

# Old and new developments on the Erdős-Szekeres convex polygon problem

Andrew Suk (UC San Diego)

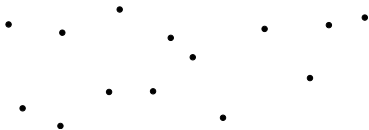
July 6, 2024

# Origins of Ramsey theory

“A combinatorial problem in geometry,” by Paul Erdős and George Szekeres (1935)

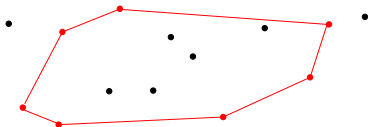


1) Rediscovered Ramsey's theorem, 2) Monotone subsequence theorem, 3) Convex polygon theorem.



### Problem (Esther Klein 1933)

*Given an integer  $n$ , is there a minimal integer  $ES(n)$ , such that any set of at least  $ES(n)$  points in the plane in general position, contains  $n$  members in convex position?*

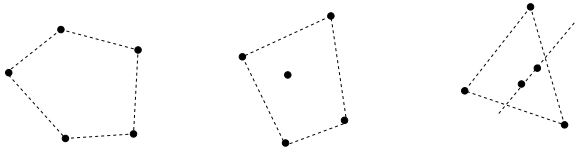


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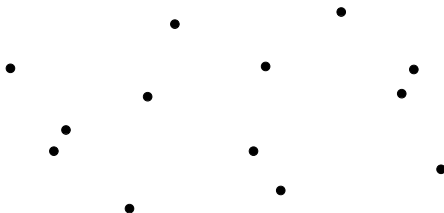
Theorem (Erdős-Szekeres, 1935)

*For every  $n \geq 3$ ,  $ES(n)$  exists.*



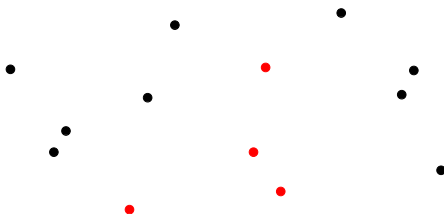
Theorem (Erdős-Szekeres, 1935)

*For every  $n \geq 3$ ,  $ES(n) \leq R_4(5, n)$ . Set  $N = R_4(5, n)$ .*



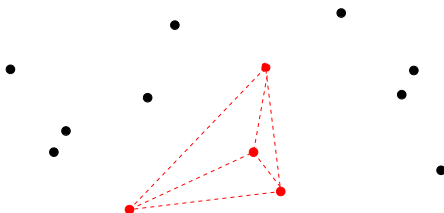
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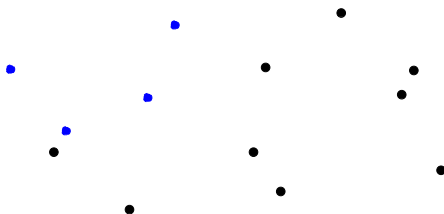
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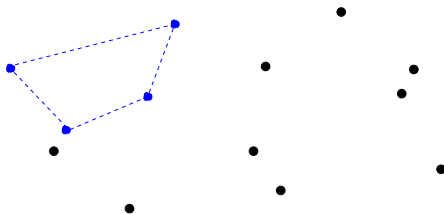
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# First proof

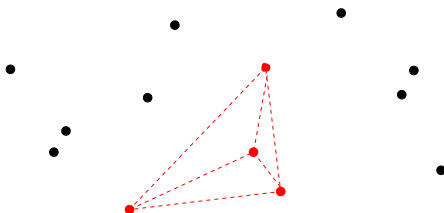
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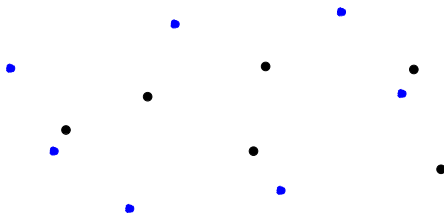
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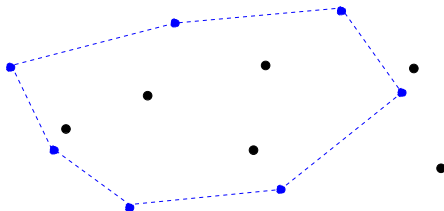
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Theorem (Erdős-Rado, 1952)

$$ES(n) \leq R_4(5, n) < 2^{2^{cn^4}}.$$

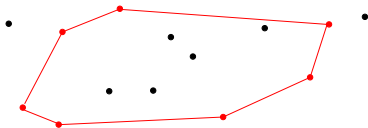
Theorem (Colon-Fox-Sudakov, 2008)

$$ES(n) \leq R_4(5, n) < 2^{2^{cn^2 \log n}}.$$

Theorem (Mubayi-S., 2018)

$$R_4(5, n) > 2^{n^{c \log n}}.$$

## Second Proof



Theorem (Erdős-Szekeres 1935, 1960)

$$2^{n-2} + 1 \leq ES(n) \leq \binom{2n-4}{n-2} + 1 = O(4^n / \sqrt{n}).$$

**Conjecture:**  $ES(n) = 2^{n-2} + 1, n \geq 3$ .

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Trivial:  $ES(3) = 3$

Klein (1933):  $ES(4) = 5$

Makai and Turán (1935):  $ES(5) = 9$

Szekeres-Peters (2006):  $ES(6) = 17$ .

For  $n \geq 7$ ,  $ES(n)$  is still unknown.

# Towards the conjecture $ES(n) = 2^{n-2} + 1$

1935, Erdős-Szekeres:  $\binom{2n-4}{n-2} + 1$

1998, Chung-Graham:  $\binom{2n-4}{n-2}$

1998, Kleitman-Pachter:  $\binom{2n-4}{n-2} - 2n + 7$

1998, Tóth-Valtr:  $\binom{2n-5}{n-2} + 2 \sim \frac{1}{2} \binom{2n-4}{n-2}$

2005, Tóth-Valtr:  $\binom{2n-5}{n-2} + 1$

2015, Norin-Yuditsky and Mojarrad-Vlachos:  $\limsup_{n \rightarrow \infty} \frac{ES(n)}{\binom{2n-4}{n-2}} \leq \frac{7}{16}$ .

$2^{n-2} + 1 \leq ES(n) \leq 4^{n-o(n)}.$

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**2017, S.:**  $ES(n) = 2^{n+o(n)}$ .

