New developments in hypergraph Ramsey theory

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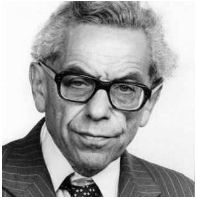
June 5, 2018

Origins of Ramsey theory

"A combinatorial problem in geometry," by Paul Erdős and George Szekeres (1935)



Erdős-Szekeres 1935

Theorem (Monotone subsequence)

Any sequence of $(n-1)^2+1$ *integers contains a monotone subsequence of length* n.

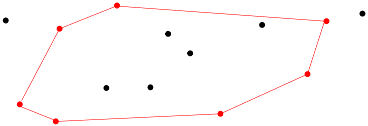
Theorem (Convex polygon)

For any $n > 0$*, there is a minimal* $ES(n)$*, such that every set of* $ES(n)$ *points in the plane in general position contains* n *members in convex position.*

Theorem (Ramsey numbers)

New proof of Ramsey's theorem.

Convex polygon theorem

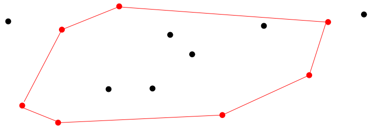


Theorem (Erdős-Szekeres 1935, 1960)

$$2^{n-2}+1 \le ES(n) \le \binom{2n-4}{n-2}+1 = O(4^n/\sqrt{n}).$$

Conjecture: $ES(n) = 2^{n-2}+1,\ n \ge 3.$

Convex polygon theorem



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$$2^{n-2} + 1 \leq ES(n) \leq \binom{2n-4}{n-2} + 1 = O(4^n/\sqrt{n}).$$

Theorem (S. 2016)

$$ES(n) = 2^{n+o(n)}$$

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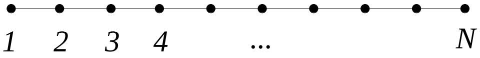
Theorem (Ramsey numbers)

New proof of Ramsey's theorem.

Formal definition: For any integers $k \geq 1$, $s, n \geq k$, there is a minimum $r_k(s, n) = N$, such that for every red/blue coloring of the k-tuples of $\{1, 2, \ldots, N\}$,



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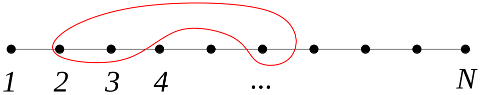

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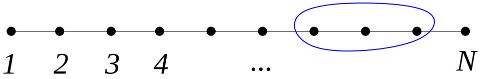

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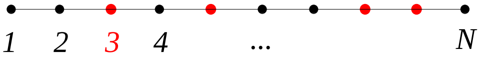

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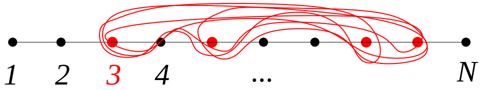

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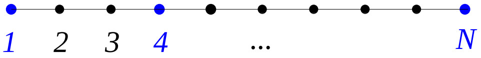

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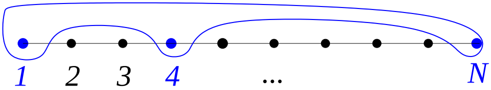

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$r_k(s, n)$ = Ramsey numbers

Graph Ramsey theorem

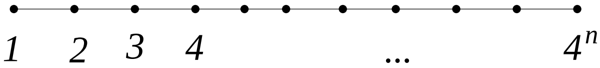
Theorem (Erdős-Szekeres 1935)

$$r_2(s,n) \leq \binom{n+s-2}{s-1}$$

$$r_2(n,n) \leq \binom{2n-2}{n-1} \approx \frac{4^n}{\sqrt{n}}$$

Graph Ramsey number $r_2(n, n) \leq 4^n$

Let G be the complete graph with 4^n vertices, every edge has color red or blue.



