# Geometric Ramsey Theory

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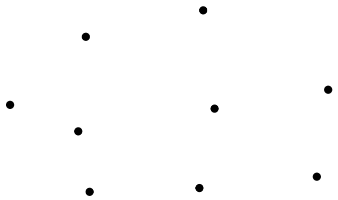
A $k$-uniform hypergraph $H = (V, E)$, $V$ is the vertex set, and edge set $E \subset \binom{V}{k}$. We define a clique in $H$ to be a sub-hypergraph for which all $k$-tuples belong to $E$, and an independent set in $H$ to be a sub-hypergraph for which all $k$-tuples not in $E$.



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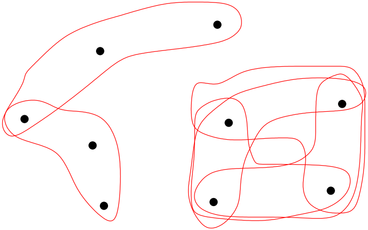

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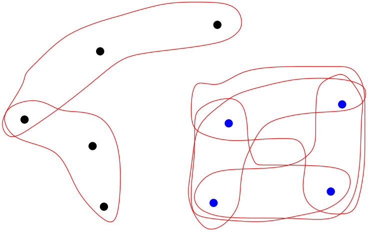

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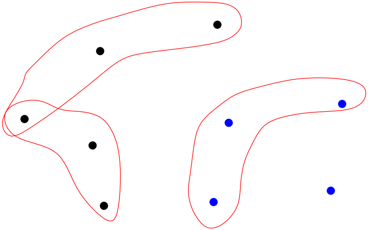

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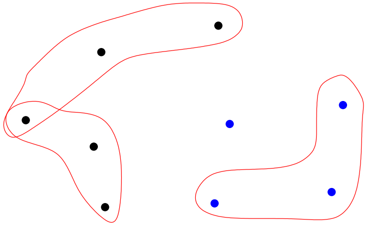

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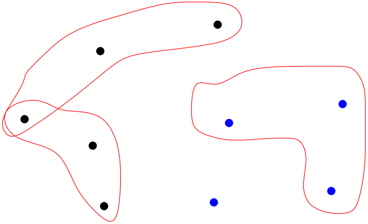

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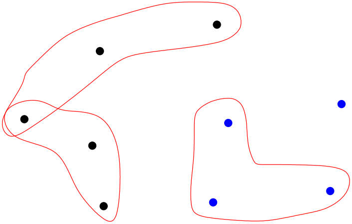

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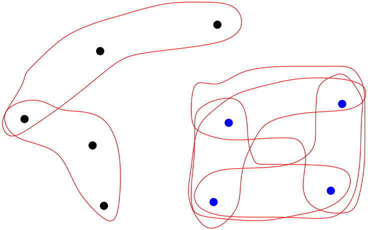

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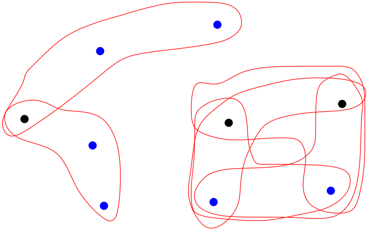

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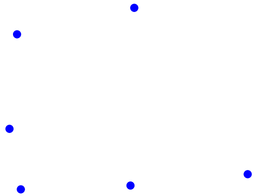

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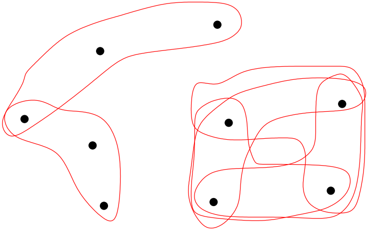


# Introduction

**Definition**

We define the *Ramsey number* $R_k(n)$ to be the minimum integer $N$ such that any $N$-vertex $k$-uniform hypergraph $H$ contains either a clique or an independent set of size $n$.

**Theorem (Ramsey 1930)**

*For all $k, n$, the Ramsey number $R_k(n)$ is finite.*

Estimate $R_k(n)$, $k$ fixed and $n \to \infty$.

Known estimates

**Theorem (Erdős-Szekeres 1935, Erdős 1947)**

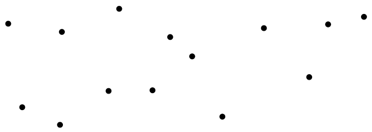
$$2^{n/2} \leq R_2(n) \leq 2^{2n}.$$

**Theorem (Erdős-Rado 1952, Erdős-Hajnal 1960's)**

$$2^{cn^2} \leq R_3(n) \leq 2^{2^{c'n}}.$$

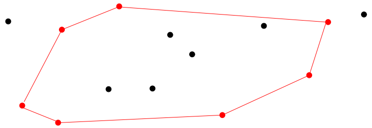
$$t_{k-1}(cn^2) \leq R_k(n) \leq t_k(c'n).$$

*Tower function* $t_i(x)$ is given by $t_1(x) = x$ and $t_{i+1}(x) = 2^{t_i(x)}$.



