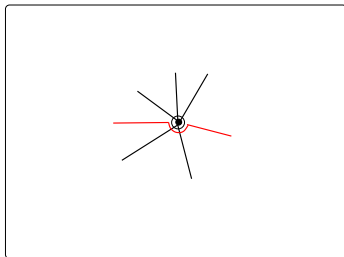


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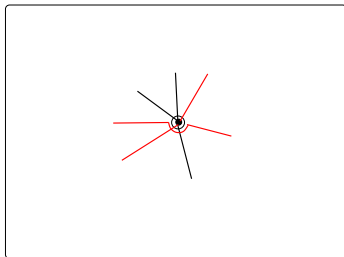


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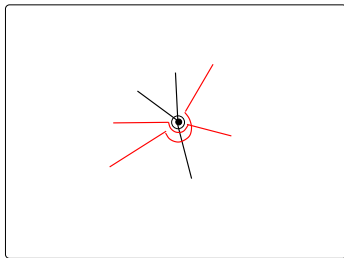


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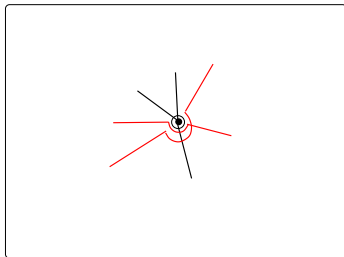


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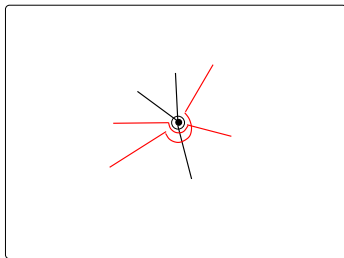
$$n \leq 1.4k$$

$$cr(K_k) \leq cr(G) + k^3/3$$

Crossings at a branch vertex

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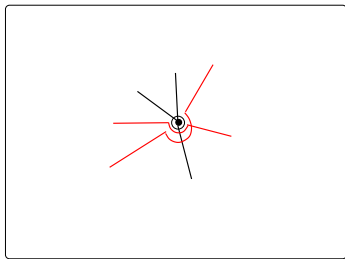


$$cr(K_k) - k^3/3 \leq cr(G)$$

Crossings at a branch vertex

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$$cr(K_k) - k^3/3 \leq cr(G)$$

Idea. Use the crossing lemma to find more crossings among the other edges.

$$cr(G) \geq cr(K_k) - k^3/3$$

Theorem (Fox-Pach-S. 2025+)

If G is a graph with $\chi(G) = k$ and at most $1.4k$ vertices, then G contains K_k as a weak immersion. Moreover, each path in the immersion has length at most 4.

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$$cr(G) \geq cr(K_k) - k^3/3 + \delta k^3(n - k)$$

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For k sufficiently large, if G is a graph on $n \leq 1.4k$ vertices and is k -critical, then $cr(G) \geq cr(K_k) + \epsilon(n - k)k^3$.

Theorem (Fox-Pach-S. 2025+)

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Open problems

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Conjecture (Fox-Pach-S. 2025+)

If G is a graph with $\chi(G) = k$ and at most $2k$ vertices, then G contains K_k as a weak immersion.

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Ackerman: conjecture holds for $n \geq 3.03k$.

New result: Journal version

Theorem (Fox-Pach-S. 2025+)

If G is a graph with $\chi(G) = k$ and at most $1.64k$ vertices, then G contains K_k as a weak immersion.

Theorem (Fox-Pach-S. 2025+)

For k sufficiently large, if G is a graph on $n \leq 1.64k$ vertices and is k -critical, then $cr(G) \geq cr(K_k)$.

Ackerman: conjecture holds for $n \geq 3.03k$.

Thank you!