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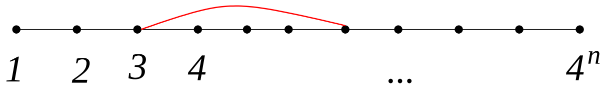

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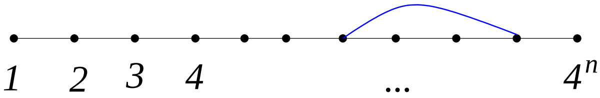

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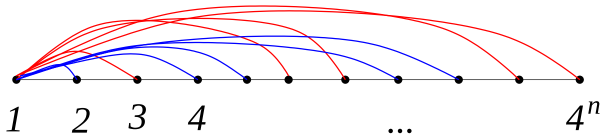

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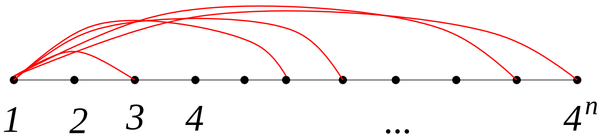

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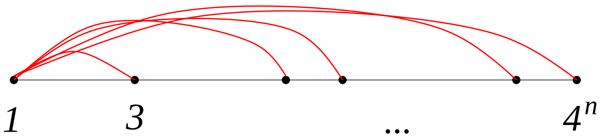

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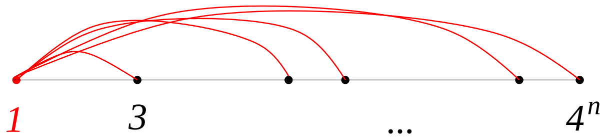

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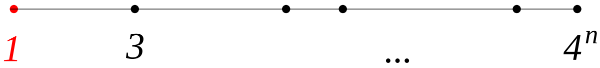

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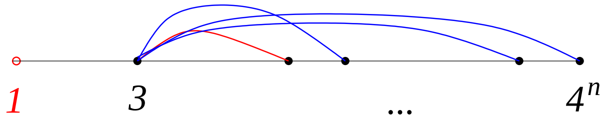

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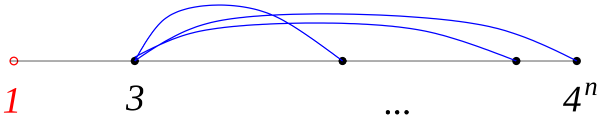

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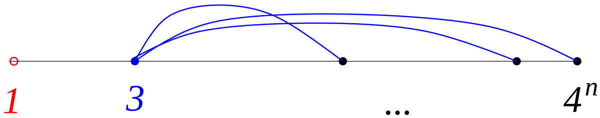

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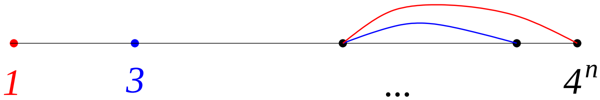

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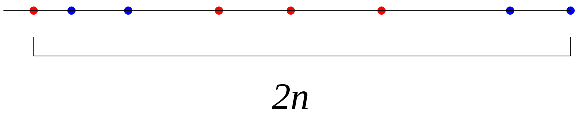

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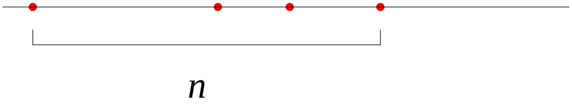

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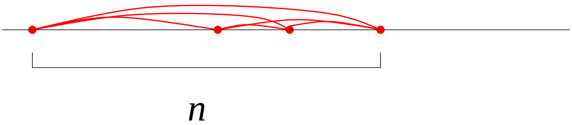


Diagonal graph Ramsey numbers

Theorem (Erdős 1947, Erdős-Szekeres 1935)

$$(1 + o(1))\frac{n}{e}2^{n/2} < r_2(n, n) < \frac{4^n}{\sqrt{n}}.$$

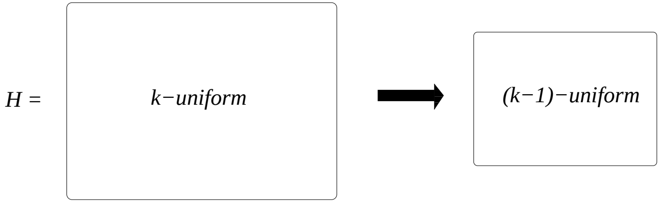
Theorem (Spencer 1977, Conlon 2008)

$$(1 + o(1))\frac{\sqrt{2}}{e}n2^{n/2} < r_2(n, n) < \frac{4^n}{n^{c \log n / \log \log n}}$$

