Hasse diagrams with large chromatic number

Andrew Suk* István Tomon†

Abstract

‘ For every positive integer n, we construct a Hasse diagram with n vertices and independence number $O(n^{3/4})$. Such graphs have chromatic number $\Omega(n^{1/4})$, which significantly improves the previously best known constructions of Hasse diagrams having chromatic number $\Theta(\log n)$. In addition, if we also require girth of at least $k \geq 5$, we construct such Hasse diagrams with independence number at most $n^{1-\frac{1}{2k-4}+o(1)}$. The proofs are based on the existence of point-line arrangements in the plane with many incidences and avoids certain forbidden subconfigurations, which we find of independent interest.

These results also have the following surprising geometric consequence. They imply the existence of a family $\mathcal{C}$ of n curves in the plane such that the disjointness graph G of $\mathcal{C}$ is triangle-free (or has high girth), but the chromatic number of G is polynomial in n. Again, the previously best known construction, due to Pach, Tardos and Tóth, had only logarithmic chromatic number.

1 Introduction

Let G be a graph. The independence number of G is denoted by $\alpha(G)$, the clique number of G is $\omega(G)$, and the chromatic number of G is $\chi(G)$. Also, the *girth* of G is the length of its shortest cycle.

Given a triangle-free graph on n vertices, how large can its chromatic number be? This question is closely related to the problem of determining the asymmetric Ramsey number $R(3, m)$, which is the maximum number of vertices in a triangle-free graph with no independent set of size m. It was proved by Ajtai, Komlós and Szemerédi [1] that $R(3, m) = O(\frac{m^2}{\log m})$, and Kim [17] proved a matching lower bound with the help of probabilistic tools. Finding explicit constructions of triangle-free graphs with no independent set of size m has also received a lot of attention over the past several decades. The best known explicit constructions of such graphs have $\Theta(m^{3/2})$ vertices, see [2, 8, 20]. Moreover, explicit constructions for other asymmetric Ramsey numbers $R(s, m)$, where s is viewed as a fixed constant, are considered by Alon and Pudlák in [3].

The aforementioned results imply that the independence number of a triangle-free graph on n vertices is at least $n^{1/2+o(1)}$, the chromatic number of such a graph is at most $n^{1/2+o(1)}$, and these bounds are best possible. However, the question whether these bounds can be improved for specific families of triangle-free graphs has been extensively studied, and one such family of particular interest is the family of Hasse diagrams.

*UC San Diego. *e-mail*: **asuk@ucsd.edu**, Research supported by an NSF CAREER award, NSF award DMS-1952786, and an Alfred Sloan Fellowship.

†ETH Zurich. *e-mail*: **istvan.tomon@math.ethz.ch**, Research supported by SNSF grant 200021_196965.

Hasse diagrams were introduced by Vogt [35] at the end of the 19th century for concise representation of partial orders. Today, they are widely used in graph drawing algorithms. Given a partially ordered set $(P,<)$, its *Hasse diagram* (or *cover graph*) is the graph on vertex set P in which x and y are joined by an edge if $x<y$ and there exists no $z\in P$ such that $x<z<y$. It is easy to see that if G is a Hasse diagram, then G is triangle-free. It is already not trivial that the chromatic number of Hasse diagrams can be arbitrarily large, but the following construction of Erdős and Hajnal [10] shows just that. The *shift graph of order* N is the graph G with vertex set $\{(i,j): 1\leq i<j\leq N\}$ in which (i,j) and (i',j') are joined by an edge if $j=i'$. Then G is the Hasse diagram of the poset defined as $(i,j)\leq(i',j')$ if $j\leq i'$; also it is a nice exercise to show that $\chi(G)=\lfloor \log_2 N\rfloor$. Bollobás [6] proved that there exist Hasse diagrams (in particular, Hasse diagrams of lattices) with arbitrarily large girth and chromatic number. Nešetřil and Rödl [26] gave an alternative proof of this result. The construction of Bollobás, when optimized, implies the existence of Hasse diagrams with n vertices, girth at least k (where k is any fixed integer) and chromatic number $\Theta(\frac{\log n}{\log\log n})$. Recently, under the same conditions, the bound on the chromatic number was improved to $\Omega(\log n)$ by Pach and Tomon [28], and they showed that this bound is optimal for the Hasse diagrams of so-called uniquely generated posets. In [21], it is proved that there exist Hasse diagrams of 2-dimensional posets with arbitrarily large chromatic number, and in [7], it is proved that the chromatic number of Hasse diagrams of 2-dimensional posets can be as large as $\Omega(\frac{\log n}{(\log\log n)^2})$, where n is the number of vertices. Felsner et al. [12] considered Hasse diagrams of interval orders and N-free posets of height h, and proved that the maximum chromatic number of such Hasse diagrams is logarithmic in h.

All of the aforementioned constructions of n vertex Hasse diagrams have chromatic number $O(\log n)$. Therefore, one might suspect that the chromatic number of Hasse diagrams is at most logarithmic. Our main result shows that this intuition is false, by constructing Hasse diagrams whose independence number is very small.

Theorem 1. *For every positive integer n, there exists a Hasse diagram with n vertices and independence number $O(n^{3/4})$.*

As an immediate corollary, we have the following.

Corollary 2. *For every positive integer n, there exists a Hasse diagram with n vertices and chromatic number $\Omega(n^{1/4})$.*

Unlike in the triangle-free graph case, probabilistic tools are not suitable to construct Hasse diagrams with small independent sets. Indeed, not only do Hasse diagrams avoid triangles, but they avoid an infinite family of ordered cycles (see Claim 11 for further details). However, surprisingly, the explicit constructions of Ramsey graphs in [8, 20] are almost tailor-made for this problem. Roughly, in these constructions, they consider the so-called super-line graph of point-line incidence graphs over finite planes. Here, we show that if one considers the super-line graph of point-line incidence graphs over the real plane instead, one also gets a Hasse diagram.