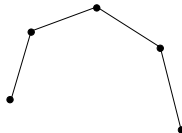
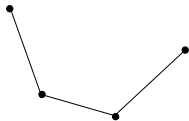
# The original upper bound

$$ES(n) \leq \binom{2n-4}{n-2} + 1.$$

## Theorem (Cups-Caps Theorem)

Let  $f(k, \ell)$  be the smallest integer  $N$  such that any  $N$ -element point set in the plane in general position contains either a  $k$ -cup or an  $\ell$ -cap. Then

$$f(k, \ell) = \binom{k + \ell - 4}{k - 2} + 1.$$



# Cups-caps construction

$$f(k, \ell) \geq f(k-1, \ell) + f(k, \ell-1) - 1$$

$$f(k, \ell-1) - 1$$

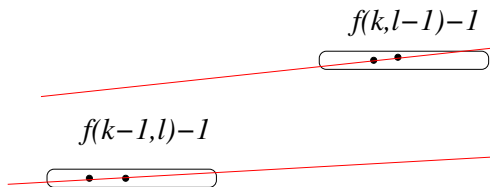


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## Theorem (S. 2017)

For  $n \geq n_0$ , where  $n_0$  is a large absolute constant

$$ES(n) \leq 2^{n+2n^{3/4}}.$$

### Basic Idea:

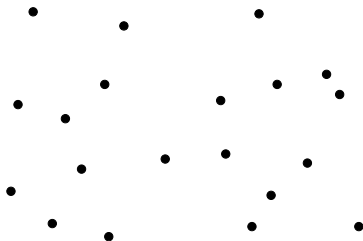
- 1 Find large clusters of points in convex position (cyclic polytope)
- 2 Glue together cups and caps between the clusters.

## Theorem (S. 2017)

For  $n \geq n_0$ , where  $n_0$  is a large absolute constant

$$ES(n) = 2^{n+o(n)}.$$

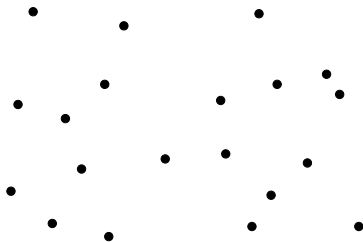
**Proof.**  $|P| = 2^{n+2n^{3/4}}.$



## Theorem (Pach-Solymosi 1998, Pór-Valtr 2002)

For  $|P| \geq 16^k$ , there is a  $k$ -element subset  $X \subset P$  such that  $X$  is either a  $k$ -cup or a  $k$ -cap, and the regions  $T_1, \dots, T_{k-1}$  from the support of  $X$  satisfies  $|T_i \cap P| \geq \frac{|P|}{2^{40k}}$ .

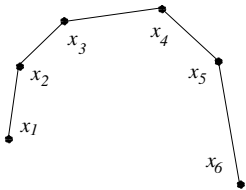
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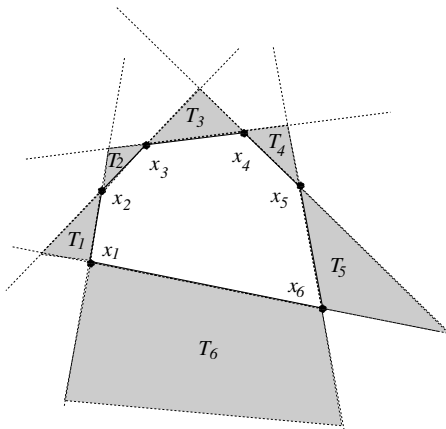
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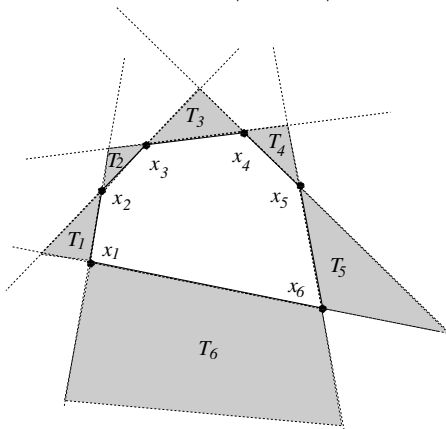
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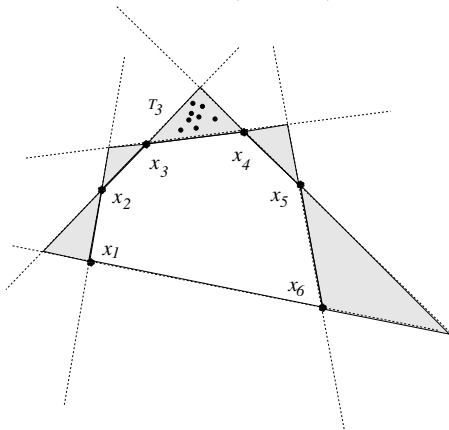
**Proof.**  $|P| = 2^{n+2n^{3/4}}$ .  $k = n^{2/3}$ ,  $|T_i \cap P| \geq 2^{n+2n^{3/4}-40n^{2/3}}$ .



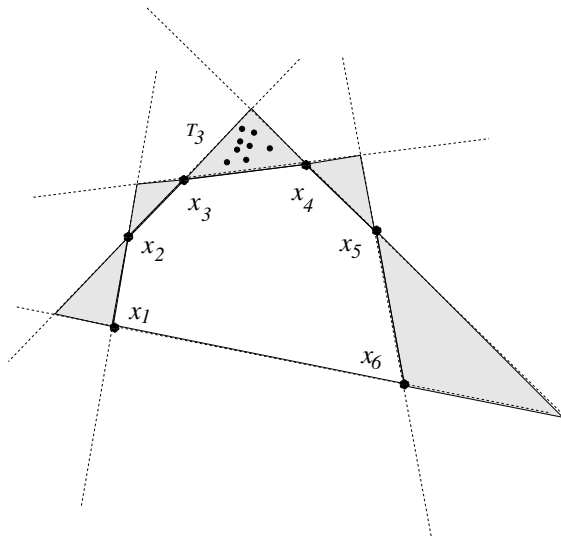
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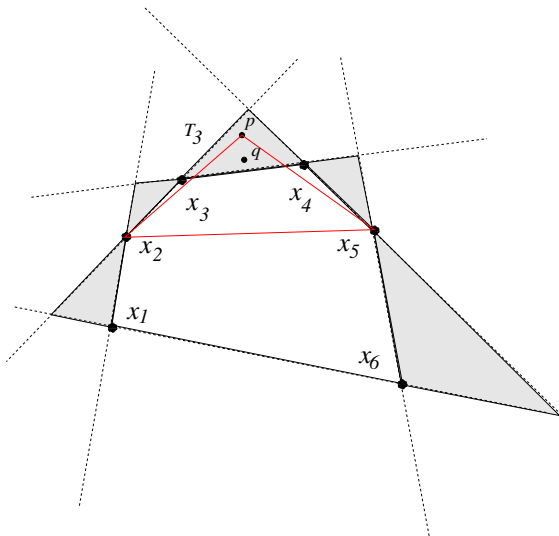
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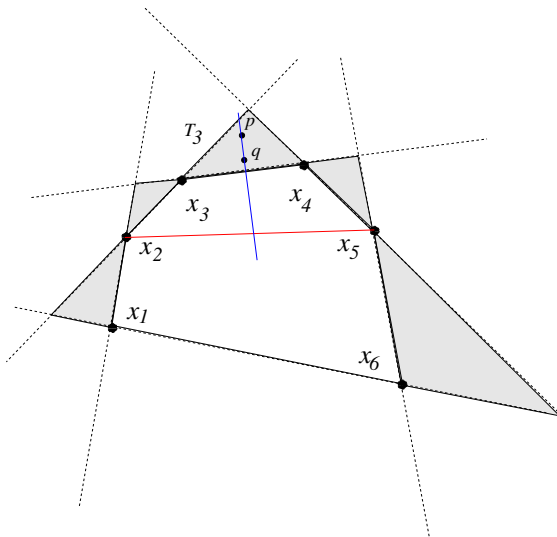
Define partial order of  $P_i = T_i \cap P$ , where  $p \prec q$  iff  $q \in \text{conv}(p \cup x_{i-1}x_{i+2})$