Upper bounds for $r_k(n, n)$

Generalize the greedy argument by considering edges emanating out of $(k-1)$-tuples.

Erdős-Rado upper bound argument



Greedy argument to reduce the problem to a $(k-1)$-uniform hypergraph problem. Argument shows

$$r_k(s,n) \leq 2^{(r_{k-1}(s-1,n-1))^{k-1}}.$$

Upper bounds for diagonal hypergraph Ramsey numbers

Applying $r_k(s,n) \leq 2^{(r_{k-1}(s-1,n-1))^{k-1}}$.

Conlon (2008): $r_2(n,n) < \frac{4^n}{n^{c \log n / \log \log n}}$

Upper bounds for diagonal hypergraph Ramsey numbers

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Erdős-Rado (1952): $r_3(n,n) < 2^{2^{cn}}$

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Erdős-Rado (1952): $r_k(n,n) < \mathrm{twr}_k(cn)$

$\mathrm{twr}_1(x) = x$ and $\mathrm{twr}_{i+1}(x) = 2^{\mathrm{twr}_i(x)}$.

Lower bounds for diagonal hypergraph Ramsey numbers

Random constructions

Spencer (1977): $\frac{\sqrt{2}}{e}n2^{n/2} < r_2(n,n) < \frac{4^n}{n^{c\log n/\log\log n}}$

Erdős (1947): $2^{c'n^2} < r_3(n,n) < 2^{2^{cn}}$

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Hypergraph Ramsey numbers

Theorem (Erdős-Rado 1952)

$$2^{cn^2} < r_3(n,n) < 2^{2^{c'n}}$$

Conjecture (Erdős, \$500)

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Lower bounds for diagonal hypergraph Ramsey numbers

Erdős-Hajnal stepping up lemma: $k \geq 3$, $r_{k+1}(n,n) > 2^{r_k(n/4,n/4)}$.

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Erdős-Hajnal: $\mathrm{twr}_{k-1}(c' n^2) < r_k(n,n) < \mathrm{twr}_k(cn)$

$\mathrm{twr}_1(x) = x$ and $\mathrm{twr}_{i+1}(x) = 2^{\mathrm{twr}_i(x)}$.

Conjecture: $r_3(n,n) > 2^{2^{cn}}$

Theorem (Erdős-Hajnal-Rado 1952/1965)

$$2^{cn^2} < r_3(n,n) < 2^{2^{c'n}}$$

Theorem (Erdős-Hajnal)

$$r_3(n,n,n,n) > 2^{2^{cn}}$$

Off-diagonal Ramsey numbers

$r_k(s, n)$ where s is fixed, and $n \to \infty$.

Graphs:

Theorem (Ajtai-Komlós-Szemerédi 1980, Kim 1995)

$r_2(3, n) = \Theta\left(\frac{n^2}{\log n}\right)$

Theorem

For fixed $s > 3$

$$n^{(s+1)/2+o(1)} < r_2(s, n) < n^{s-1+o(1)}$$

Upper bounds for off-diagonal Ramsey numbers

3-uniform hypergraphs:

Theorem (Erdős-Hajnal-Rado)

For fixed $s \geq 4$,

$$2^{csn} < r_3(s,n) < 2^{c' n^{2s-4}}.$$

Theorem (Conlon-Fox-Sudakov 2010)

For fixed $s \geq 4$,

$$2^{csn \log n} < r_3(s,n) < 2^{c' n^{s-2} \log n}.$$

Fixed $s \geq k + 1$.