Furthermore, we prove that even if one requires a Hasse diagram with large girth, we can achieve small independence number and polynomial chromatic number. Indeed, we get this almost immediately by randomly sparsifying our construction from Theorem 1.

Theorem 3. *For every positive integer $k \geq 5$, there exists a Hasse diagram on n vertices with girth at least k and independence number at most $n^{1-\frac{1}{2k-4}+o(1)}$.*

Corollary 4. *For every positive integer $k \geq 5$, there exists a Hasse diagram on n vertices with girth at least k and chromatic number at least $n^{\frac{1}{2k-4}+o(1)}$.*

Let us remark that our constructions are also semi-algebraic with bounded complexity, which might be of independent interest.

1.1 Forbidden configurations in point-line arrangements

In Section 4, we adapt the proof of Theorem 3 to show the existence of point-line arrangements in the plane with many incidences, that avoids certain forbidden subconfigurations.

Given a set of points P and a set of lines $\mathcal{L}$ in the plane, we say that $(P, \mathcal{L})$ contains a $k \times k$ *grid* if there are k^2 distinct points $p_{i,j} \in P$, where $(i, j) \in [k] \times [k]$, and two sets of distinct lines $l_1, \dots, l_k \in \mathcal{L}$ and $l'_1, \dots, l'_k \in \mathcal{L}$ such that $p_{i,j} \in l_i \cap l'_j$. In [24], it was shown that any set of n points and n lines in the plane that does not contain a $k \times k$-grid has at most $O(n^{4/3-\varepsilon})$ incidences, where ε depends on k. In the other direction, we prove the following.

Theorem 5. *For every positive integer $k \geq 2$, there exists a set of n points and n lines in the plane that does not contain a $k \times k$ grid, and determines at least $n^{4/3-\Theta(1/k)}$ incidences.*

We say that $(P, \mathcal{L})$ contains a *k-fan* if there are $k + 1$ distinct points $p_0, p_1, \dots, p_k \in P$ and a set of $k+1$ distinct lines $l_0, l_1, \dots, l_k \in \mathcal{L}$ such that $p_0 \in l_1 \cap l_2 \cap \cdots \cap l_k$, $p_i \in l_0 \cap l_i$ for $i = 1, \dots, k$ and $p_0 \notin l_0$. In [33], Solymosi showed that every set of n points and n lines in the plane that does not contain a 2-fan determines at most $o(n^{4/3})$ incidences. However, for $k \geq 3$, the following conjecture remains open (see [5]).

Conjecture 6. *For fixed $k \geq 3$, every set of n points and n lines in the plane that does not contain a k-fan determines at most $o(n^{4/3})$ incidences.*

In the other direction, we prove the following.

Theorem 7. *There exists a set of n points and n lines in the plane that does not contain a 2-fan and determines at least $n^{7/6+o(1)}$ incidences. In general, for $k \geq 3$, there exists a set of n points and n lines in the plane that does not contain a k-fan and determines at least $n^{4/3-\Theta(1/k)}$ incidences.*

Finally, we say that $(P, \mathcal{L})$ contains a *k-cycle* if there are k distinct points $p_1, \dots, p_k \in P$ and k distinct lines $l_1, \dots, l_k \in \mathcal{L}$ such that $p_0 \in l_i \cap l_{i+1}$ for $i = 1, \dots, k$, where the indices are meant modulo k. In [25], it is proved that for every r there exists a configuration of n points and n lines with at least $\Omega(n^{1+\frac{4}{r^2+6r-3}})$ incidences, and no $(r + 5)$-cycle. We can improve this as follows.

Theorem 8. *For every positive integer $k \geq 2$, there exists a set of n points and n lines in the plane that does not contain a k-cycle, and determines at least $n^{1+\Theta(1/k)}$ incidences.*

1.2 Coloring geometric intersection and disjointness graphs

A family of graphs $\mathcal{G}$ is *χ-bounded* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{G}$. Such a function f is called *χ-bounding* for $\mathcal{G}$. The *intersection graph* of a family of geometric objects $\mathcal{C}$ is the graph whose vertices correspond to the elements of $\mathcal{C}$, and two vertices are joined by an edge if the corresponding sets have a nonempty intersection. Also, the *disjointness graph* of $\mathcal{C}$ is the complement of the intersection graph. A *curve* is the image of a continuous function $\phi : [0, 1] \to \mathbb{R}^2$, and a curve is *$x$-monotone*, if every vertical line intersects it in at most one point.

Results about the χ-boundedness of intersection and disjointness graphs of geometric objects have a vast literature. One of the first such results is due to Asplund and Grünbaum [4], who proved that the family of intersection graphs of axis-parallel rectangles in the plane is χ-bounded. In [14, 19], it is proved that intersection graphs of chords of a circle are χ-bounded, and recently Davies and McCarty [9] proved that $f(x) = O(x^2)$ is χ-bounding for this family. However, the family of intersection graphs of segments is not χ-bounded: Pawlik et al. [30] gave a construction of n segments in the plane whose intersection graph is triangle-free and has chromatic number $\Omega(\log \log n)$. On the other hand, it follows from a result of Rok and Walczak [31] that $\chi(G) = O_{\omega(G)}(\log n)$ if G is the intersection graph of n segments, or more generally x-monotone curves, and Fox and Pach [13] showed that $\chi(G) = (\log n)^{O(\log \omega(G))}$ if G is the intersection graph of curves. In contrast, if we assume that G is the intersection graph of curves, and the girth of G is at least 5, then $\chi(G)$ is bounded by a constant [13].