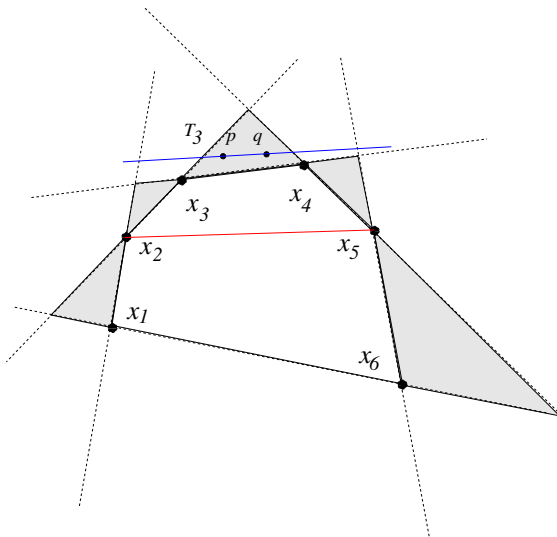
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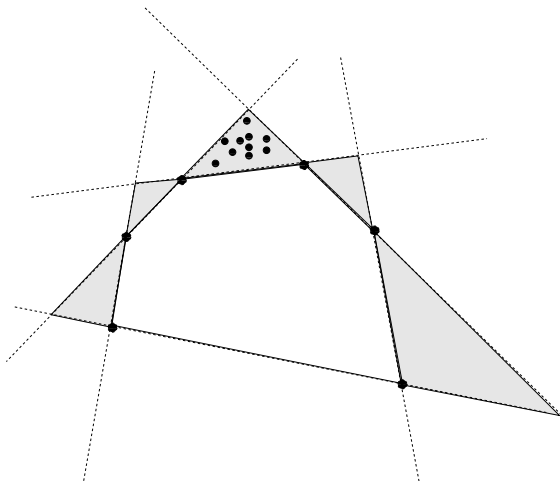
$p \not\sim q$



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Dilworth's Theorem: Each  $P_i$  contains either

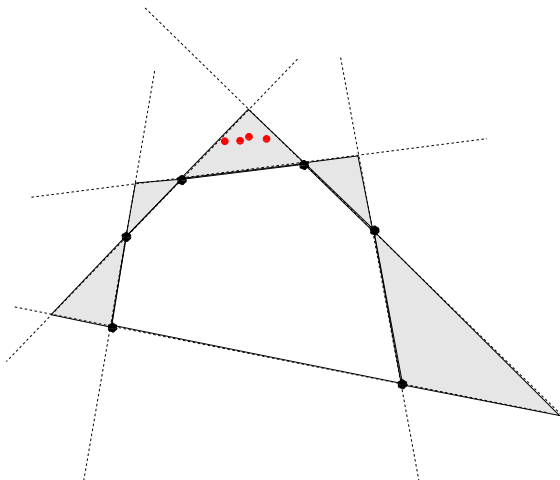
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- ② Chain size  $\left(\frac{N}{2^{40k}}\right)^{1-\alpha} = 2^{n+n^{3/4}-40n^{2/3}-2n^{1/2}}$



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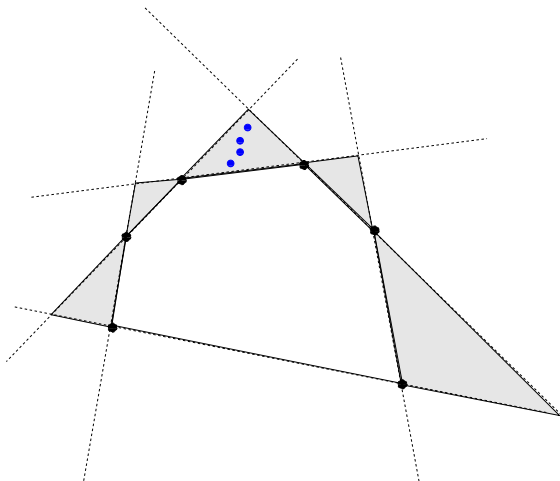
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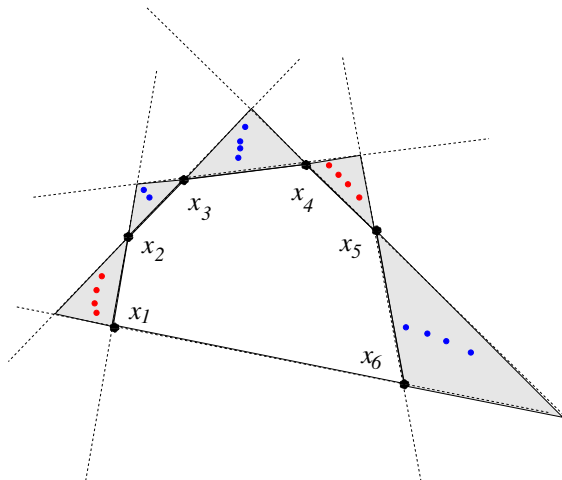
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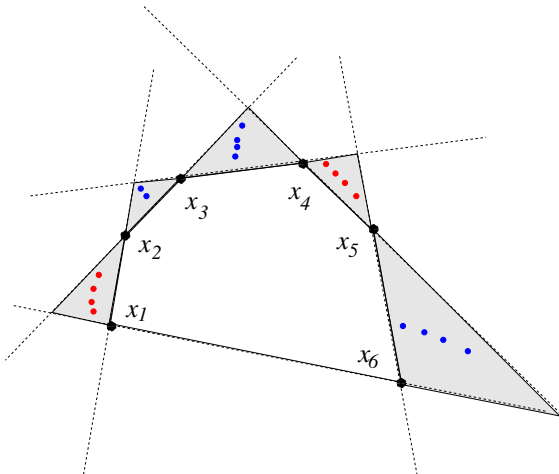
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$k = n^{2/3}$  regions.

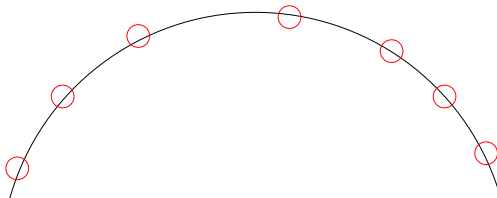
By Pigeonhole:

- 1  $\frac{\sqrt{k}}{2}$  nonadjacent regions have **antichains** of size  $\left(\frac{N}{2^{40k}}\right)^\alpha$ .
- 2  $\sqrt{k}$  consecutive regions have **chains** of size  $\left(\frac{N}{2^{40k}}\right)^{1-\alpha}$