**Problem (Esther Klein 1930's)**

*Given an integer $n$, does there exist a number $ES(n)$, such that any set of at least $ES(n)$ points in the plane in general position, contains $n$ members in convex position?*



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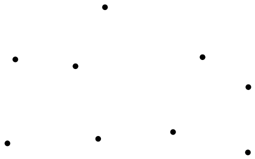


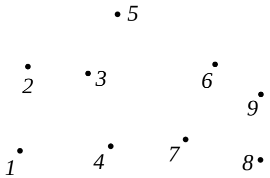
$V = \{N$ points in the plane$\}$,
$E = \{$triples having a clockwise orientation$\}$.

**Observation:** Any subset of points for which every triple has the same orientation, must be in convex position.

$$ES(n) \leq R_3(n) \leq 2^{2^{c'n}}.$$

Can we do better?

# $ES(n)$ exists

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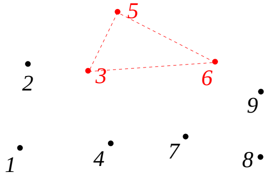


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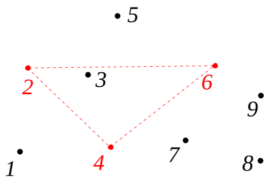


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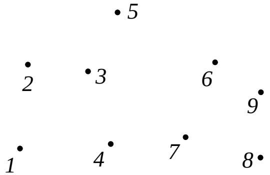


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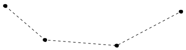
Can we do better?

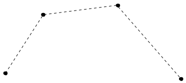
**Theorem (Erdős-Szekeres 1935)**

$$2^{n-2}+1 \leq ES(n) \leq \binom{2n-4}{n-2}+1 = O(4^n/\sqrt{n}).$$

(a) 4-cup.



(b) 4-cap.



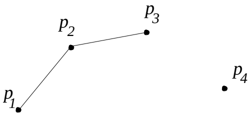
**Theorem (Erdős-Szekeres 1935)**

*For any positive integers $k$ and $l$, there exists an integer $f(k,l)$, such that any set of at least $f(k,l)$ points in the plane in general position, contains either a $k$-cup or an $l$-cap. Moreover*

$$f(k,l) = \binom{k+l-4}{k-2} + 1$$

Proof is very combinatorial. The only geometric fact used was the following: Order the points from left to right $\{p_1, ..., p_N\}$

**transitive property:** If $(p_1, p_2, p_3)$ is a cap (cup), and $(p_2, p_3, p_4)$ is a cap (cup), then $p_1, p_2, p_3, p_4$ is a 4-cap (4-cup).



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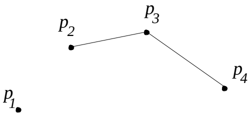

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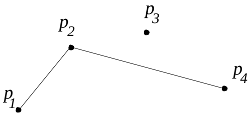

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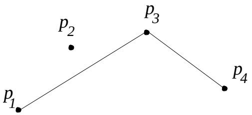

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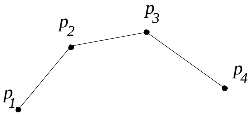

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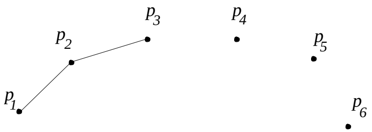

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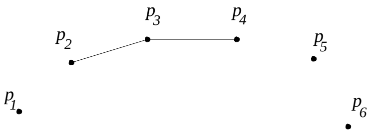

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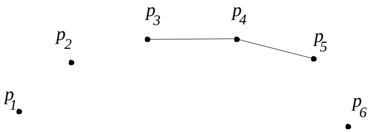

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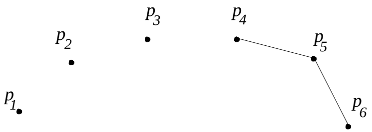

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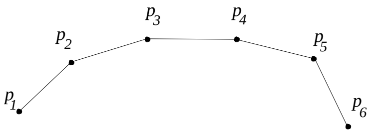
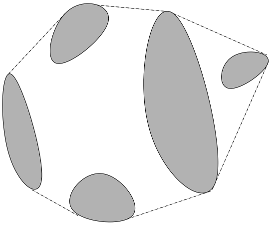
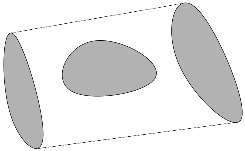


# Generalizing to convex bodies

## Definition

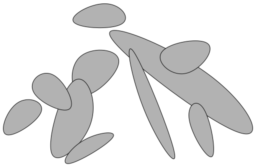
A family $\mathcal{C}$ of convex bodies (compact convex sets) in the plane is said to be in *convex position* if none of its members is contained in the convex hull of the union of the others. We say that $\mathcal{C}$ is in *general position* if every three members are in convex position.





## Definition

We say that a family of convex bodies in the plane is *noncrossing* if any two members share at most two boundary points.