In the case of disjointness graphs, it follows from an argument of Larman, Matoušek, Pach, and Törőcsik [22] that the family of disjointness graphs of convex sets and x-monotone curves is χ-bounded with χ-bounding function $f(x) = x^4$. Recently, Pach and Tomon [29] proved that in the case of x-monotone curves, this χ-bounding function is optimal up to a constant factor. However, the family of disjointness graphs of curves is not χ-bounded. Pach, Tardos and Tóth [27] constructed a family of n curves, each being a polygonal line of 4 segments, whose disjointness graph is a shift graph, therefore triangle-free and has chromatic number $\Theta(\log n)$. After the aforementioned results, it seems reasonable to conjecture that the chromatic number of triangle-free (or large girth) disjointness graphs of curves is also at most poly-logarithmic. Surprisingly, this is completely false.

It turns out that Hasse diagrams and disjointness graphs are closely related. A curve is *grounded* if it is contained in the nonnegative half-plane, and one of its endpoints lies on the y-axis. It is known that a graph G is a Hasse diagram if and only if there exists a family of grounded curves whose disjointness graph is triangle-free and isomorphic to G; the forward implication follows from Corollary 2.7 in [23], and the backward implication is a consequence of Theorem 1 in [32]. But then Theorem 1 and Theorem 3 immediately imply the following result.

Theorem 9. *For every positive integer n, there exists a family of n curves, whose disjointness graph G is triangle-free and $\chi(G) \geq \Omega(n^{1/4})$. Also, for every $k \geq 5$, there exists a family of n curves, whose disjointness graph G has girth k and $\chi(G) \geq n^{\frac{1}{2k-4}+o(1)}$.*

Hasse diagrams also appear in connection to another geometric problem. Given a set of points P in the plane, its *Delaunay graph* with respect to axis-parallel rectangles is the graph whose vertices are P, and p and q are joined by an edge if there exists an axis-parallel rectangle whose intersection with P is exactly $\{p, q\}$. Even et al. [11] asked whether there exists a constant $c > 0$ such that the Delaunay graph of any set of n points contains an independent set of size cn. Chen et al. [7] proved that the answer is no by showing that a random set of n points has no independent set of size larger than $O(n\frac{(\log\log n)^2}{\log n})$. On the other hand, it remains open whether every such Delaunay graph contains an independent set of size $n^{1-o(1)}$. However, given a set of points P, one can consider the partial ordering $<$ on P, where $(a, b) < (c, d)$ if $a < c$ and $b < d$. Then $(P, <)$ is a 2-dimensional poset (in particular, every 2-dimensional poset can be defined this way). If G is the Hasse diagram of $(P, <)$, then G is a subgraph (on the same vertex set) of the Delaunay graph of P. Therefore, if one could construct a 2-dimensional poset with n vertices whose Hasse diagram has independence number at most $n^{1-\epsilon}$, it would imply the existence of an n-element point set in the plane whose Delaunay graph has no independent set larger than $n^{1-\epsilon}$.

Problem 10. *Are there 2-dimensional posets on n vertices whose Hasse diagram has independence number $n^{1-\varepsilon}$?*

Our construction for Theorem 1 gives some hope that there might exist such 2-dimensional posets, but so far we were unable to construct one.

2 Constructing Hasse diagrams with large chromatic number

In this section, we outline our general construction and prove Theorem 1.

An ordered graph is a pair $(G, \prec)$, where G is a graph and $\prec$ is a total order on $V(G)$. A *monotone cycle* of length k in an ordered graph $(G, \prec)$ is a k-element subset of the vertices $v_1 \prec \cdots \prec v_k$ such that v_1v_k and v_iv_{i+1} are edges of G for $i = 1, \ldots, k-1$. The following simple characterization of Hasse diagrams will come in handy.

Claim 11. *A graph G is a Hasse diagram if and only if there exists an ordering $\prec$ on $V(G)$ such that $(G, \prec)$ does not contain a monotone cycle.*

Proof. Let G be a graph and let $\prec$ be an ordering on $V(G)$ such that $(G, \prec)$ does not contain a monotone cycle. Define the relation $<$ on $V(G)$ such that $x < y$ if there exists a sequence of vertices $x = v_1 \prec v_2 \prec \cdots \prec v_m = y$ such that v_iv_{i+1} is an edge for $i = 1, \ldots, m-1$. Then it is easy to check that $<$ is a partial ordering on $V(G)$, and if $(G, \prec)$ has no monotone cycles, then the Hasse diagram of $(V(G), <)$ is G.

On the other hand, if G is a Hasse diagram of some poset $(V(G), <)$, then let $\prec$ be any *linear extension* of $<$, that is, a total order which satisfies $x \prec y$ if $x < y$. Then $(G, \prec)$ does not contain a monotone cycle. □

Let P be a set of N points, and $\mathcal{L}$ be a set of N lines on the plane, where we specify N later. We denote by $I(P, \mathcal{L})$ the set of incidences between P and $\mathcal{L}$, that is, the set of pairs $(p, l) \in P \times \mathcal{L}$ such that $p \in l$. Also, let $x(p)$ denote the x-coordinate of a point p, and let $s(l)$ denote the slope of a line l (here, a is the *slope* of l if l is given by the equation $ax + b = y$). By applying a projection to $(P, \mathcal{L})$, we can assume that the points have different x-coordinates, and the lines have different slopes, without changing the incidence structure.

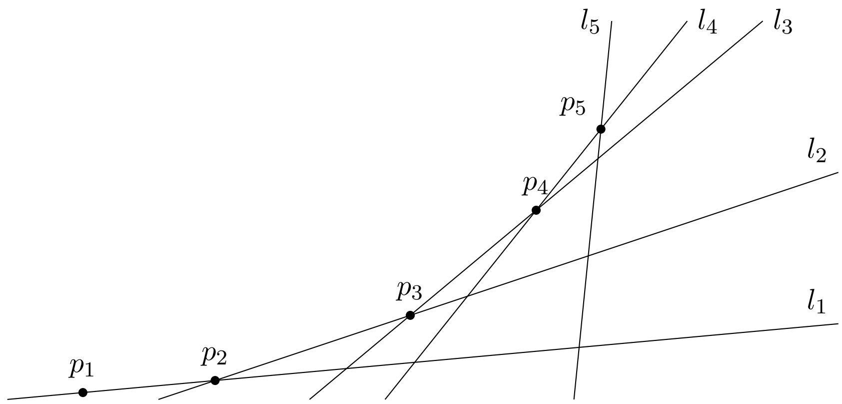

Figure 1: A monotone path $(p_1, l_1), \ldots, (p_5, l_5)$.