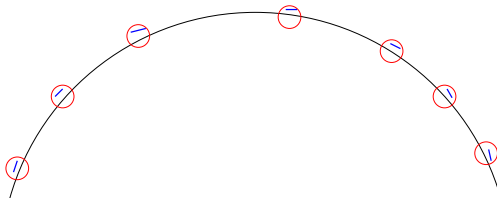
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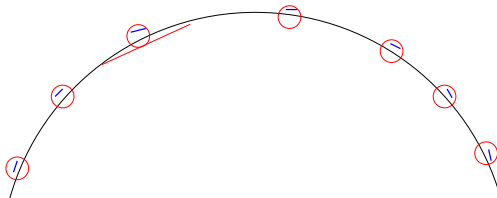
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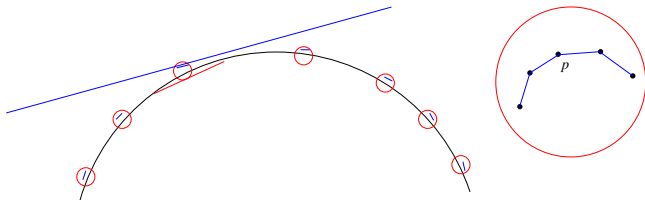
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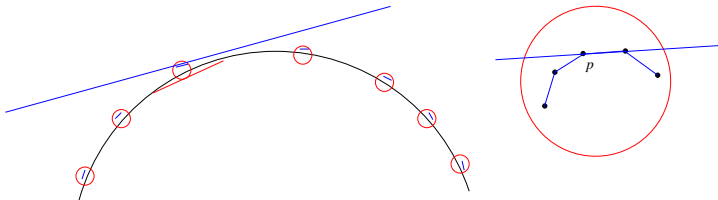
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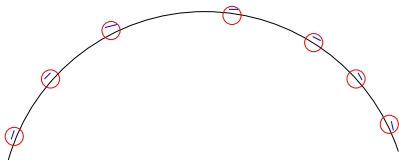
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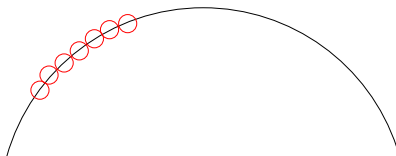


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# Many large mutually avoiding sets

Case 2:  $\sqrt{k} = n^{1/3}$  of the  $P_i$ -s are consecutive chains.