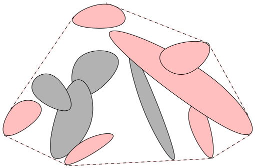
**Theorem (Pach and Tóth 2000)**

*For any positive integer n, there exists an integer NC(n), such that any set of at least NC(n) noncrossing convex bodies in the plane in general position must contain n members in convex position. Moreover*

$$2^{n-2} + 1 \leq NC(n) \leq 2^{2^{2^{2^n}}}.$$

$NC(n) \leq 2^{2^{cn}}$, Hubard-Montejano-Mora-S. 2011

$NC(n) \leq 2^{c' n^2 \log n}$, Fox-Pach-Sudakov-S. 2012.



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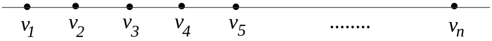
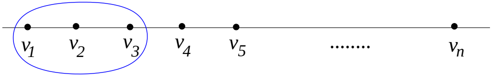
For an ordered 3-uniform hypergraph $H = ([N], E)$, a monotone 3-path of length $n$ are edges $(v_1, v_2, v_3), (v_2, v_3, v_4), (v_3, v_4, v_5), \ldots, (v_{n-2}, v_{n-1}, v_n)$.




In general, for an ordered $k$-uniform hypergraph $H = ([N], E)$, a monotone $k$-path of length $n$ are edges $(v_1, v_2, \ldots, v_k), (v_2, v_3, \ldots, v_{k+1}) \ldots, (v_{n-k+1}, \ldots, v_n)$.

# Finding monochromatic paths in ordered hypergraphs

For an ordered 3-uniform hypergraph $H = ([N], E)$, a monotone 3-path of length $n$ are edges $(v_1, v_2, v_3), (v_2, v_3, v_4), (v_3, v_4, v_5), ..., (v_{n-2}, v_{n-1}, v_n)$.

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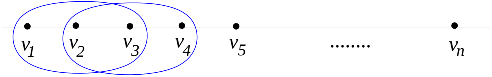


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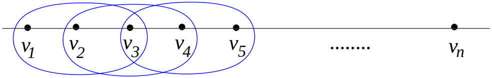


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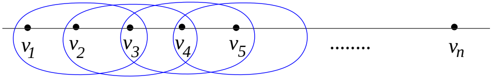


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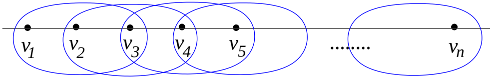
# Finding monochromatic paths in ordered hypergraphs

For an ordered 3-uniform hypergraph $H = ([N], E)$, a monotone 3-path of length $n$ are edges $(v_1, v_2, v_3), (v_2, v_3, v_4), (v_3, v_4, v_5), \ldots, (v_{n-2}, v_{n-1}, v_n)$.



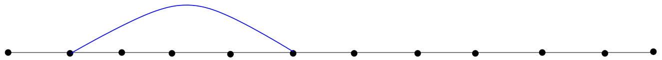

In general, for an ordered $k$-uniform hypergraph $H = ([N], E)$, a monotone $k$-path of length $n$ are edges $(v_1, v_2, \ldots, v_k), (v_2, v_3, \ldots, v_{k+1}) \ldots, (v_{n-k+1}, \ldots, v_n)$.

# Ordered hypergraphs

**Definition**

Let $N_k(q,n)$ denote the smallest integer $N$ such that for every $q$ coloring on the $k$-tuples of the set $[N]$ contains a monochromatic monotone $k$-path of length $n$.

$N_2(q,n) = (n-1)^q + 1$ by Dilworth's theorem.



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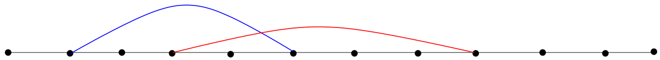

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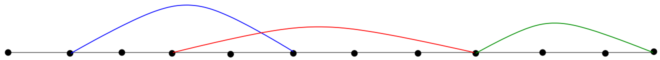

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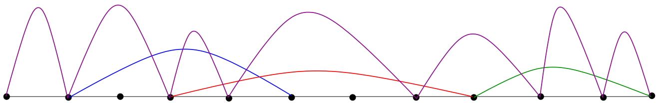
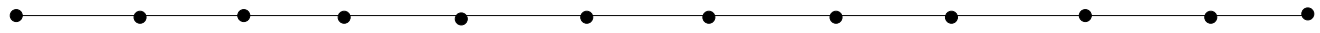


$N_3(2, n) = \binom{2n-4}{n-2} + 1$ by the Erdős-Szekeres cups-caps (red-blue) argument.