Define the ordered graph $(G, \prec)$ as follows. Let $I(P, \mathcal{L})$ be the vertex set of G, and define the ordering $\prec$ such that $(p, l) \prec (p', l')$ if $x(p) < x(p')$, or $p = p'$ and $s(l) < s(l')$. Now define the edge set of G as follows. Join (p, l) and (p', l') by an edge if $x(p) < x(p')$, $s(l) < s(l')$ and $p' \in l$. The key observation is the following.

Claim 12. *G is a Hasse diagram.*

Proof. By Claim 11, it is enough to show that $(G, \prec)$ does not contain a monotone cycle. Indeed, suppose that $(p_1, l_1), \ldots, (p_k, l_k)$ are the vertices of a monotone cycle. Then

- $x(p_1) < \cdots < x(p_k)$,
- $s(l_1) < \cdots < s(l_k)$,
- $p_1 \in l_1$, and $l_{i-1} \cap l_i = \{p_i\}$ for $i = 2, \ldots, k$.

These properties imply that $p_1, \ldots, p_k$ and $l_1, \ldots, l_k$ form a convex sequence, see Figure 1 for an illustration. But then p_k is strictly above the line l_1, so $p_k \notin l_1$, contradicting that (p_1, l_1) and (p_k, l_k) are joined by an edge. □

Next, we show that G contains no large independent sets.

Claim 13. *G has no independent set larger than $2N$.*

Proof. Let B be the incidence graph of $(P, \mathcal{L})$, that is, B is the bipartite graph with parts P and $\mathcal{L}$, and $I(P, \mathcal{L})$ is the edge set of B. Also, define the ordering $\prec$ on $V(B)$ such that for $p, p' \in P$, $p \prec p'$ if $x(p) < x(p')$, for $l, l' \in \mathcal{L}$, $l \prec l'$ if $s(l) < s(l')$, and $p \prec l$ for every $p \in P$, $l \in \mathcal{L}$. If $J \subset I(P, \mathcal{L})$ is an independent set of G, then there exists no $p, p' \in P$ and $l, l' \in \mathcal{L}$ such that $x(p) < x(p')$, $s(l) < s(l')$ and $(p, l), (p', l), (p', l') \in J$ (note that this is actually a weaker condition than J being an independent set). But this configuration of 3 edges in B is equivalent to a copy of the ordered path S, which has 4 vertices $x_1 \prec x_2 \prec x_3 \prec x_4$ and 3 edges x_1x_3, x_2x_3, x_2x_4.

Therefore, it is enough to show that if $(H, \prec)$ is a bipartite ordered graph with parts $X \prec Y$ of size N, and H does not contain a copy of the ordered path S, then H has at most $2N$ edges.

This statement is well known in different forms, but due to its simplicity, let us present the proof. Suppose that H has at least $2N+1$ edges. For each $x \in X$, let y_x denote the largest element of Y (with respect to $\prec$) such that (x, y_x) is an edge of H, and if x is an isolated vertex, do not define y_x. For each $x \in X$, delete the edge (x, y_x) from H, and let H' be the resulting graph. Then H' still has at least $N+1$ edges. Therefore, there exists $y \in Y$ of degree at least 2, let $x \prec x'$ be two neighbors of y. But then setting $y' = y_{x'}$, the edges $xy, x'y, x'y'$ form a copy of S. $\square$

We finish the proof by noting the well known result that there exists a point-line configuration $(P, \mathcal{L})$ such that $|P| = |\mathcal{L}| = N$ and $|I(P, \mathcal{L})| = \Theta(N^{4/3})$, which is the maximum number of incidences by the celebrated Szemerédi-Trotter theorem [34]. Indeed, setting $n = |I(P, \mathcal{L})|$, such a configuration gives a Hasse diagram G with n vertices and independence number $\alpha(G) \leq 2N = O(n^{3/4})$.

As we will also need it later, let us describe a point-line configuration with $\Theta(N^{4/3})$ incidences, which we shall refer to as the *standard configuration.* Take

$$P = \{(a, b) \in \mathbb{N}^2 : a < N^{1/3}, b < N^{2/3}\}$$

and

$$\mathcal{L} = \{ax + b = y : a, b \in \mathbb{N}, a < N^{1/3}, b < N^{2/3}\}.$$

3 Large girth

In this section, we prove Theorem 3. Let $(P, \mathcal{L})$ be the standard point-line configuration with N points and N lines, let B_0 be the incidence graph of $(P, \mathcal{L})$, and let G_0 be the Hasse diagram defined in the previous section with respect to $(P, \mathcal{L})$. Note that the edges of B_0 are the vertices of G_0. Using probabilistic techniques, we show that G_0 has a large induced subgraph with large girth. (Here, we remark that a subgraph of a Hasse diagram is also a Hasse diagram, which is not obvious at first glance, but follows easily from Claim 11.)

Let B be a subgraph of B_0 and let G be the subgraph of G_0 induced on $E(B)$. Then the girth of B and the girth of G are related as follows.

Claim 14. *If the girth of B is at least $2k-2$, then the girth of G is at least k.*

Proof. Suppose that G contains a cycle of length r for some $r \leq k-1$, and let the vertices of such a cycle be $(p_1, l_1), \dots, (p_r, l_r)$. Then $l_1, p_2, l_2, \dots, p_{r-1}, l_{r-1}, p_r, l_1$ is a closed walk of length $2r-2$ in B. But no two consecutive edges of this walk are the same, so this walk contains a cycle of length at most $2r-2 \leq 2k-4$ in B, contradiction. $\square$

Therefore, our task is reduced to finding a large subgraph B of B_0 with girth at least $2k-2$. Most of this section is devoted to the proof of the following lemma.