Erdős-Szekeres (1935): $r_2(s, n) < n^{s-1+o(1)}$

Conlon-Fox-Sudakov (2010): $r_3(s, n) < 2^{cn^{s-2} \log n}$

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Erdős-Rado: $r_k(s,n) < \mathrm{twr}_{k-1}(n^c)$

Fixed $s \geq k+1$.

Bohman-Keevash (2010): $n^{(s+1)/2+o(1)} < r_2(s,n) < n^{s-1+o(1)}$

Conlon-Fox-Sudakov (2010): $2^{c' n \log n} < r_3(s,n) < 2^{cn^{s-2}\log n}$

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Erdős-Rado: $r_k(s,n) < \mathrm{twr}_{k-1}(n^c)$

Tower growth rate for $r_4(5,n)$ is unknown.

Lower bounds for off-diagonal hypergraph Ramsey numbers

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Conlon-Fox-Sudakov (2010): $2^{c' n \log n} < r_3(s,n) < 2^{cn^{s-2}\log n}$

Erdős-Hajnal: $2^{2^{c'n}} < r_4(7,n) < 2^{2^{n^c}}$

Erdős-Rado (1952): $r_5(s,n) < 2^{2^{2^{n^c}}}$

$\vdots$

Erdős-Rado: $r_k(s,n) < \mathrm{twr}_{k-1}(n^c)$

Tower growth rate for $r_4(5,n)$ is unknown.

Fixed $s \geq k + 1$.

Bohman-Keevash (2010): $n^{(s+1)/2+o(1)} < r_2(s, n) < n^{s-1+o(1)}$

Conlon-Fox-Sudakov (2010): $2^{c' n \log n} < r_3(s, n) < 2^{cn^{s-2} \log n}$

Erdős-Hajnal: $2^{2^{c' n}} < r_4(7, n) < 2^{2^{n^c}}$

MS and CFS (2015): $2^{2^{2^{c' n}}} < r_5(8, n) < 2^{2^{2^{n^c}}}$

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MS and CFS (2015): $\mathrm{twr}_{k-1}(c'n) < r_k(k+3,n) < \mathrm{twr}_{k-1}(n^c)$

Tower growth rate for $r_4(5,n)$ is unknown.

Lower bounds for off-diagonal hypergraph Ramsey numbers

Fixed $s \geq k+1$.

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MS and CFS (2015): $\mathrm{twr}_{k-1}(c'n) < r_k(k+3,n) < \mathrm{twr}_{k-1}(n^c)$

What is the tower growth rate of $r_k(k+1,n)$ and $r_k(k+2,n)$?

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What is the tower growth rate of $r_4(5,n)$ and $r_4(6,n)$?

Towards the Erdős-Hajnal conjecture

Conjecture (Erdos-Hajnal)

$$r_4(5, n), r_4(6, n) > 2^{2^{cn}}$$

Erdős-Hajnal (1972): $r_4(5, n), r_4(6, n) > 2^{cn}$

Mubayi-S. (2017): $r_4(5, n) > 2^{n^2}$ $\quad r_4(6, n) > 2^{n^{c \log n}}$

New lower bounds for off-diagonal hypergraph Ramsey numbers

Theorem (Mubayi-S., 2018)

$$r_4(5,n) > 2^{n^{c\log n}} \qquad r_4(6,n) > 2^{2^{cn^{1/5}}}.$$

for fixed $k > 4$

$$r_k(k+1,n) > \mathrm{twr}_{k-2}(n^{c\log n}) \qquad r_k(k+2,n) > \mathrm{twr}_{k-1}(cn^{1/5}).$$

$$r_k(k+2,n) = \mathrm{twr}_{k-1}(n^{\Theta(1)})$$

Open problems

Diagonal Ramsey problem ($500 Erdős):

$$2^{cn^2} < r_3(n,n) < 2^{2^{cn}}.$$

Off-diagonal Ramsey problem:

$$2^{n^{c \log n}} < r_4(5,n) < 2^{2^{cn}}.$$

Theorem (Mubayi-S. 2017)

Showing $r_3(n,n) > 2^{2^{cn}}$ *implies that* $r_4(5,n) > 2^{2^{c'n}}$.