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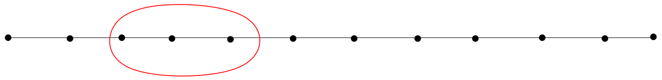

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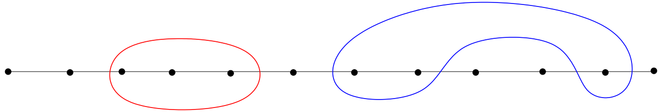

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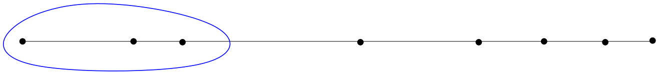

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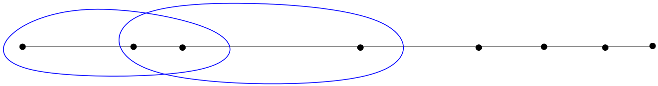

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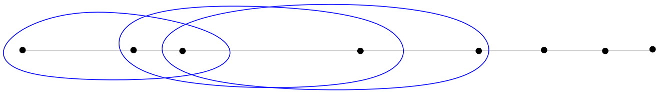

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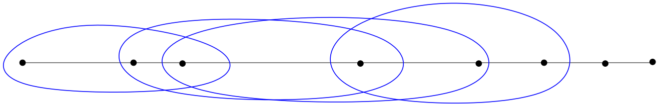

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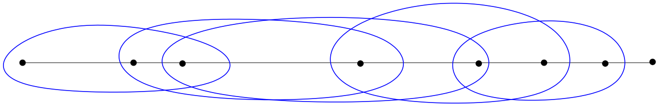

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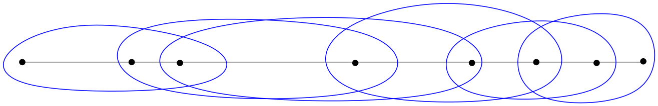


For more colors.

**Theorem (Fox, Pach, Sudakov, S. 2012)**

*For $q \geq 3$, we have*

$$2^{(n/q)^{q-1}} \leq N_3(q,n) \leq 2^{n^{q-1}\log n},$$

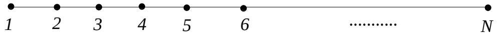
**Application:** Noncrossing convex bodies problem, $NC(n)$.

Obtain the **transitive property** on triples of convex bodies.

$$NC(n) \leq N_3(3,n) \leq 2^{n^2 \log n}$$

**Proof of** $N_3(q,n) \le N_2(n^{q-1}, n) \le n^{n^{q-1}} = 2^{n^{q-1}\log n}$:

1. Set $N = N_2(n^{q-1}, n)$
2. $\chi : \binom{[N]}{3} \to [q]$ be $q$-coloring on the triples of $[N]$.
3. Then define $\phi : \binom{[N]}{2} \to [n]^{q-1}$ as follows. We color $(i,j) \in \binom{[N]}{2}$ with color $(a_1, a_2, ..., a_{q-1})$ where $a_t$ denotes the length of the longest monotone 3-path ending with vertices $i, j$ in color $t$.



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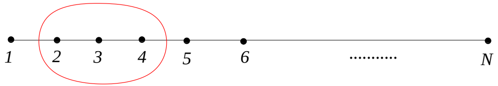

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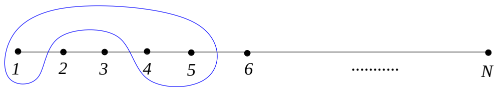

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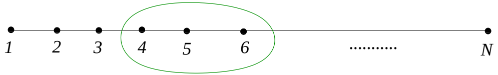

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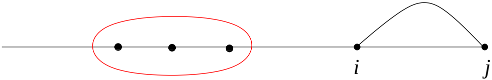

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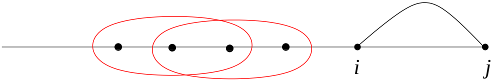

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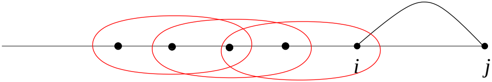

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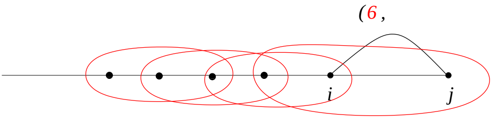

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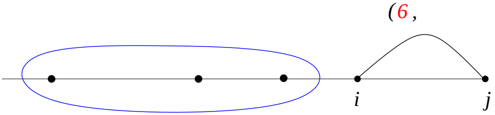

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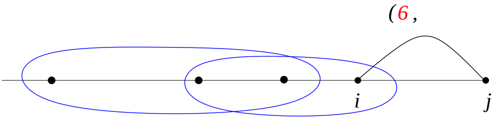

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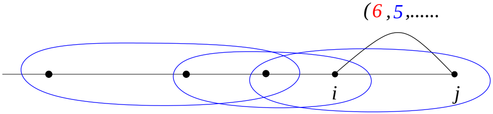
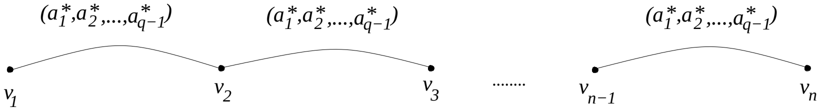


By definition of $N_2(n^{q-1}, n)$, there is monochromatic 2-path on vertices $v_1 < v_2 < ... < v_n$ with color $(a_1^*, ..., a_{q-1}^*)$.