Lemma 15. *B_0 has a subgraph B such that the girth of B is at least $2k-2$, and $|E(B)| \geq N^{1+\frac{1}{2k-5}+o(1)}$.*

Note that every vertex of B_0 has degree at most $N^{1/3}$, and at least $\frac{N}{2}$ lines and at least $\frac{N}{2}$ points have degree at least $\frac{N^{1/3}}{2}$. Let H be the graph on P in which p and p' are joined by an edge if

$p, p' \in l$ for some $l \in \mathcal{L}$. Note that by the previous observation, the degree of each vertex of H is at most $N^{2/3}$. Moreover, we have the following claim.

Claim 16. *Let $p, p' \in P$ be distinct vertices. The number of common neighbors of p and p' in H is at most $N^{1/3+o(1)}$.*

Proof. Translate our configuration so that $p = (0,0)$, let $p' = (a,b)$, and let $q = (u,v) \in P$ be a common neighbor of p and p'. Then $u \notin \{0,a\}$ as $\mathcal{L}$ contains no vertical lines. Recall that the slope of every line $l \in \mathcal{L}$ is an integer. Therefore, as p and q are on the same line of $\mathcal{L}$, we have $u \mid v$ and $0 \leq \frac{v}{u} < N^{1/3}$. Also, as p' and q are on the same line of $\mathcal{L}$, we have $(u-a) \mid (v-b)$. Set $t = \frac{v}{u} \in \mathbb{N}$, which implies

$$v - b = t(u-a) + ta - b.$$

Then by dividing both sides by $(u-a)$, we can see that $(u-a) \mid (ta-b)$. We have $ta - b = O(N^{2/3})$, so if $ta - b \neq 0$, then the number of divisors (including both positive and negative) of $ta-b$ is $N^{o(1)}$ (see, e.g., [15]). Therefore, if t is fixed and $t \neq \frac{b}{a}$, then u can take at most $N^{o(1)}$ distinct values. Also, t can take at most $N^{1/3}$ distinct values. On the other hand, if $t = \frac{b}{a}$, then u can take at most $N^{1/3}$ distinct values. Thus, p and p' have at most $N^{1/3+o(1)}$ common neighbors in H. □

Now we will use this claim to count the number of cycles of given length in B_0.

Claim 17. *For every $r \geq 3$, the number of cycles of length $2r$ in B_0 is at most $N^{2r/3+o(1)}$.*

Proof. Note that each cycle of length $2r$ in B_0 gives rise to a unique cycle of length r in H (using that B_0 is C_4-free), so it is enough to count the cycles of length r in H.

Consider two cases. If r is even, count the cycles $p_1, \dots, p_r$ in H as follows. We have at most $N^{r/2}$ ways to choose the vertices p_{2i} for $i = 1, \dots, \frac{r}{2}$. After p_{2i-2} and p_{2i} are fixed (indices are meant modulo r), we have at most $N^{1/3+o(1)}$ choices for p_{2i-1} by Claim 16. Therefore, we have at most $N^{r/6+o(1)}$ further choices for the vertices p_{2i-1} for $i = 1, \dots, r/2$. In total, there are at most $N^{2r/3+o(1)}$ cycles of length r in H.

If r is odd, we proceed similarly. We have at most $N^{(r-1)/2}$ ways to choose the vertices p_{2i} for $i = 1, \dots, \frac{r-1}{2}$. Then, we have at most $N^{2/3}$ further choices for p_1 as it is a neighbor of p_2. For each vertex p_{2i-1}, where $i = 2, \dots, (r+1)/2$, there are at most $N^{1/3+o(1)}$ choices by Claim 16, which gives at most $N^{(r-1)/6+o(1)}$ further choices. Thus, in total, there are at most $N^{2r/3+o(1)}$ cycles of length r in H. □

Let us remark that in general, the $o(1)$ error term is needed in Claim 17. Indeed, a result of Klavík, Král' and Mach [18] gives that the number of cycles of length 6 in B_0 is $\Omega(N^2 \log\log N)$.

Proof of Lemma 15. For $3 \leq r \leq k-2$, let C_r be the number of cycles of length $2r$ in B_0. In what follows, let $q \in (N^{-1/3}, 1)$, and let B' be a subgraph of B_0 in which each edge of B_0 is present independently with probability q. Also, let X_r be the number of cycles of length $2r$ in B', and let $X = \sum_{r=3}^{k-2} X_r$. Then $\mathbb{E}(|E(B')|) = q|E(B_0)| = \Theta(qN^{4/3})$ and $\mathbb{E}(X_r) = q^{2r}C_r$.

As $C_r \leq N^{2r/3+o(1)}$ by Claim 17, we have

$$\mathbb{E}(X) = \sum_{r=3}^{k-2} \mathbb{E}(X_r) = \sum_{r=3}^{k-2} q^{2r} C_r < N^{o(1)} \left(\sum_{r=3}^{k-2} (qN^{1/3})^{2r} \right) = N^{o(1)} (qN^{1/3})^{2k-4},$$

where the last equality holds by the assumption that $q > N^{-1/3}$. Hence, we can choose some $q = N^{\frac{1}{2k-5} - \frac{1}{3} + o(1)}$ such that

$$\mathbb{E}(X) < \frac{q|E(B_0)|}{2}.$$

Therefore, there exists a choice for B' such that

$$|E(B')| - X \geq \mathbb{E}(|E(B')| - X) \geq \frac{q|E(B_0)|}{2} = N^{1+\frac{1}{2k-5}+o(1)}.$$

Pick such a B', and from each cycle of length at most $2k-4$, delete an arbitrary edge. Let B be the resulting graph. Then the girth of B is $2k-2$, and B has at least $|E(B')| - X = N^{1+\frac{1}{2k-5}+o(1)}$ edges. □

Now we are ready to prove Theorem 3.