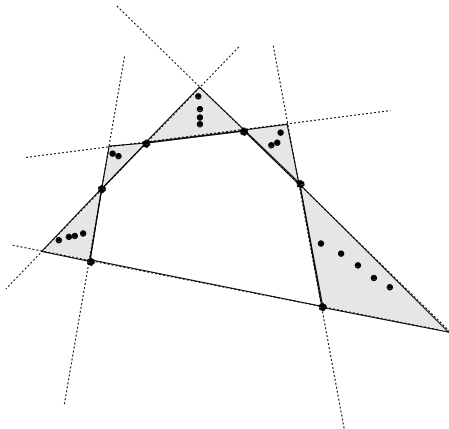
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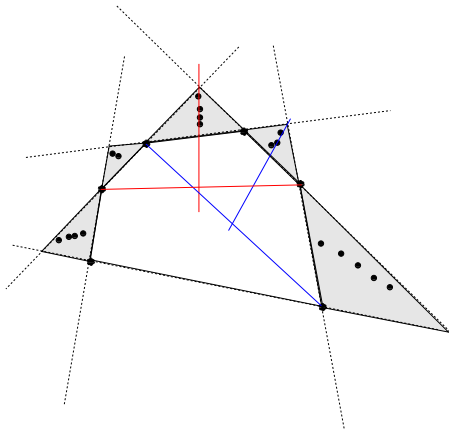
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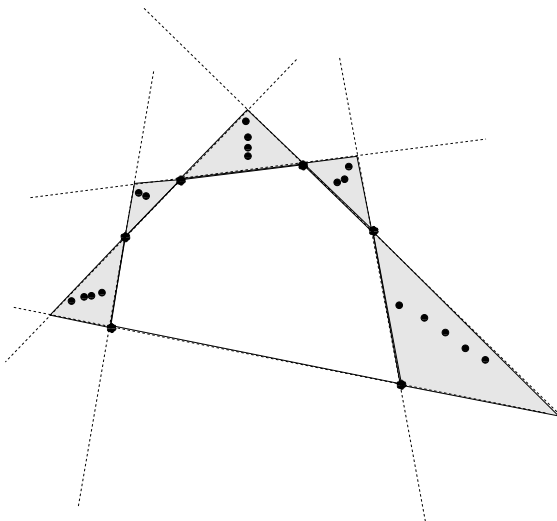
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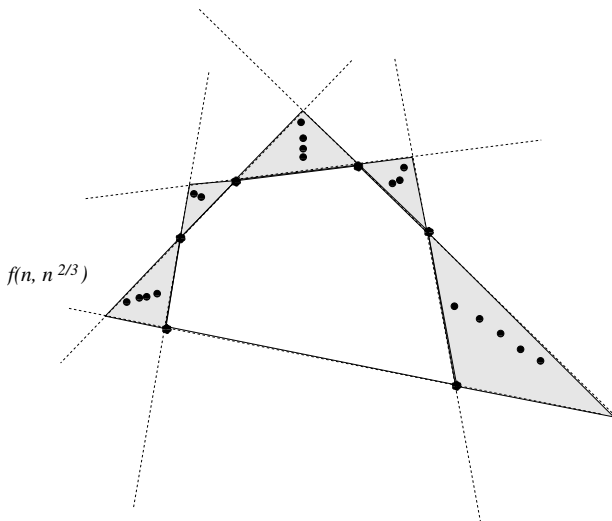
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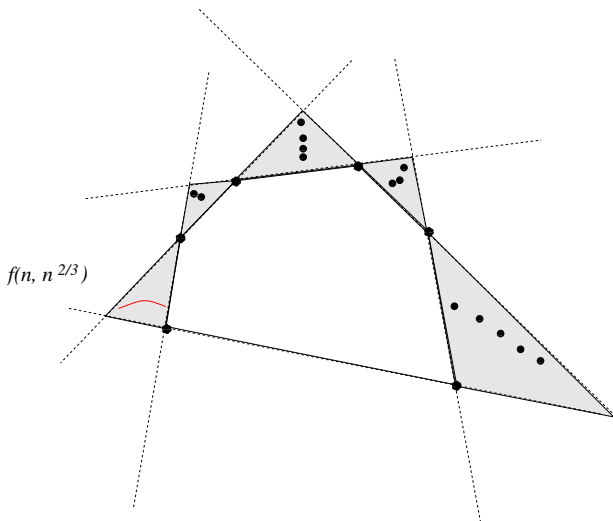
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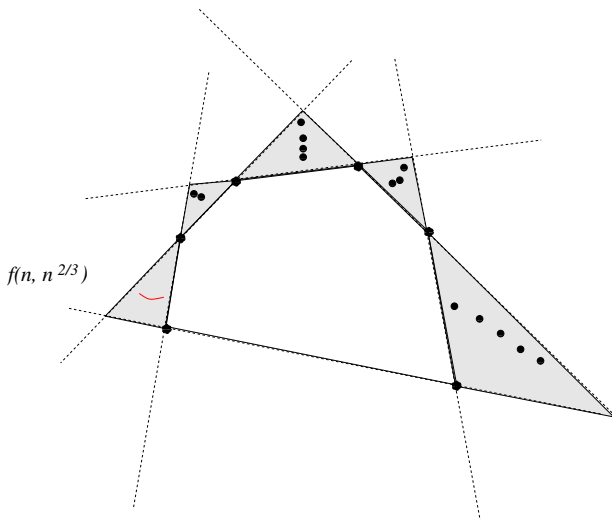
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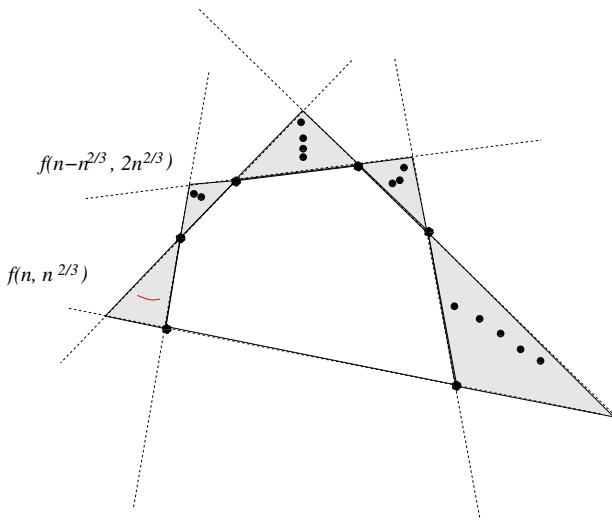
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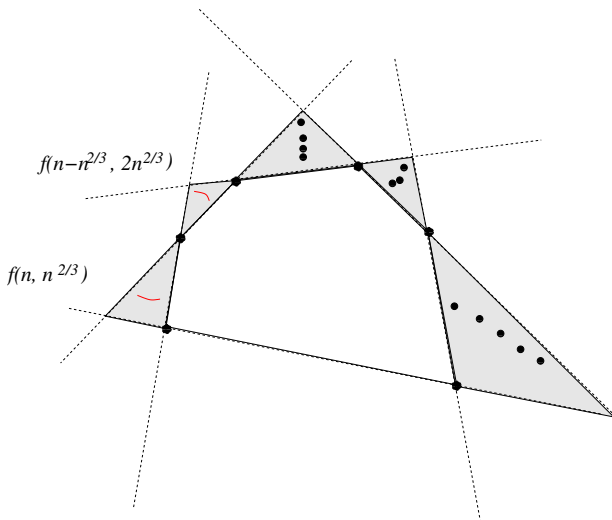
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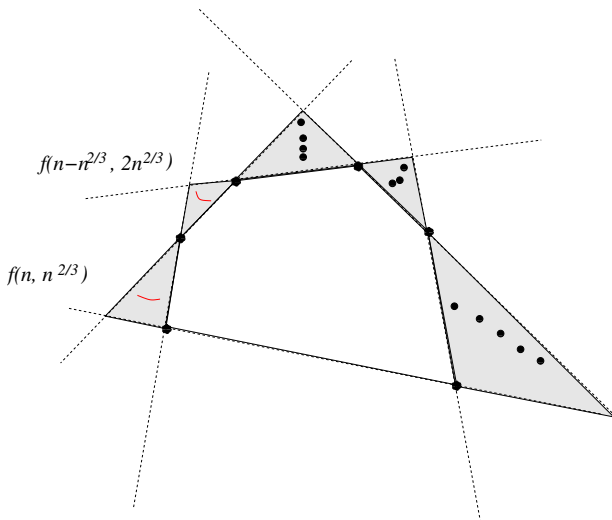
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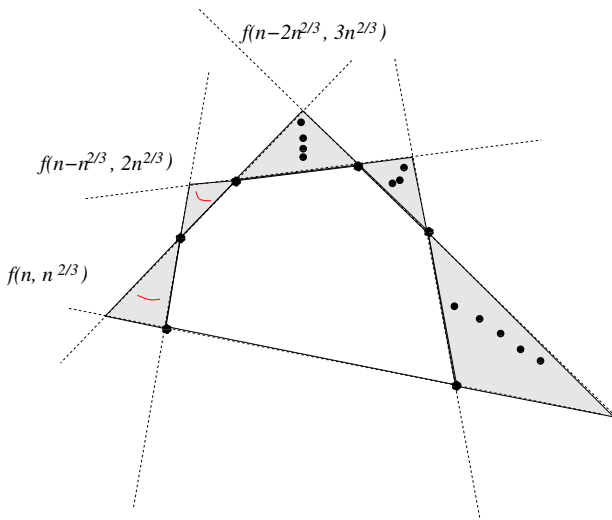
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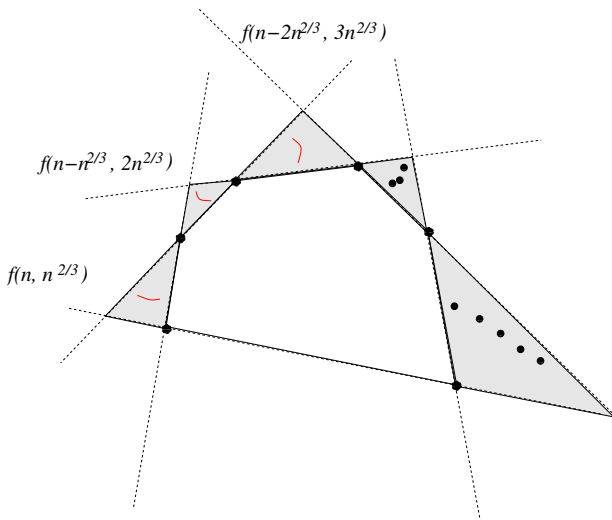
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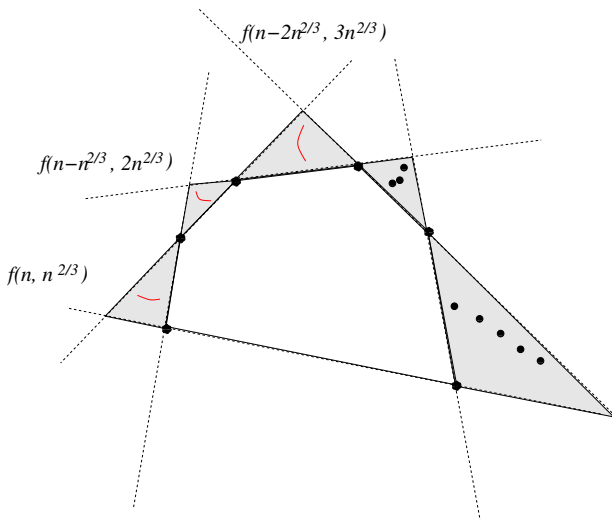
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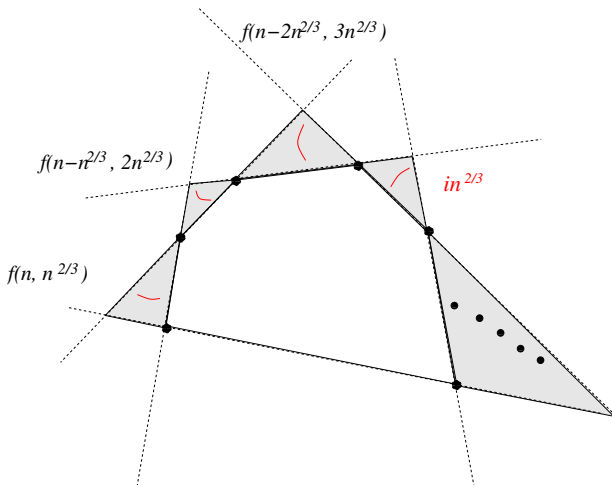