**Claim:** $(v_1, ..., v_n)$ is a monochromatic 3-path (with color $q$)!
Indeed, Assume $(v_i, v_{i+1}, v_{i+2})$ has color $j \neq q$.

1. Longest $j$th-colored 3-path ending with vertices $(v_i, v_j)$ must be shorter than the longest $j$th-colored 3-path ending with vertices $(v_{j+1}, v_{j+2})$.
2. Contradicts $\phi(v_i, v_{i+1}) = \phi(v_{i+1}, v_{i+2})$.
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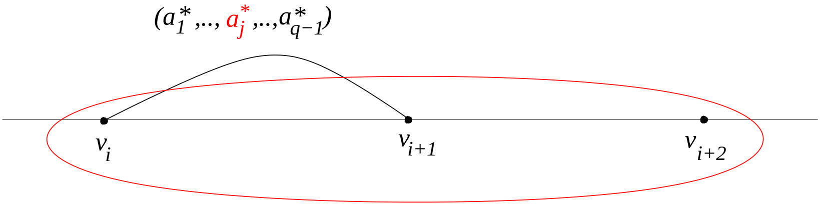

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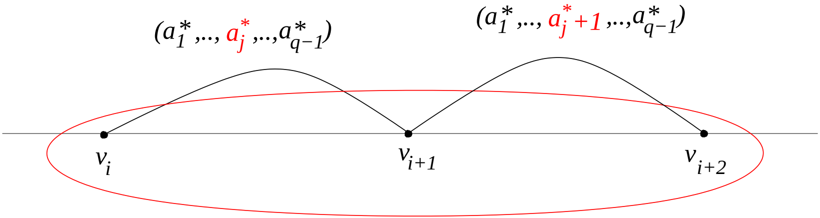


The upper bound proof can easily be generalized to show

$$N_k(q, n) \leq N_{k-1}((n - k + 1)^{q-1}, n)$$

Using the stepping-up approach we have

**Theorem (Fox-Pach-Sudakov-S. 2012)**

*Define $t_1(x) = x$ and $t_{i+1}(x) = 2^{t_i(x)}$. Then for $k \geq 4$ we have*

$$t_{k-1}(cn^{q-1}) \leq N_k(q, n) \leq t_{k-1}(c' n^{q-1} \log n).$$

Recall Ramsey numbers:

$$t_{k-1}(cn^2) \leq R_k(n) \leq t_k(c' n).$$

# Recent development

**Theorem (Moshkovitz and Shapira 2013+)**

*For $k \geq 3$ we have*

$$t_{k-1}(cn^{q-1}) \leq N_k(q,n) \leq t_{k-1}(c'n^{q-1}).$$

Noncrossing convex bodies problems:

$$NC(n) \leq 2^{n^2 \log n} \quad \Rightarrow \quad NC(n) \leq 2^{n^2}$$

$$NC(n) \overset{?}{\leq} 2^n$$

For $k \geq 3$,

$$t_{k-1}(cn^2) \leq R_k(n) \leq t_k(c'n).$$

# Combinatorial Problem

$$2^{cn^2} \leq R_3(n) \leq 2^{2^{c'n}}.$$

**Problem**

*Close the gap on* $R_3(n)$

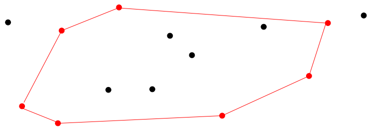
**Conjecture (Erdős, $500 problem)**

$$2^{2^{cn}} \leq R_3(n)$$

**Erdős-Hajnal Stepping Up Lemma:** $x < R_k(n)$, then $2^x \lesssim R_{k+1}(n)$ for $k \geq 3$

Would imply $R_4(n) = 2^{2^{2^{\Theta(n)}}}$, and $R_k(n) = t_k(\Theta(n))$.

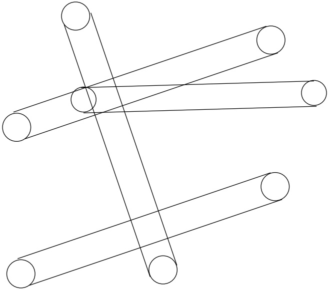
Is there a geometric construction showing $2^{2^{cn}} \leq R_3(n)$?



$V = \{N$ points in the plane in general position$\}$
$E = \{$triples with a clockwise orientation$\}$

Many graphs and hypergraphs defined geometrically.

$V = \{N$ tubes of length $l$ and radius 1 in $\mathbb{R}^d\}$
$E = \{$pairs that intersect$\}$.



**Semi-algebraic hypergraphs**.

# semi-algebraic sets (Tarski cell)

**Definition**

A set $A \subset \mathbb{R}^d$ is called *semi-algebraic* if there are polynomials $f_1, f_2, ..., f_r \in \mathbb{R}[x_1, ..., x_d]$ and a Boolean formula $\Phi(X_1, X_2, ..., X_r)$, where $X_1, ..., X_r$ are variables attaining values "true" and "false", such that

$$A = \left\{ x \in \mathbb{R}^d : \Phi\left(f_1(x) \geq 0, ..., f_r(x) \geq 0\right) \right\}.$$

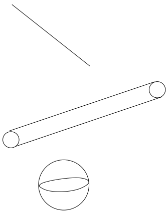
$\Phi$ involves unions, intersections, and complementations. Assume Quantifier-free (Tarski's Theorem).