Proof of Theorem 3. Let B be a subgraph of B_0 such that the girth of B is at least $2k-2$, and B has at least $N^{1+\frac{1}{2k-5}+o(1)}$ edges. By Lemma 15, such a subgraph exists. Let $n = |E(B)|$, and let G be the subgraph of G_0 induced by B. Then G is a Hasse diagram on n vertices, the independence number of G is at most $2N = n^{1-\frac{1}{2k-4}+o(1)}$ by Claim 13, and the girth of G is at least k by Claim 14. □

4 Proof of Theorems 5, 7, and 8

The proof of Theorems 5, 7 and 8 follows the probabilistic arguments as in the proof of Lemma 15. Let N be a positive integer and let $(P_0, \mathcal{L}_0)$ be the standard point-line configuration with N points and N lines. We select each element of P_0 and $\mathcal{L}_0$ independently with probability q. In the case of k-grids and k-fans, we show that if q is chosen appropriately, then the resulting point-line configuration $(P', \mathcal{L}')$ still has many incidences, but does not contain a k-grid or a k-fan. In the case of k-cycles, we choose q such that $(P', \mathcal{L}')$ contains only few k-cycles, and we destroy these k-cycles by removing one of their points.

In order to execute these ideas, we first need to count the number of k-fans, k-grids and k-cycles in $(P_0, \mathcal{L}_0)$.

Lemma 18. *In $(P_0, \mathcal{L}_0)$, there are at most*

1. *N^{2k} k-grids,*

2. *$N^{2+o(1)}$ k-fans,*

3. *$N^{2k/3+o(1)}$ k-cycles.*

Proof. 1. The $2k$ lines $l_1, \ldots, l_k$ and $l'_1, \ldots, l'_k$ uniquely determine a k-grid, so the number of k-grids is at most N^{2k}.

2. Given $p \in P_0$ and $l \in \mathcal{L}_0$ such that $p \notin l$, let $m(p,l)$ be the number of points $p' \in P_0 \cap l$ such that the line containing p and p' is in $\mathcal{L}_0$. First, we show that $m(p,l) = N^{o(1)}$. Translate our configuration so that $p = (0,0)$, and suppose that l is given by the equation $ax + b = y$. Then $b \neq 0$ as $p \notin l$. If $p' = (u,v) \in l$ and the line through p and p' is contained in $\mathcal{L}_0$, then $au + b = v$ and $u \mid v$. But then $u \mid b$ and $|b| \leq N^{2/3}$, so u can take at most $N^{o(1)}$ different values. As u uniquely determines v, this implies that $m(p,l) = N^{o(1)}$.

Given $p_0 \in P_0$ and $l_0 \in \mathcal{L}_0$ such that $p_0 \notin l_0$, the number of k-fans with points $p_0, p_1, \ldots, p_k$ and lines $l_0, l_1, \ldots, l_k$ is at most $m(p_0, l_0)^k = N^{o(1)}$. As there are at most N^2 choices for (p_0, l_0), this gives that the number of k-fans is at most $N^{2+o(1)}$.

3. This is the consequence of Claim 17. □

We will also need the following multiplicative form of Chernoff's inequality, see e.g. Theorem 2.8 in [16].

Claim 19. *(Chernoff's inequality) Let $X_1, \ldots, X_n$ be independent random variables such that $\mathbb{P}(X_i = 1) = q$ and $\mathbb{P}(X_i = 0) = 1 - q$, and let $X = \sum_{i=1}^n X_i$. Then for $0 < \delta \leq 1$, we have*

$$\mathbb{P}(X \geq (1+\delta)qn) \leq e^{-\frac{\delta^2 qn}{3}},$$

and

$$\mathbb{P}(X \leq (1-\delta)qn) \leq e^{-\frac{\delta^2 qn}{3}}.$$

Proof of Theorems 5, 7, 8. Let $\epsilon > 0$, $q \in (N^{-1/3+\epsilon}, 1)$, and select each element of P_0 and $\mathcal{L}_0$ independently with probability q. Let P' and $\mathcal{L}'$ be the sets of selected elements. Let $\mathcal{A}$ be the event that the configuration $(P', \mathcal{L}')$ satisfies the following properties:

1. $\frac{qN}{2} < |P'|, |\mathcal{L}'| < 2qN$,
2. there are at least $\frac{qN}{4}$ points in P' that are incident to at least $\frac{qN^{1/3}}{4}$ lines in $\mathcal{L}'$.

It follows from Chernoff's inequality that $\mathcal{A}$ holds with high probability (probability tending to 1 as N tends to infinity). Indeed, let us show this for the second property, as a similar argument follows for the first property. Let $Q \subset P_0$ be the set of points which are incident to at least $\frac{N^{1/3}}{2}$ lines in $\mathcal{L}_0$. Then we have $|Q| \geq \frac{N}{2}$ since all points $(a,b) \in P_0$ satisfying $b \geq \frac{1}{2}N^{2/3}$ belong to Q. Each element of Q is present in P' independently with probability q, so by Chernoff's inequality,

$$\mathbb{P}\left(|Q \cap P'| \leq \frac{qN}{4}\right) \leq \mathbb{P}\left(|Q \cap P'| \leq \frac{q|Q|}{2}\right) \leq e^{-\frac{q|Q|}{12}} \leq e^{-\frac{qN}{24}}.$$

Also, if $p \in P' \cap Q$, let d be the the number of lines in $\mathcal{L}_0$ incident to p, and let d' be the number of lines in $\mathcal{L}'$ incident to p. Then $d \geq \frac{N^{1/3}}{2}$, so by Chernoff's inequality

$$\mathbb{P}\left(d' \leq \frac{qN^{1/3}}{4}\right) \leq \mathbb{P}\left(d' \leq \frac{qd}{2}\right) \leq e^{-\frac{qd}{12}} \leq e^{-N^{\epsilon}/24}.$$

Therefore, with high probability, we have $|P' \cap Q| \geq \frac{qN}{4}$, and each $p \in P' \cap Q$ is incident to at least $\frac{qN^{1/3}}{4}$ lines in $\mathcal{L}'$, so the second property holds.

