Semi-algebraic hypergraph Ramsey numbers

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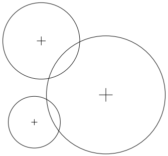
What is a semi-algebraic hypergraph?

Roughly speaking: Hypergraph $H = (V, E)$

$V = \{n$ *simple* geometric objects in $\mathbb{R}^d\}$

$E = \{k$-tuples that satisfy a *simple* algebraic relation$\}$.

$V = \{A_1, ..., A_N\}$, N disks in the plane. $E = \{$pairs of disks that intersect$\}$.



Example

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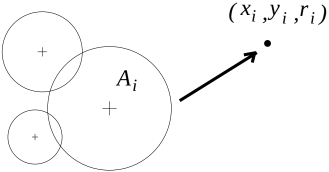

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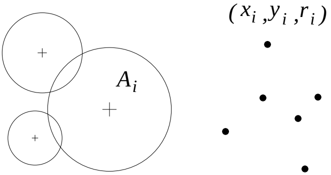

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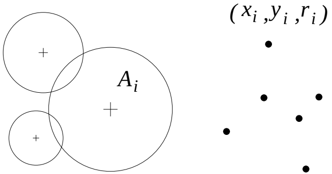

$A_i \to p_i = (x_i, y_i, r_i)$, $A_j \to p_j = (x_j, y_j, r_j)$. A_i and A_j cross if and only if

$$-x_i^2 + 2x_ix_j - x_j^2 - y_i^2 + 2y_iy_j - y_j^2 + r_i^2 + 2r_ir_j + r_j^2 \geq 0.$$

Example

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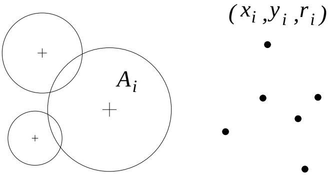

(V, E) is semi-algebraic graph,

$$f(z_1, ..., z_6) = -z_1^2 + 2z_1z_4 - z_4^2 - z_2^2 + 2z_2z_5 - z_5^2 + z_3^2 + 2z_3z_6 + z_6^2.$$

$(p_i, p_j) \in E \Leftrightarrow f(p_i, p_j) \geq 0.$

We say that $H = (V, E)$ is a **semi-algebraic k-uniform hypergraph in d-space** if

$V = \{n \text{ points in } \mathbb{R}^d\}$

E defined by polynomials $f_1, ..., f_t$ and a Boolean formula Φ such that

$$(p_{i_1}, ..., p_{i_k}) \in E$$

$$\Leftrightarrow \Phi(f_1(p_{i_1}, ..., p_{i_k}) \geq 0, ..., f_t(p_{i_1}, ..., p_{i_k}) \geq 0) = \text{yes}$$

Complexity of H is at most C if d, t and degree of the f_i-s is at most C (constant).

Main problem

Definition

We define the *Ramsey number* $R_k^{semi}(s, n)$ to be the minimum integer N such that any N-vertex k-uniform **semi-algebraic** hypergraph H (in $\mathbb{R}^d$) contains either a clique of size s or an independent set of size n.

Problem: Estimate $R_k^{semi}(s, n)$.

Diagonal Semi-algebraic Ramsey numbers ($s = n$)

Theorem (Alon, Pach, Pinchasi, Radoičić, Sharir 2005)

$$R_2^{semi}(n, n) \leq n^{c_1}.$$

Hypergraphs $k \geq 3$

Theorem (Conlon, Fox, Pach, Sudakov, S. 2012)

Let $t_1(x) = x$, *and* $t_k(x) = 2^{t_{k-1}(x)}$. *Then for* $k \geq 3$,

$$t_{k-1}(c_2 n) \leq R_k^{semi}(n, n) \leq t_{k-1}(n^{c_1}).$$

Classical Ramsey numbers:

$$2^{n/2} \leq R_2(n, n) \leq 2^{2n}.$$

$$t_{k-1}(n^2) \leq R_k(n, n) \leq t_k(cn).$$

Off diagonal Ramsey numbers (s is fixed)

Graphs:
$R_2(3,n) = \Theta(n^2 \log n)$ (Ajtai-Komlós-Szemerédi 1980, Kim 1995)
$R_2(s,n) = n^{\Theta(1)}$ (Spencer 1978, Ajtai, Komlós, Szemerédi 1980)
For hypergraphs

Theorem (Erdős-Hajnal-Rado 1952, 1965)

Fixed s

$$2^n \leq R_3(s,n) \leq 2^{n^{2s}}.$$

$$\mathrm{twr}_{k-1}(n) \leq R_k(s,n) \leq \mathrm{twr}_{k-1}(n^{2s}).$$

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For hypergraphs

Theorem (Conlon-Fox-Sudakov 2010)