*A* has *complexity at most t* if $d, r \leq t$ and each $deg(f_i) \leq t$.

Examples: hyperplanes, balls, boxes, tubes, etc. in $\mathbb{R}^d$.

# Encode sets to points

Let $V = \{A_1, ..., A_N\}$ be a family of $N$ semi-algebraic sets in $\mathbb{R}^d$, each set with complexity at most $t$.



$$A_i = \left\{ x \in \mathbb{R}^d : \Phi\left(f_1(x) \geq 0, ..., f_r(x) \geq 0\right) \right\}.$$

**Encode each set:** $A_i \to p_i \in \mathbb{R}^q$ for $q = q(t)$.

$V = \{p_1, ..., p_N\}$, $N$ points in $\mathbb{R}^q$.

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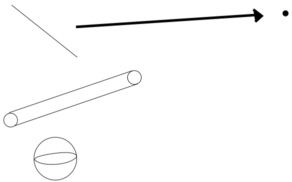

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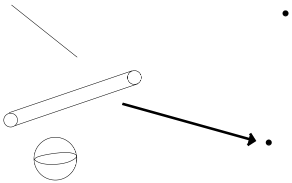

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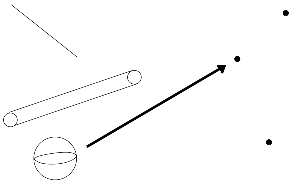

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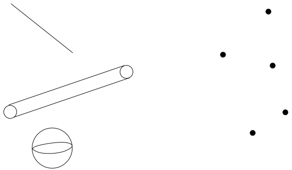

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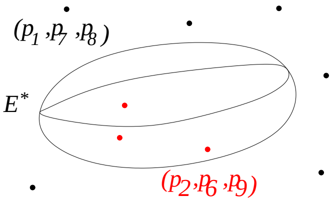
# Semi-algebraic relation

For $V = \{p_1, ..., p_N\} \subset \mathbb{R}^q$, the edge set $E \subset \binom{V}{k}$ is semi-algebraic if $E$ can be described with a constant number of polynomial equations and inequalities (each of bounded degree), and a boolean formula $\Phi$.

# Semi-algebraic relation

For $V = \{p_1, ..., p_N\} \subset \mathbb{R}^q$, the edge set $E \subset \binom{V}{k}$ is semi-algebraic if there exists a semi-algebraic set $E^* \subset \mathbb{R}^{kq}$ with bounded description complexity, such that for $i_1 < \cdots < i_k$

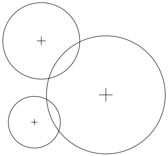
$$(p_{i_1}, ..., p_{i_k}) \in E \Leftrightarrow (p_{i_1}, ..., p_{i_k}) \in E^* \subset \mathbb{R}^{kq}.$$




**Example:** For $k = 3$ look at all triples $(p_{i_1}, p_{i_2}, p_{i_3})$ in $\mathbb{R}^{3q}$.

Call the pair $(V, E)$ a semi-algebraic $k$-uniform hypergraph (with bounded description complexity).

$V = \{A_1, ..., A_N\}$, $N$ disks in the plane. $E = \{$pairs of disks that intersect$\}$.



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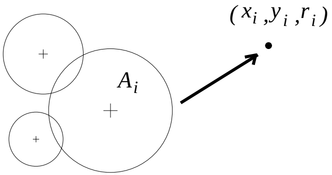

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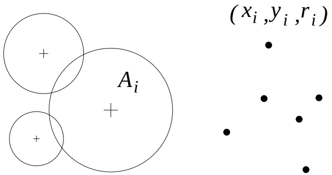

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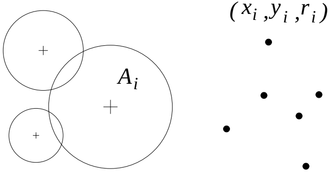
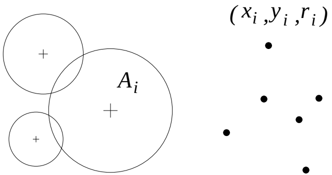



$A_i \to p_i = (x_i, y_i, r_i)$, $A_j \to p_j = (x_j, y_j, r_j)$. $A_i$ and $A_j$ cross if and only if

$$-x_i^2 + 2x_ix_j - x_j^2 - y_i^2 + 2y_iy_j - y_j^2 + r_i^2 + 2r_ir_j + r_j^2 \geq 0.$$

$V = \{A_1, \ldots, A_N\}$, $N$ disks in the plane. $E = \{$ pairs of disks that intersect$\}$.




