Off-Diagonal Hypergraph Ramsey Numbers

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June 6, 2016

Introduction

For k-uniform hypergraphs.

Definition

We define the *Ramsey number* $r_k(s, n)$ to be the minimum integer N such that any N-vertex k-uniform hypergraph H contains either a clique of size s or an independent set of size n.

Theorem (Ramsey 1930)

For all k, s, n, the Ramsey number $r_k(s, n)$ is finite.

Estimate $r_k(s, n)$.

Diagonal Ramsey numbers

Where $s = n$, $r_k(n,n)$.

Theorem (Erdős-Szekeres 1935, Erdős 1947)

$$(\sqrt{2})^n \leq r_2(n,n) \leq 4^n.$$

Theorem (Erdős-Rado 1952, Erdős-Hajnal 1960's)

$$2^{cn^2} \leq r_3(n,n) \leq 2^{2^{c'n}}.$$

$$\mathrm{twr}_{k-1}(cn^2) \leq r_k(n,n) \leq \mathrm{twr}_k(c'n).$$

Tower function: $\mathrm{twr}_1(x) = x$ and $\mathrm{twr}_{i+1}(x) = 2^{\mathrm{twr}_i(x)}$.

Erdős conjecture: $r_3(n,n) = 2^{2^{\Theta(n)}}$ (offered \$500).

Off-diagonal Ramsey numbers

Where $s =$ fixed, $r_k(s, n)$

Theorem (Ajtai-Komlós-Szemerédi 1980, Kim 1995)

$$r_2(3, n) = \Theta(\frac{n^2}{\log n}).$$

$$n^{\frac{s+1}{2}+o(1)} < r_2(s, n) < n^{s-1+o(1)}$$

Theorem (Erdős-Rado 1952)

For $s \geq k+1$ fixed,

$$r_k(s, n) \leq \mathrm{twr}_{k-1}(n^c).$$

Conjecture (Erdős-Hajnal 1972)

For $s \geq k+1$,

$$r_3(s,n) > 2^{cn}$$

$$r_4(s,n) > 2^{2^{cn}}$$

$$r_k(s,n) > \mathrm{twr}_{k-1}(cn)$$

Theorem (Erdős-Hajnal 1972)

Lower bound holds for $k = 3$, *and for* $s \approx 2^k$

Theorem (Conlon-Fox-Sudakov 2009)

Lower bound holds for $s = \lceil 5k/2 \rceil - 3$

Theorem (Mubayi-S. and Conlon-Fox-Sudakov 2015)

Lower bound holds for $s \geq k+3$:

$$r_k(s,n) \geq \mathrm{twr}_{k-1}(cn)$$

Conjecture (Erdős-Hajnal 1972)

For $s \geq k+1$,

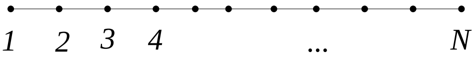
$$r_3(s,n) > 2^{cn}$$

$$r_4(s,n) > 2^{2^{cn}}$$

$$r_k(s,n) > \mathrm{twr}_{k-1}(cn)$$

Ordered Path vs Clique

$V = \{1, 2, \ldots, N\}$



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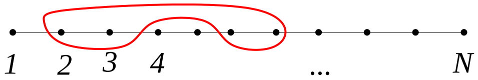

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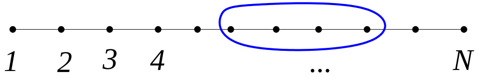

Ordered Path vs Clique

$V = \{1, 2, \ldots, N\}$

Red Ordered Path: P_s, $i_1 < \cdots < i_s$, $(i_j, \ldots, i_{j+k-1}) \in E(G)$.

Ordered Path vs Clique

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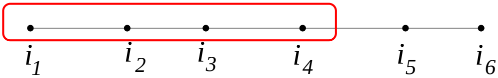

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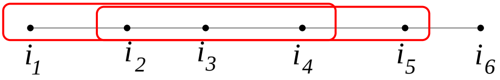

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Ordered Path vs Clique

$V = \{1, 2, \ldots, N\}$

Definition: $r_k(P_s, n)$ be the minimum N such that every red/blue coloring of the k-tuples in $[N]$ contains a red P_s or a blue K_n.

$r_k(P_s, n) \leq r_k(s, n)$

Theorem (Mubayi-S. 2015)

For $q = s - k + 1$,

$$r_{k-1}(\underbrace{\frac{n}{q}, \ldots, \frac{n}{q}}_{q \text{ times}}) \leq r_k(P_s, n) \leq r_{k-1}(\underbrace{n, \ldots, n}_{q \text{ times}}).$$

Theorem (Mubayi-S. 2015)

For $s = k + 3$,

$$r_{k-1}(\frac{n}{4}, \frac{n}{4}, \frac{n}{4}, \frac{n}{4}) \leq r_k(P_{k+3}, n) \leq r_{k-1}(n, n, n, n).$$

By definition: $r_k(P_{k+3}, n) \leq r_k(k + 3, n)$

Theorem (Mubayi-S. 2015)

$$r_k(k+3,n) \geq r_k(P_{k+3},n) > r_{k-1}(\frac{n}{4},\frac{n}{4},\frac{n}{4},\frac{n}{4}) > \mathrm{twr}_{k-1}(cn).$$

$$r_k(k+2,n) \geq r_k(P_{k+2},n) > r_{k-1}(\frac{n}{3},\frac{n}{3},\frac{n}{3}) > \mathrm{twr}_{k-1}(c\log^2 n)$$

$$r_k(k+1,n) \geq \mathrm{twr}_{k-2}(n^{c\log\log n})$$

Conjecture (Mubayi-S. 2015)

For all $s \geq k+1$

$$r_k(P_s,n) \geq \mathrm{twr}_{k-1}(cn).$$

Theorem (Mubayi-S. 2015)

$$r_3(n/2, n/2) \leq r_4(P_5, n) \leq r_3(n, n).$$

Theorem (Mubayi-S. 2015)

$r_3(n,n) \geq 2^{2^{cn}}$ *holds if and only if* $r_4(P_5, n) \geq 2^{2^{cn}}$.

Very off-diagonal Ramsey numbers

Definition: $r_k(k+1, t; n)$ be the smallest integer N such that every N-vertex k-uniform hypergraph contains either an independent set of size n, or $k+1$ vertices that induces at least t edges.

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Conjecture (Erdős-Hajnal 1964)

For $t \leq k$

$$r_k(k+1, t; n) = \mathrm{twr}_{t-1}(cn)$$

Ramsey approach

Theorem (Mubayi-S. 2016)

For $t \leq k-2$, and $k-t$ even

$$r_k(k+1, t; n) = \mathrm{twr}_{t-1}(n^{k-t+1+o(1)})$$

For $t \leq k-2$, and $k-t$ odd

$$\mathrm{twr}_{t-1}(n^{(k-t+1)/2}) \leq r_k(k+1, t; n) \leq \mathrm{twr}_{t-1}(n^{k-t+1+o(1)})$$

Thank you!