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Off diagonal Ramsey numbers for Semi-algebraic hypergraphs

Graphs: (Fox-Pach-Suk 2015+, Alon et al. 2005)

$$n^{4/3} \leq R_2^{semi}(s,n) \leq n^C.$$

$C = 2^d$

3-uniform hypergraphs: (Conlon-Fox-Pach-Sudakov-S. 2013)

$$n^{c'} \leq R_3^{semi}(s,n) \leq 2^{n^c}.$$

$$\mathrm{twr}_{k-2}(n^{c'}) \leq R_k^{semi}(s,n) \leq \mathrm{twr}_{k-1}(n^c).$$

$$n^c \leq R_3^{semi}(s,n) \leq 2^{n^{c'}}.$$

Conjecture (Conlon-Fox-Pach-Sudakov-S. 2014)

For fixed s,

$$R_3^{semi}(s,n) \leq n^C$$

Question: How do you find cliques or independent sets in a 3-uniform hypergraph?

$$n^c \leq R_3^{semi}(s,n) \leq 2^{n^{c'}}.$$

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For fixed s,

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Question: How do you find cliques or independent sets in a 3-uniform hypergraph?

Usual Answer: Apply the Erdős-Rado greedy argument.

Erdős-Rado Greedy argument

Example for graphs:

$G =$

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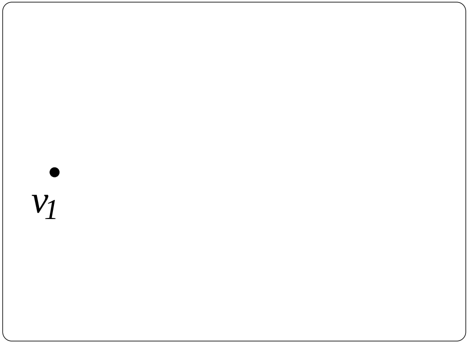
$G =$ 

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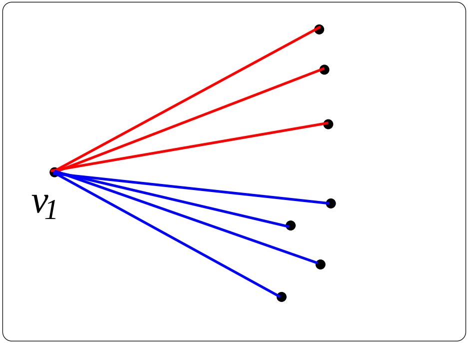

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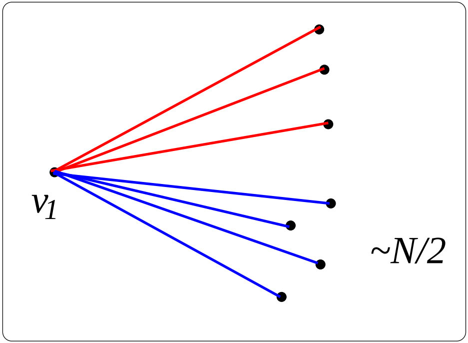
$G =$ 

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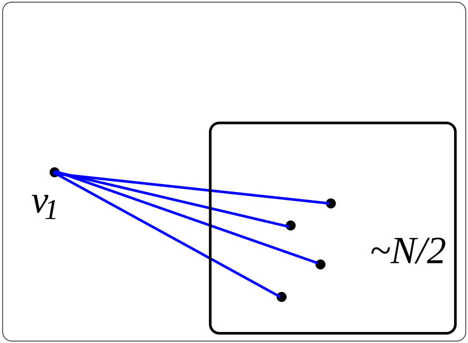
$G =$ 

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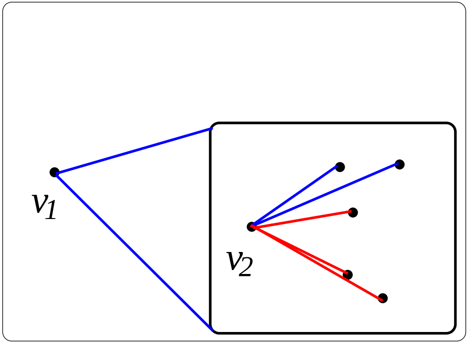

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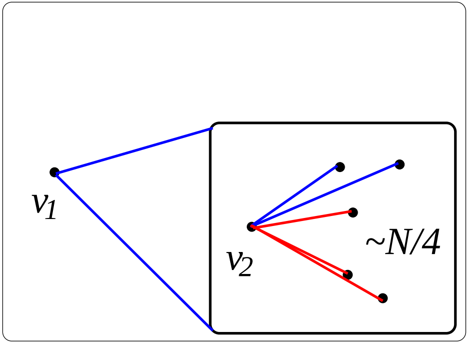