$(V, E)$ is semi-algebraic graph,

$E^* = \{(z_1, \ldots, z_6) \in \mathbb{R}^6 : f(z_1, \ldots, z_6) \geq 0\}$, where

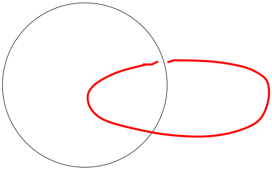
$$f(z_1, \ldots, z_6) = -z_1^2 + 2z_1z_4 - z_4^2 - z_2^2 + 2z_2z_5 - z_5^2 + z_3^2 + 2z_3z_6 + z_6^2.$$

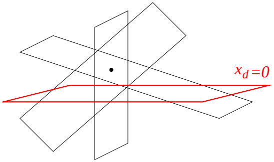
$(p_i, p_j) \in E \Leftrightarrow (p_i, p_j) \in E^*$.

**Examples**

1. $V = \{N \text{ circles in } \mathbb{R}^3\}$
   $E = \{\text{pairs that are linked}\}$.

2. $V = \{N \text{ hyperplanes in } \mathbb{R}^d \text{ in general position}\}$,
   $E = \{d\text{-tuples whose intersection point is above the hyperplane } x_d = 0\}$.





# Results

**Definition:** Let $R_k^{semi}(n)$ be the minimum integer $N$ such that any $N$-vertex semi-algebraic $k$-uniform hypergraph $H = (V, E)$ contains either a clique or an independent set of size $n$. $R_k^{semi}(n) \leq R_k(n)$.

**Theorem (Alon, Pach, Pinchasi, Radoičić, Sharir 2005)**

$$R_2^{semi}(n) \leq n^{c_1}.$$

Applying Milnor-Thom Theorem and Cutting Lemma:

**Theorem (Conlon, Fox, Pach, Sudakov, S. 2012)**

*for* $k \geq 3$,

$$t_{k-1}(c_2 n) \leq R_k^{semi}(n) \leq t_{k-1}(n^{c_1}).$$

**Recall:** for $k \geq 3$, $t_{k-1}(cn^2) \leq R_k(n) \leq t_k(c'n)$.

**Theorem (Conlon, Fox, Pach, Sudakov, S. 2012)**

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**Several applications...**

## Problem (Matoušek-Welzl 1992, Dujmović-Langerman 2011, Matoušek-Eliáš 2012.)

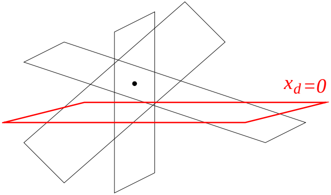
*Determine the minimum integer $OSH_d(n)$, such that any family of at least $OSH_d(n)$ hyperplanes in $\mathbb{R}^d$ in general position, must contain n members such that every d-tuple intersects on one-side of the hyperplane $x_d = 0$.*

$$OSH_2(n) = \Theta(n^2), \qquad OSH_d(n) \leq R_d(n) \leq t_d(c'n).$$

$V = \{\text{N hyperplanes}\}$,

$E = \{d\text{-tuples that intersect above } x_d = 0 \text{ hyperplane}\}$.

**New bound:** $OSH_d(n) \leq R_d^{semi}(n) \leq t_{d-1}(n^{c_1})$



# Combinatorial Problem

Ramsey number of 3-uniform hypergraphs.

$$2^{cn^2} \leq R_3(n) \leq 2^{2^{c'n}}.$$

**Conjecture (Erdős)**

$$2^{2^{cn}} \leq R_3(n)$$

Is there a geometric construction showing $2^{2^{cn}} \leq R_3(n)$?

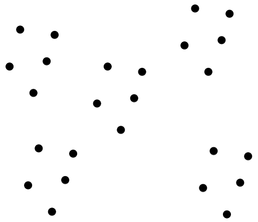
**Our Result:** $R_3^{semi}(n) \leq 2^{n^{c_1}}$.

# Future work

Regularity lemma for semi-algebraic graphs (and hypergraphs).

**Lemma (Regularity Lemma, Szemerédi)**

*Let $G = (V, E)$ be an $N$-vertex graph with $\epsilon N^2$ edges. Then there exists a partition $V = \{V_1, ..., V_M\}$ into $M(\epsilon)$ equal parts, such that all but at most $\epsilon\binom{M}{2}$ pairs of parts are* **regular**.



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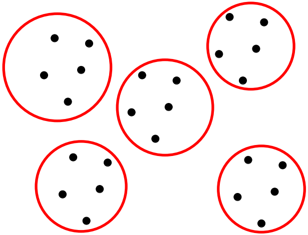

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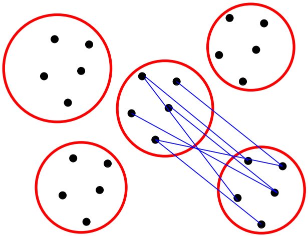

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**Szemerédi:** $M(\epsilon) \leq t_{\frac{1}{\epsilon^5}}(2)$.

**Semi-algebraic graphs:**

1. regular $\stackrel{?}{\rightarrow}$ complete or empty.
2. $M(\epsilon) \stackrel{?}{\leq} \frac{1}{\epsilon^C}$.