Let X be the number of incidences of $(P', \mathcal{L}')$. Note that if $\mathcal{A}$ holds, then we have $X \geq \frac{q^2 N^{4/3}}{16}$.

k-grids: Let Y be the number of k-grids. Each k-grid of $(P_0, \mathcal{L}_0)$ survives with probability q^{-2k-k^2}, so $\mathbb{E}(Y) \leq N^{2k} q^{-2k-k^2}$ by Lemma 18. By Markov's inequality, we then have

$$\mathbb{P}(Y \geq 1) \leq N^{2k} q^{-k^2-2k},$$

which is the probability that $(P', \mathcal{L}')$ contains a k-grid. Setting $q = N^{-2/k}$, we get $\mathbb{P}(Y \geq 1) \leq N^{-4}$, which tells us that with high probability $\mathcal{A}$ holds and $(P', \mathcal{L}')$ contains no k-grid. Hence, there exists a point-line configuration $(P', \mathcal{L}')$ satisfying these properties. Let $n = \min\{|P'|, |\mathcal{L}'|\} \geq \frac{qN}{2} = \frac{N^{1-2/k}}{2}$. Then by a simple averaging argument, we can find $P \subset P'$ and $\mathcal{L} \subset \mathcal{L}'$ such that $|P| = |\mathcal{L}| = n$, and

$$|I(P, \mathcal{L})| \geq \frac{q^2 N^{4/3}}{64} = \frac{N^{4/3-4/k}}{64} = n^{4/3-\Theta(1/k)}.$$

k-fans: Let Y be the number of k-fans. Each k-fan of $(P_0, \mathcal{L}_0)$ survives with probability q^{-2k}, so $\mathbb{E}(Y) \leq N^{2+o(1)} q^{-2k}$ by Lemma 18. By Markov's inequality, we then have

$$\mathbb{P}(Y \geq 1) \leq N^{2+o(1)} q^{-2k},$$

which is the probability that $(P', \mathcal{L}')$ contains a k-fan. Setting $q = N^{-2/k}$, we get $\mathbb{P}(Y \geq 1) \leq N^{-2+o(1)}$, which tells us that with high probability $\mathcal{A}$ holds and $(P', \mathcal{L}')$ contains no k-fan. Hence, there exists a point-line configuration $(P', \mathcal{L}')$ satisfying these properties. Let $n = \min\{|P'|, |\mathcal{L}'|\} \geq \frac{qN}{2} = \frac{N^{1-2/k}}{2}$. Then by a simple averaging argument, we can find $P \subset P'$ and $\mathcal{L} \subset \mathcal{L}'$ such that $|P| = |\mathcal{L}| = n$, and

$$|I(P, \mathcal{L})| \geq \frac{q^2 N^{4/3}}{64} = \frac{N^{4/3-4/k}}{64} = n^{4/3-\Theta(1/k)}.$$

k-cycles: Let Y be the number of k-cycles. Each k-cycle of $(P_0, \mathcal{L}_0)$ survives with probability q^{-2k}, so $\mathbb{E}(Y) \leq N^{2k/3+o(1)} q^{-2k}$ by Lemma 18. We choose $q = N^{-\frac{2k-3}{6k-3}+o(1)}$ so that $\mathbb{E}(Y) \leq \frac{qN}{16}$. By Markov's inequality, we have $\mathbb{P}(Y \geq \frac{qN}{8}) \leq \frac{1}{2}$, so there is a choice for $(P', \mathcal{L}')$ such that $\mathcal{A}$ holds and $Y \leq \frac{qN}{8}$. From each k-cycle in $(P', \mathcal{L}')$, delete an arbitrary point, and let $(P'', \mathcal{L}')$ be the resulting point-line configuration. Then $|Q \cap P''| \geq |Q \cap P'| - Y > \frac{qN}{4}$, and as $\mathcal{A}$ holds, we have

$$|I(P'', \mathcal{L}')| \geq |Q \cap P''| \frac{qN^{1/3}}{4} \geq \frac{qN}{8} \cdot \frac{qN^{1/3}}{4} = \frac{q^2 N^{4/3}}{32}.$$

Let $n = \min\{|P''|, |\mathcal{L}'|\} \geq \frac{qN}{4} = N^{\frac{4k}{6k-3}+o(1)}$. Then by a simple averaging argument, we can find $P \subset P''$ and $\mathcal{L} \subset \mathcal{L}'$ such that $|P| = |\mathcal{L}| = n$ and

$$|I(P, \mathcal{L})| \geq \frac{q^2 N^{4/3}}{128} = N^{\frac{4k+2}{6k-3}+o(1)} = n^{1+\Theta(1/k)}.$$

$\square$

5 Concluding remarks

We proved that the chromatic number of a Hasse diagram on n vertices can be as large as $\Omega(n^{1/4})$. However, it would be interesting to decide whether it can be $n^{1/2+o(1)}$, which is the general upper bound for all-triangle free graphs. We suspect that the answer is no.

Conjecture 20. *There exists $\epsilon > 0$ such that if G is a Hasse diagram on n vertices, then $\chi(G) \leq n^{1/2-\epsilon}$.*

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References

[1] M. Ajtai, J. Komlós, E. Szemerédi. "A note on Ramsey numbers." J. Combinatorial Theory Ser. A 29 (1980), 354–360.