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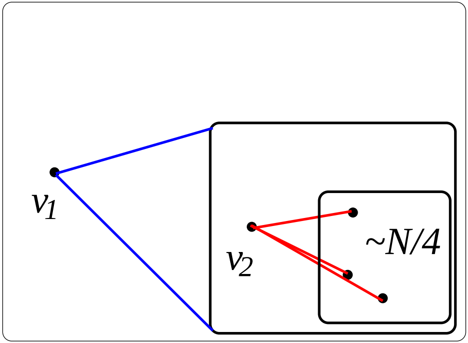
$G =$ 

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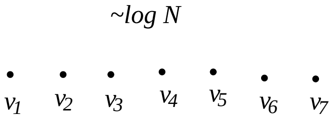

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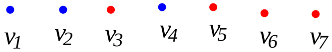

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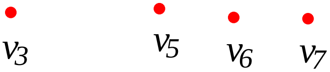

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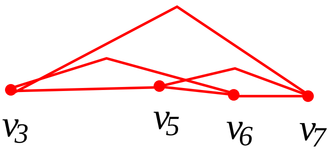

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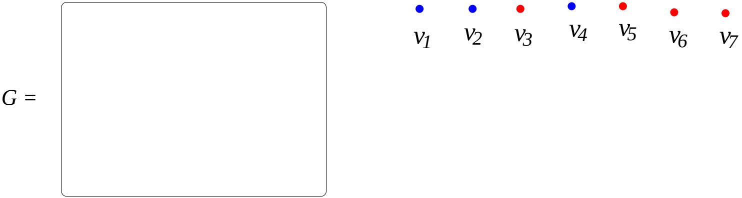


Argument shows (roughly) $R_2(s,n) \leq 2^{R_1(s,n)}$, where $R_1(s,n)$ is just the pigeonhole principle.

Generalize the argument (Erdős-Rado) to show (roughly) $R_k(s,n) \leq 2^{R_{k-1}(s,n)}$

$$n^c \leq R_3^{semi}(s,n) \leq 2^{n^{c'}}.$$

Conjecture (Conlon-Fox-Pach-Sudakov-S. 2014)

For fixed s,

$$R_3^{semi}(s,n) \leq n^C$$

The Erdős-Rado greedy argument will not work!

$$R_3^{semi} \leq 2^{R_2^{semi}(s,n)}$$

Finally the main result

Using different methods (Not Erdős-Rado)

Theorem (S. 2014+)

For fixed k, s

$$R_3^{semi}(s,n) \leq 2^{n^{o(1)}}.$$

Now apply the greedy argument: $R_k^{semi}(s,n) \leq 2^{R_{k-1}^{semi}(s,n)}$

Corollary (S. 2014+)

For fixed k, s

$$R_k^{semi}(s,n) \leq \mathrm{twr}_k(n^{o(1)}).$$

Finally the main result

Using different methods (Not using the Erdős-Rado greedy argument)

Theorem (S. 2014+)

For fixed k, s

$$R_3^{semi}(s, n) \leq 2^{n^{o(1)}}.$$

More exactly

$$R_3^{semi}(s, n) \leq 2^{2^{c\sqrt{(\log n)(\log \log n)}}}.$$

Application

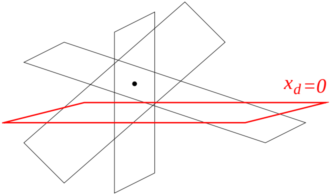
Problem

Determine the minimum integer $OSH_d(s,n)$, such that any family of at least $OSH_d(s,n)$ hyperplanes in $\mathbb{R}^d$ in general position, must contain either s members such that every d-tuple intersects above the $x_d = 0$ hyperplane, or n members such that every d-tuple intersects below the $x_d = 0$ hyperplane .

$OSH_2(s,n) = O(sn)$, $OSH_3(s,n) \leq 2^{OSH_2(s,n)}$

Conlon-Fox-Pach-Sudakov-S.: $OSH_3(s,n) \leq 2^{O(sn)}$

S., 2014+ $OSH_3(s,n) \leq 2^{n^{o(1)}}$.



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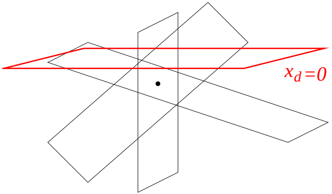

Thank you!