Attack other problems in discrete geometry in a semi-algebraic setting.

# Future work

Unit distance problem in $\mathbb{R}^2$ and $\mathbb{R}^3$.

**Conjecture (Erdős, $500)**

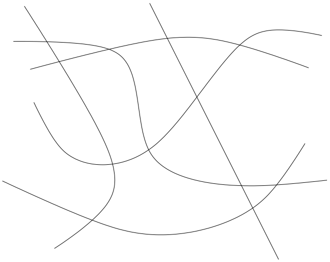
*Given N points in the plane, no more than $N^{1+c/\log\log N}$ pairs can be unit distance apart.*

$V = \{$N points in the plane$\}$
$E = \{$Pairs that are a unit distance apart$\}$

**Thank you!**

1. **Milnor-Thom theorem**: $M$ bounded degree surfaces partitions $\mathbb{R}^q$ into $O(M^q)$ cells.

2. **Cutting lemma** (Chazelle, Edelsbrunner, Guibas, Sharir): Given $M$ bounded degree surfaces $\Sigma$ in $\mathbb{R}^q$ and integer $r$, we can partition $\mathbb{R}^q$ into $O(r^{2q})$ "simple" regions (cells) such that each cell is "crossed" by $O(M/r)$ surfaces from $\Sigma$.



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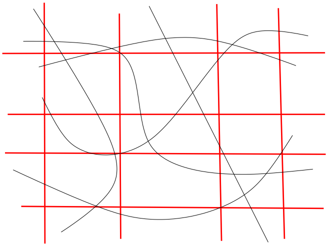

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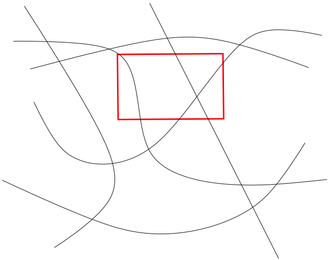


# Key Theorem

Getting an exponential improvement.

$$R_k^{semi}(n) \leq t_{k-1}(n^c).$$

Combining a combinatorial argument + the Milnor-Thom theorem + cutting lemma,

**Theorem (Conlon, Fox, Pach, Sudakov, S. 2012)**

$$R_{k+1}^{semi}(n) \leq 2^{\widetilde{O}(R_k^{semi}(n))}$$

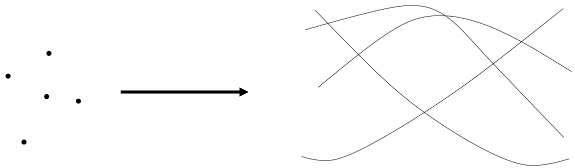
$$R_2^{semi}(n) \leq n^c \qquad R_3^{semi}(n) \leq 2^{n^{c_1}}$$

$$R_4^{semi}(n) \leq 2^{2^{n^{c_1}}}, \ldots.$$

Semi-algebraic $k$-uniform hypergraph $H = (V, E)$,
$V = \{p_1, ..., p_N\} \subset \mathbb{R}^q$,

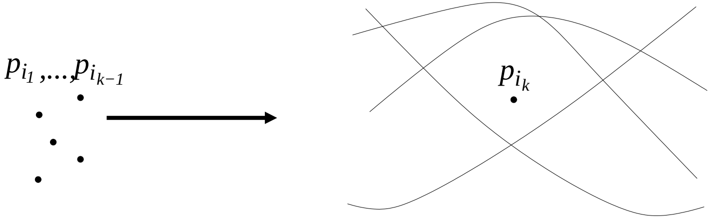
$$E^* = \{(x_1, ..., x_k) \subset \mathbb{R}^{kq} : \Phi(f_1(x_1, ..., x_k) > 0, ..., f_t(x_1, ..., x_k) > 0)\}$$

Every $k-1$-tuple of points, $p_{i_1}, ..., p_{i_{k-1}}$, gives rise to $t$ bounded degree surfaces in $\mathbb{R}^q$.



$$\{f_1(p_{i_1}, ..., p_{i_{k-1}}, x_k) = 0\}, ..., \{f_t(p_{i_1}, ..., p_{i_{k-1}}, x_k) = 0\} \subset \mathbb{R}^q.$$

$(p_{i_1}, p_{i_2}, ..., p_{i_k}) \in E$??




sign pattern $(f_1(p_{i_1}, p_{i_2}, ..., p_{i_k}), ..., f_t(p_{i_1}, p_{i_2}, ..., p_{i_k}))$. I.e. $(+,-,+,0,+,+)$.

$$E^* = \{(x_1, ..., x_k) \subset \mathbb{R}^{2q} : \Phi(f_1(x_1, ..., x_k) > 0, ..., f_t(x_1, ..., x_k) > 0)\}$$

**Our problem is about:** $N$ points in $\mathbb{R}^q$ and $M = t\binom{N}{k-1}$ bounded degree surfaces in $\mathbb{R}^q$.