[2] N. Alon. "Explicit Ramsey graphs and orthonormal labelings." The Electronic Journal of Combinatorics, 1 (1994), R12, 8pp.

[3] N. Alon, P. Pudlák. "Constructive lower bounds for off-diagonal Ramsey numbers." Israel Journ. of Math. 122 (2001), 243–251.

[4] E. Asplund, B. Grünbaum. "On a coloring problem." Math. Scand. 8 (1960), 181–188.

[5] P. Brass, W. Moser, J. Pach. "Research Problems in Discrete Geometry." Springer, 2000.

[6] B. Bollobás. "Colouring lattices." Algebra Universalis 7 (1977), 313–314.

[7] X. Chen, J. Pach, M. Szegedy, G. Tardos. "Delaunay graphs of point sets in the plane with respect to axis-parallel rectangles." Random Structures & Algorithms 34.1 (2009), 11–23.

[8] B. Codenotti, P. Pudlák, J. Resta. "Some structural properties of low rank matrices related to computational complexity." Theoretical Computer Sci. 235 (2000), 89–107.

[9] J. Davies, R. McCarty. "Circle graphs are quadratically χ-bounded." arXiv preprint (2019), arXiv:1905.11578.

[10] P. Erdős, A. Hajnal. "Some remarks on set theory." IX. Combinatorial problems in measure theory and set theory. Michigan Math. J. 11 (2) (1964), 107–127.

[11] G. Even, Z. Lotker, D. Ron, S. Smorodinsky. "Conflict–free colorings of simple geometric regions with applications to frequency assignment in cellular networks." SIAM J.Comput. 33 (2003), 94–136.

[12] S. Felsner, J. Gustedt, M. Morvan, J.-X. Rampon. "Constructing colorings for diagrams." Discrete Applied Mathematics, 51 (1) (1994),85–93.

[13] J. Fox , J. Pach. "Applications of a new separator theorem for string graphs." Combin.Probab. Comput., 23 (2014), 66–74.

[14] A. Gyárfás. "On the chromatic number of multiple interval graphs and overlap graphs." Discrete Math. 55 (1985), 161–166.

[15] G. H. Hardy, E. Wright. "An Introduction to the Theory of Numbers, 6th Edition." Oxford University Press, 2008.

[16] S. Janson, T. Łuczak, and A. Ruciński. "Random Graphs." Wiley-Interscience, 2000.

[17] J. H. Kim. "The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$." Random Structures & Algorithms 7.3 (1995), 173–207.

[18] P. Klavík, D. Král', L. Mach. "Triangles in arrangements of points and lines in the plane (note)." Journal of Combinatorial Theory Series A 118 (2011), 1140–1142.

[19] A. Kostochka, J. Kratochvíl. "Covering and coloring polygon-circle graphs." Discrete Math 163 (1-3) (1997): 299–305.

[20] A. Kostochka, P. Pudlák, V. Rödl. "Some constructive bounds on Ramsey numbers." Journal of Combinatorial Theory, Series B 100 (2010), 439–445.

[21] I. Kříž, J. Nešetřil. "Chromatic number of Hasse diagrams, eyebrows and dimension." Order 8 (1) (1991), 41–48.

[22] D. Larman, J. Matoušek, J. Pach, J. Törőcsik. "A Ramsey-type result for convex sets." Bull. Lond. Math. Soc., 26 (1994), 132–136.

[23] M. Middendorf, F. Pfeiffer. "Weakly transitive orientations, Hasse diagrams and string graphs." in: Graph theory and combinatorics (Marseille-Luminy, 1990), Discrete Math. 111 (1-3) (1993), 393–400.

[24] M. Mirzaei, A. Suk. "On grids in point-line arrangements in the plane." 35th International Symposium on Computational Geometry, SoCG 2019, 129, Leibniz Zentrum, Dagstuhl (2019), 50:1–50:11.

[25] M. Mirzaei, A. Suk, J. Verstraëte. "Constructions of point-line arrangements in the plane with large girth." arXiv preprint (2019), arXiv:1911.11713.

[26] J. Nešetřil, V. Rödl. "A short proof of the existence of highly chromatic hypergraphs without short cycles." J. Combin. Theory Ser. B 27 (2) (1979), 225–227.

[27] J. Pach, G. Tardos, G. Tóth. "Disjointness graphs of segments." in: 33rd International Symposium on Computational Geometry, SoCG 2017, 77, Leibniz Zentrum, Dagstuhl (2017), 59:1–59:15.

[28] J. Pach, I. Tomon. "Coloring Hasse Diagrams and Disjointness Graphs of Curves." in: International Symposium on Graph Drawing and Network Visualization (2019), 244–250. Springer, Cham

[29] J. Pach, I. Tomon. "On the chromatic number of disjointness graphs of curves." to appear in J. Combin. Theory Ser. B. also in: 35th International Symposium on Computational Geometry, SoCG 2019, 129, Leibniz Zentrum, Dagstuhl (2019), 54:1–54:17.

[30] A. Pawlik, J. Kozik, T. Krawczyk, M. Lasoń, P. Micek, W. T. Trotter, B. Walczak. "Triangle-free intersection graphs of line segments with large chromatic number." J. Combin. Theory Ser. B 105 (2014), 6–10.

[31] A. Rok, B. Walczak. "Outerstring Graphs are χ-Bounded." SIAM J. Discrete Math. 33 (4) (2019), 2181–2199.

[32] F. W. Sinden. "Topology of thin film RC circuits." Bell System Technical Journal 45 (1966), 1639–1662.

[33] J. Solymosi. "Dense Arrangements are Locally Very Dense. I." SIAM J. Discrete Math. 20 (3) (2006), 623–627.

[34] E. Szemerédi, W. T. Trotter. "Extremal problems in discrete geometry." Combinatorica 3, 3-4 (1983), 381–392.

[35] H. G. Vogt. "Leçons sur la résolution algébrique des équations." Nony, (1895), 91.