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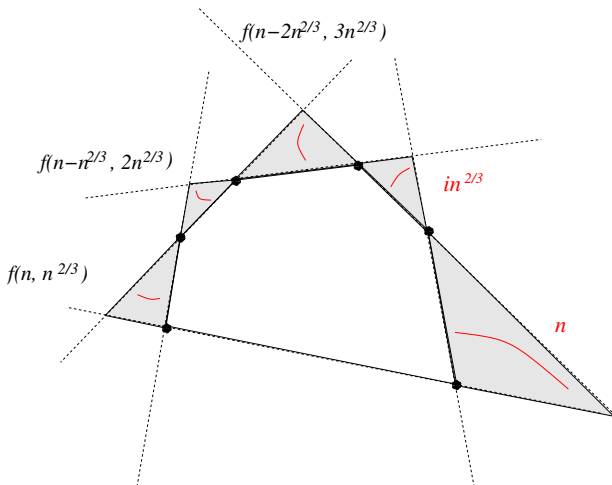


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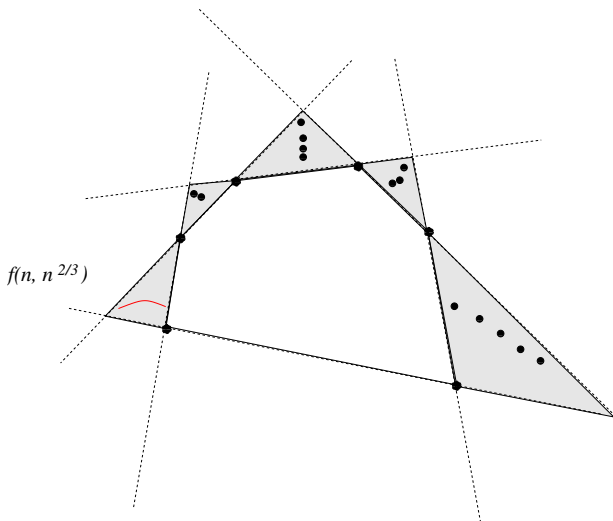
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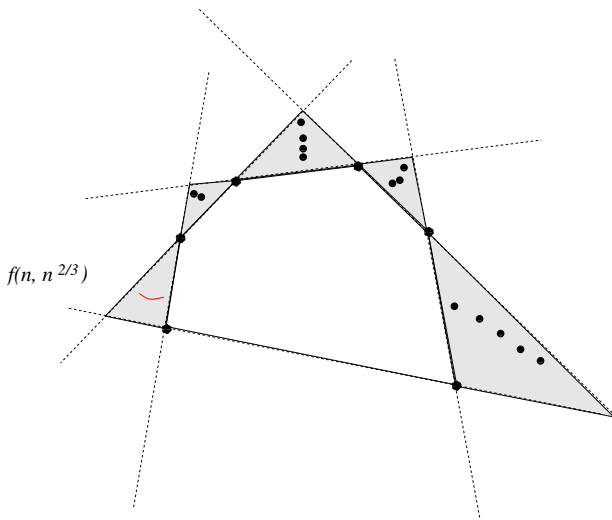
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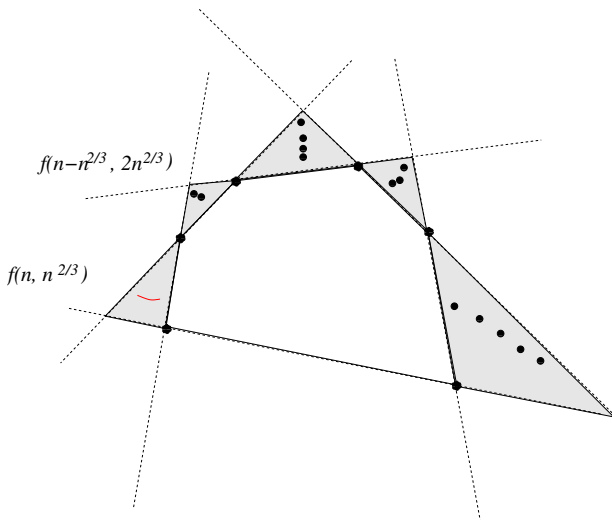
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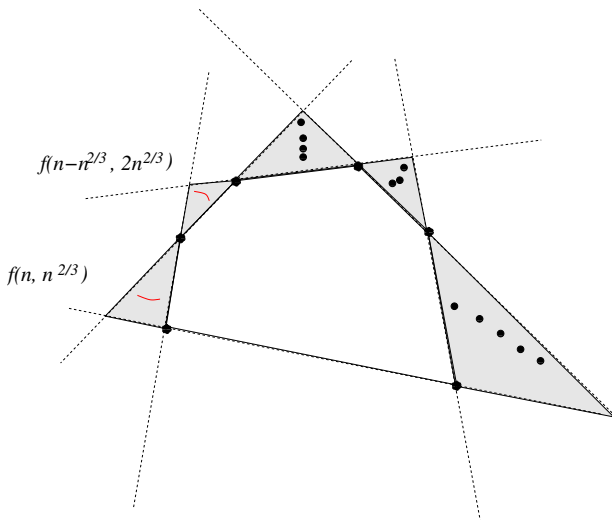
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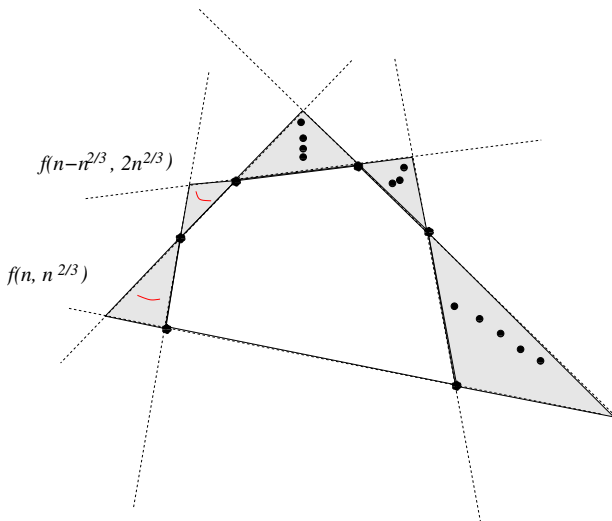
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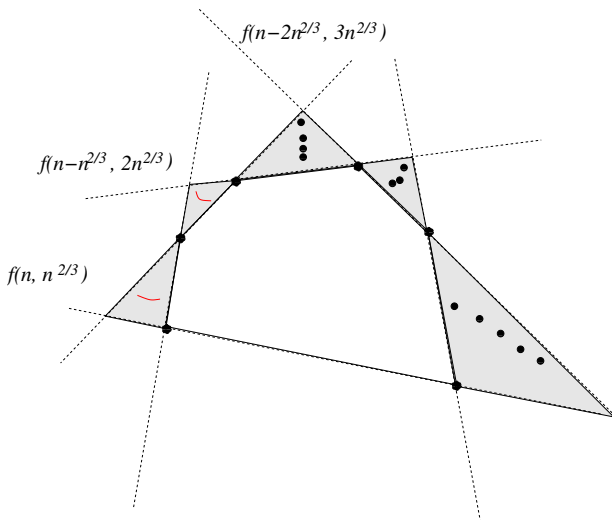
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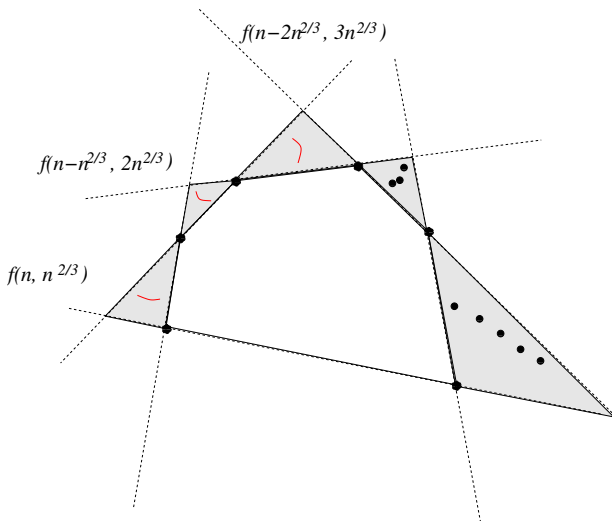
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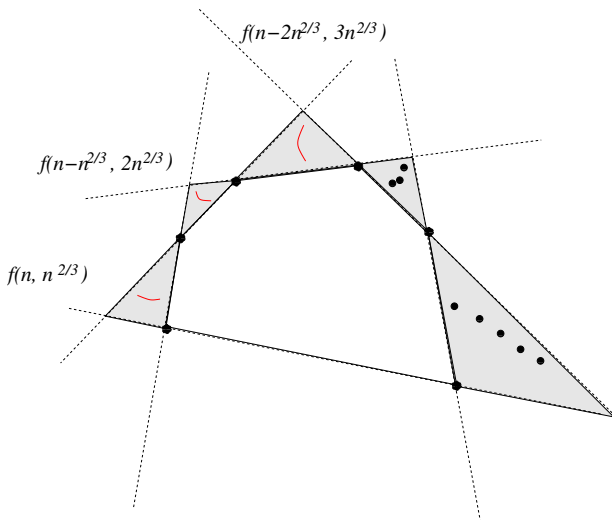
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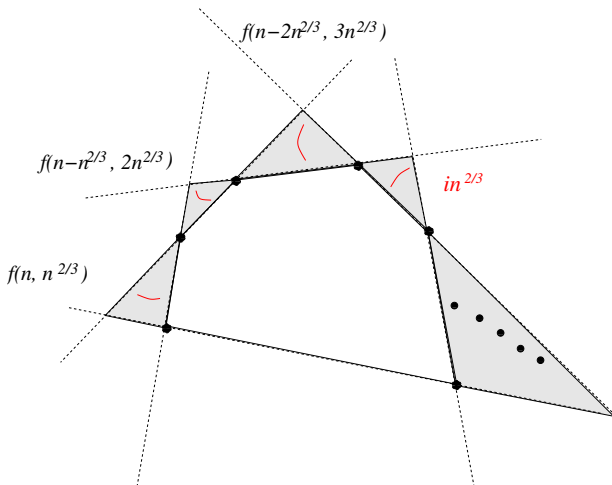