More off-diagonal?

Off diagonal hypergraph Ramsey numbers: $r_k(k+1,n)$

Red clique size $k+1$ or Blue clique of size n.

$$r_k(k,n) = n \qquad \text{(trivial)}$$

More off-diagonal?

Off diagonal hypergraph Ramsey numbers: $r_k(k+1, n)$

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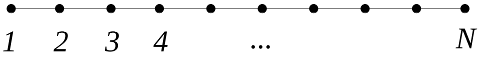
$$r_k(k, n) = n \qquad \text{(trivial)}$$

A more off diagonal Ramsey number:

Almost Red clique size $k+1$ or Blue clique of size n.

Another Ramsey function: Let $r_k(k+1, t; n)$ be the minimum N, such that for every red/blue coloring of the k-tuples of $\{1, 2, \ldots, N\}$,



1. $k+1$ integers which induces at least t red k-tuples, or
2. n integers for which every k-tuple is blue.

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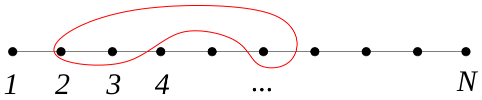

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Very off-diagonal

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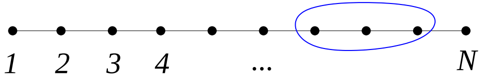

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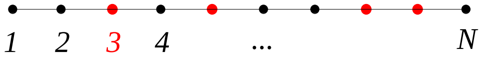

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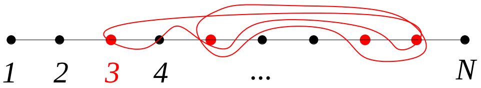

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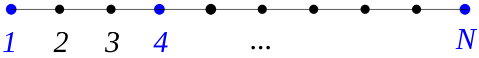

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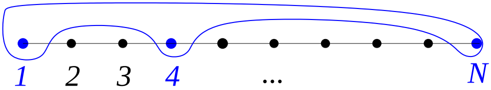

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2. n integers for which every k-tuple is blue.

An old problem of Erdős and Hajnal 1972

Problem (Erdős-Hajnal 1972)

For $k \geq 3$ and $t \in [k+1]$, estimate $r_k(k+1, t; n)$.

$$r_k(k+1, 1; n) = n$$

$$\vdots$$

$$\mathrm{twr}_{k-2}(n^{c' \log n}) \leq r_k(k+1, k+1; n) = r_k(k+1, n) \leq \mathrm{twr}_{k-1}(n^c)$$

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$$r_k(k+1, 1; n) = n$$

$$r_k(k+1, 2; n) \leq O(n^{k-1})$$

$$\vdots$$

$$\mathrm{twr}_{k-2}(n^{c' \log n}) \leq r_k(k+1, k+1; n) = r_k(k+1, n) \leq \mathrm{twr}_{k-1}(n^c)$$

Upper bounds

Erdős-Rado argument: $r_k(k+1, t; n) \leq 2^{(r_{k-1}(k, t-1; n))^{k-1}}$.

Theorem (Erdős-Hajnal 1972)

$$r_k(k+1, 2; n) < cn^{k-1}$$

Upper bounds

Erdős-Rado argument: $r_k(k+1, t; n) \leq 2^{(r_{k-1}(k, t-1; n))^{k-1}}$.

Theorem (Erdős-Hajnal 1972)

$$r_k(k+1, 2; n) < cn^{k-1}$$

$$r_k(k+1, 3; n) < 2^{n^c}$$

Upper bounds

Erdős-Rado argument: $r_k(k+1, t; n) \leq 2^{(r_{k-1}(k,t-1;n))^{k-1}}$.