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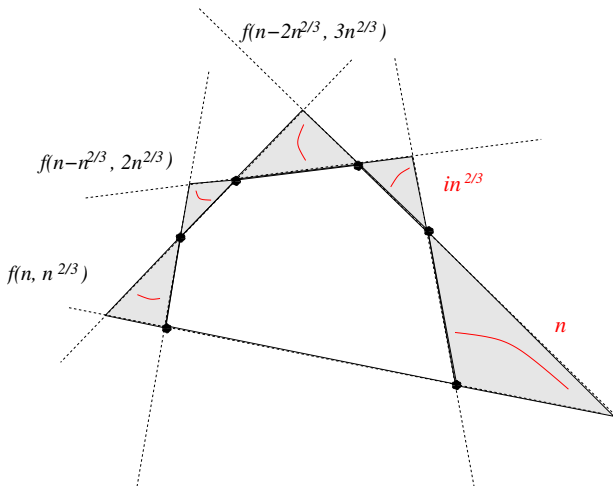


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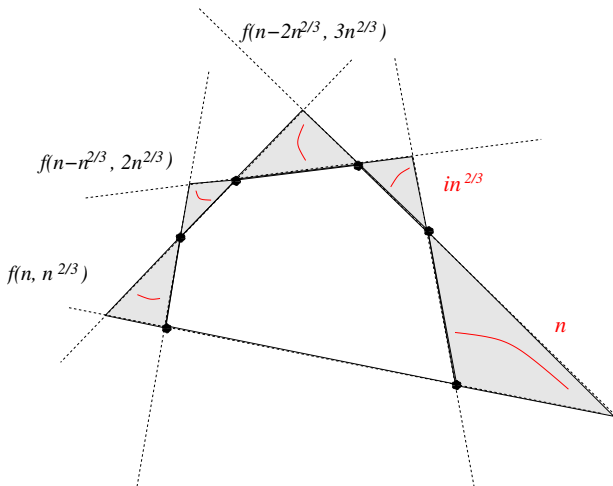


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*For  $n \geq n_0$ , where  $n_0$  is a large absolute constant*

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$$ES(n) \leq 2^{n+O(\sqrt{n \log n})}.$$

# Two extensions of the Erdős-Szekeres theorem

- 1 Point sets in higher dimensions
- 2 Point sets in the plane NOT NECESSARILY in general position

# Higher dimensions

$ES(d; n)$  = smallest integer such that any set of  $ES(d; n)$  points in  $\mathbb{R}^d$  in general position contains  $n$  members in convex position.

Theorem (Károlyi 2001)

$$ES(d; n) \leq ES(d - 1; n - 1) + 1.$$

$$ES(d; n) \leq ES(n - d + 2) + d - 2 \leq 2^{n+o(n)}.$$

Conjecture (Füredi)

$$ES(3; n) = 2^{\Theta(\sqrt{n})}.$$

$$ES(d; n) = 2^{\Theta(n^{1/(d-1)})}.$$

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$$ES(d; n) = 2^{\Theta(n^{1/(d-1)})}.$$

## Theorem (Károlyi-Valtr 2003)

$$ES(d; n) \geq 2^{cn^{1/(d-1)}}.$$

$$ES(d; n) \leq 2^{n+o(n)}.$$

# New upper bounds



Theorem (Pohoata-Zakarhov 2024+)

For  $d \geq 3$ ,

$$ES(d; n) < ES(3; n) \leq 2^{o(n)}.$$



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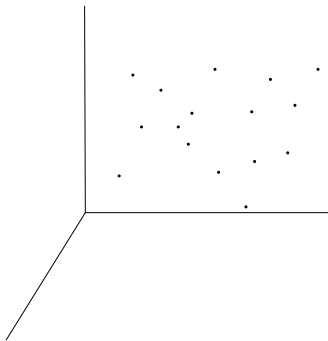
- 1 Apply cups-caps in  $\mathbb{R}^3$ .
- 2 Find large clusters of points in cyclic position.
- 3 Glue together cups and caps between the clusters.

# Rough Idea

$P = N$  points in  $\mathbb{R}^3$ .

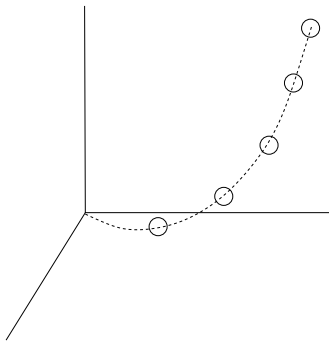
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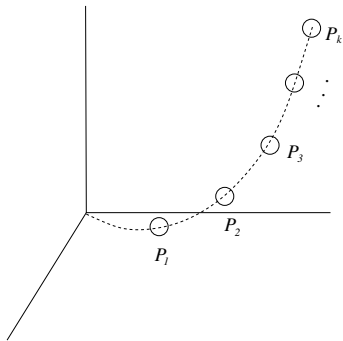
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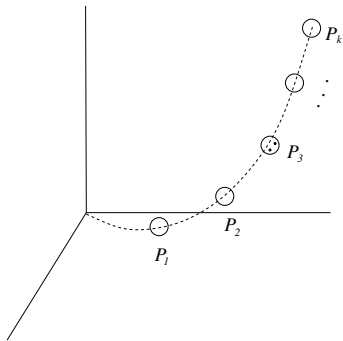
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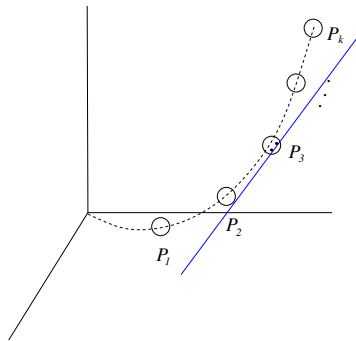
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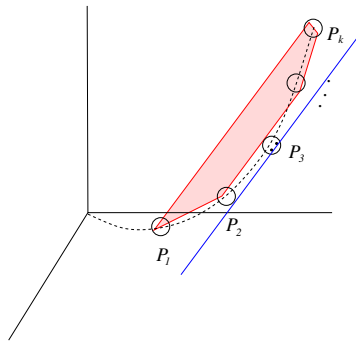
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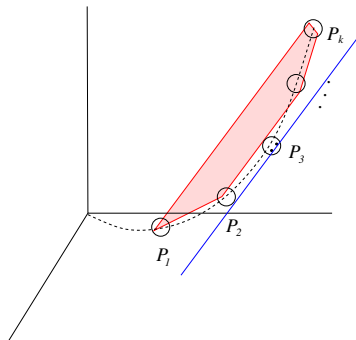
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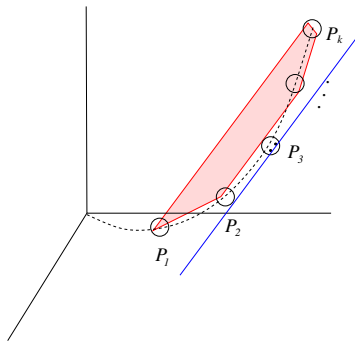
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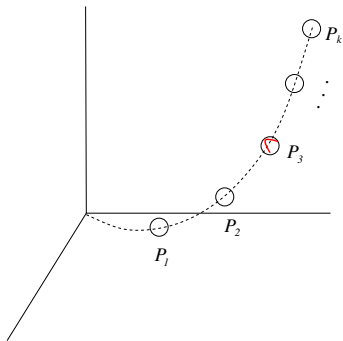
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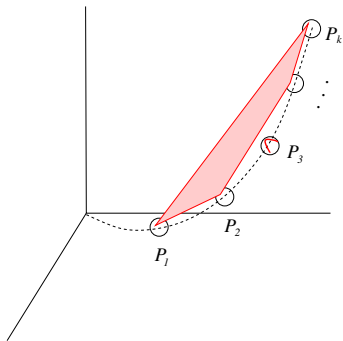
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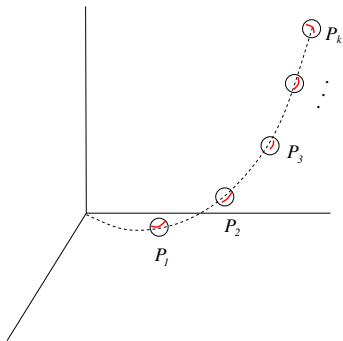
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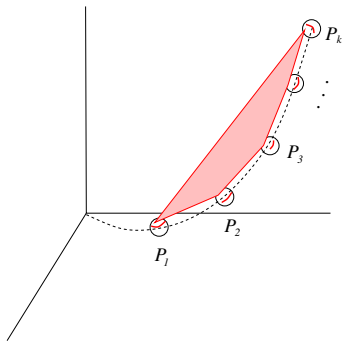
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**Current Status.** Károlyi-Valtr 2003, Zakharov-Pohoata 2024

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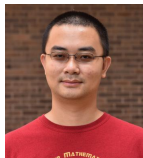
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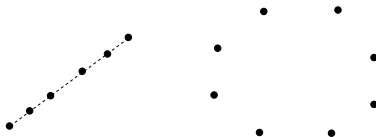
# Big line or big convex polygon

**AIM Squares workshop:** David Conlon, Jacob Fox, Xiaoyu He, Dhruv Mubayi, Jacques Verstraëte



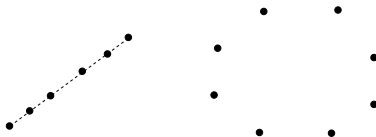
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Let  $ES_\ell(n)$  be the minimum integer  $N$  such that any set of  $N$  points in the plane contains  $\ell$  collinear points, or  $n$  points in convex position.



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Every sufficiently large point set in the plane contains  $\ell$  collinear members, or  $n$  points in convex position.

# Upper bound for $ES_\ell(n)$

$\ell = 4$  collinear points or  $n$  points in convex position.

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$ES_4(n) < (ES(n))^2 = 4^{n+o(n)}.$

# Big line or big convex polygon

Theorem (Conlon-Fox-He-Mubayi-S.-Verstraete 2024+)

*There is an absolute constant  $C$  such that for each  $n, \ell \geq 3$ ,*

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$$ES_{\ell}(n) \geq (3\ell - 1)2^{n-5} + 1.$$

# Big line or big convex polygon

Theorem (Conlon-Fox-He-Mubayi-S.-Verstraete 2024+)

*There is an absolute constant  $C$  such that for each  $n, \ell \geq 3$ ,*

$$ES_{\ell}(n) \leq \ell^2 \cdot 2^{n+C\sqrt{n\log n}}.$$

Theorem (Conlon-Fox-He-Mubayi-S.-Verstraete 2024+)

*For each  $n, \ell \geq 3$ ,*

$$ES_{\ell}(n) \geq (3\ell - 1)2^{n-5} + 1.$$

**Conjecture.**  $ES_{\ell}(n) \leq \ell \cdot 2^{n+C\sqrt{n\log n}}.$

**Thank you!**