Theorem (Erdős-Hajnal 1972)

$$r_k(k+1, 2; n) < cn^{k-1}$$

$$r_k(k+1, 3; n) < 2^{n^c}$$

$$r_k(k+1, 4; n) < 2^{2^{n^c}}$$

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$$\vdots$$

$$r_k(k+1, t; n) < \mathrm{twr}_{t-1}(n^c)$$

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$$\vdots$$

$$r_k(k+1, k; n) < \mathrm{twr}_{k-1}(n^c)$$

$$r_k(k+1, k+1; n) < \mathrm{twr}_{k-1}(n^c)$$

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Lower bounds

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Theorem

$$c' \frac{n^{k-1}}{\log n} < r_k(k+1, 2; n) < c \frac{n^{k-1}}{\log n}$$

$$r_k(k+1, 3; n) < 2^{n^c}$$

$$r_k(k+1, 4; n) < 2^{2^{n^c}}$$

$$\vdots$$

$$r_k(k+1, t; n) < \mathrm{twr}_{t-1}(n^c)$$

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Theorem

$$c' \frac{n^{k-1}}{\log n} < r_k(k+1, 2; n) < c \frac{n^{k-1}}{\log n}$$

$$2^{c' n} < r_k(k+1, 3; n) < 2^{n^c}$$

$$r_k(k+1, 4; n) < 2^{2^{n^c}}$$

$$\vdots$$

$$r_k(k+1, t; n) < \mathrm{twr}_{t-1}(n^c)$$

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Theorem

$$c' \frac{n^{k-1}}{\log n} < r_k(k+1, 2; n) < c \frac{n^{k-1}}{\log n}$$

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$$2^{c' n} < r_k(k+1, 4; n) < 2^{2^{n^c}}$$

$$\vdots$$

$$2^{c' n} < r_k(k+1, t; n) < \mathrm{twr}_{t-1}(n^c)$$

New bounds for $r_k(k+1, t; n)$

Improvement on the Erdős-Rado upper bound argument.

Theorem (Mubayi-S. 2018)

For $k \geq 3$, and $2 \leq t \leq k$, we have

$$r_k(k+1, t; n) < \mathrm{twr}_{t-1}(cn^{k-t+1} \log n)$$

For $3 \leq t \leq k-2$

Theorem (Mubayi-S. 2018)

For $k \geq 6$, and $3 \leq t \leq k-2$, we have

$$\mathrm{twr}_{t-1}(c' n^{k-t+1}) < r_k(k+1, t; n) < \mathrm{twr}_{t-1}(c n^{k-t+1} \log n)$$

when $k - t$ is even, and

New bounds for $r_k(k+1, t; n)$

For $3 \leq t \leq k-2$

Theorem (Mubayi-S. 2018)

For $k \geq 6$, and $3 \leq t \leq k-2$, we have

$$\mathrm{twr}_{t-1}(c' n^{k-t+1}) < r_k(k+1, t; n) < \mathrm{twr}_{t-1}(c n^{k-t+1} \log n)$$

when $k-t$ is even, and

$$\mathrm{twr}_{t-1}(c' n^{(k-t+1)/2}) < r_k(k+1, t; n) < \mathrm{twr}_{t-1}(c n^{k-t+1} \log n)$$

when $k-t$ is odd.

New bounds for $r_k(k+1, t; n)$

For $t = k-1, k, k+1$

Theorem (Mubayi-S. 2018)

For $k \geq 6$,

$$\mathrm{twr}_{k-3}(cn^3) < r_k(k+1, k-1; n) < \mathrm{twr}_{k-2}(c'n^2 \log n)$$

$$\mathrm{twr}_{k-3}(cn^3) < r_k(k+1, k; n) < \mathrm{twr}_{k-1}(c'n \log n)$$

$$\mathrm{twr}_{k-2}(n^{c \log n}) < r_k(k+1, k+1; n) < \mathrm{twr}_{k-1}(c'n^2 \log n).$$

Open problem for 5-uniform hypergraphs

$$2^{c'n^3} < r_5(6, 5; n) < 2^{2^{2^{n^c}}}$$

Improve the upper or lower bounds for $r_5(6, 5; n)$.

Thank you!