Ramsey numbers: combinatorial and algebraic

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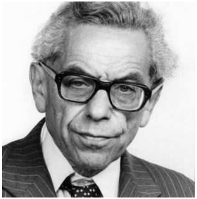
Origins of Ramsey theory

"A combinatorial problem in geometry," by Paul Erdős and George Szekeres (1935)




Erdős-Szekeres 1935

Theorem (Monotone subsequence)

Any sequence of $(n-1)^2+1$ *integers contains a monotone subsequence of length* n.

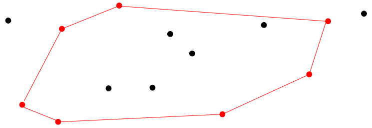
Theorem (Convex polygon)

For any $n > 0$*, there is a minimal* $ES(n)$*, such that every set of* $ES(n)$ *points in the plane in general position contains* n *members in convex position.*

Theorem (Ramsey numbers)

New proof of Ramsey's theorem.

Convex polygon theorem

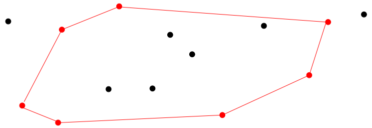


Theorem (Erdős-Szekeres 1935, 1960)

$$2^{n-2} + 1 \leq ES(n) \leq \binom{2n-4}{n-2} + 1 = O(4^n/\sqrt{n}).$$

Conjecture: $ES(n) = 2^{n-2} + 1,\ n \geq 3.$

Second Proof



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Theorem (S. 2016)

$$ES(n) = 2^{n+o(n)}$$

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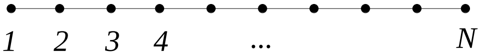
Theorem (Ramsey numbers)

New proof of Ramsey's theorem.

Formal definition: For any integers $k \geq 1$, $s, n \geq k$, there is a minimum $r_k(s, n) = N$, such that for every red/blue coloring of the k-tuples of $\{1, 2, \ldots, N\}$,



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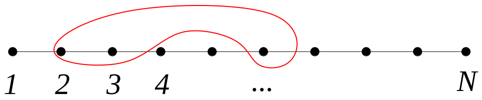

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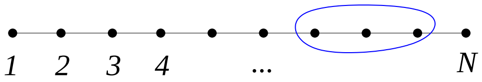

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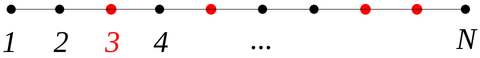

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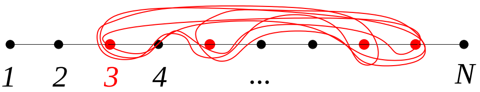

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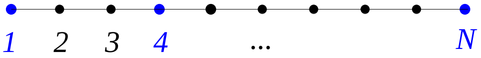

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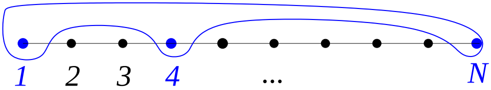

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$r_k(s,n) =$ Ramsey numbers

Graph Ramsey theorem

Theorem (Erdős-Szkeres 1935)

$$r_2(s,n) \le \binom{n+s-2}{s-1}$$

$$r_2(n,n) \le \binom{2n-2}{n-1} \approx \frac{4^n}{\sqrt{n}}$$

Diagonal graph Ramsey numbers

Theorem (Erdős 1947, Erdős-Szekeres 1935)

$$(1 + o(1))\frac{n}{e}2^{n/2} < r_2(n, n) < \frac{4^n}{\sqrt{n}}.$$

Theorem (Spencer 1977, Conlon 2008)

$$(1 + o(1))\frac{\sqrt{2}}{e}n2^{n/2} < r_2(n, n) < \frac{4^n}{n^{c \log n / \log \log n}}$$

diagonal hypergraph Ramsey numbers

$\text{twr}_1(x) = x$ and $\text{twr}_{i+1}(x) = 2^{\text{twr}_i(x)}$.

Theorem (Erdős-Hajnal-Rado 1952/1965)

$$2^{cn^2} < r_3(n,n) < 2^{2^{c'n}}$$

For $k > 3$

$$\text{twr}_{k-1}(cn^2) < r_k(n,n) < \text{twr}_k(c'n)$$

Conjecture (Erdős, \$500)

$$r_3(n,n) > 2^{2^{cn}}$$

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Theorem (Erdős-Hajnal)

$$r_3(n,n,n,n) > 2^{2^{cn}}$$

Theorem (Conlon-Fox-Sudakov 2011)

Every 3-uniform hypergraph of size $2^{cn^{2+o(1)}}$ contains n vertices with either fewer than $o\binom{n}{3}$ edges or more than $(1-o(1))\binom{n}{3}$ edges.

Off-diagonal Ramsey numbers

$r_k(s, n)$ where s is fixed, and $n \to \infty$. $r_k(s, n) \ll r_k(n, n)$

Graphs:

Theorem (Ajtai-Komlós-Szemerédi 1980, Kim 1995)

$r_2(3, n) = \Theta\left(\frac{n^2}{\log n}\right)$

Theorem

For fixed $s > 3$

$$n^{(s+1)/2+o(1)} < r_2(s, n) < n^{s-1+o(1)}$$

$$2^{n/2} < r_2(n, n) < 4^n$$

Off-diagonal hypergraph Ramsey numbers

3-uniform hypergraphs:

Theorem (Erdős-Hajnal-Rado)

For fixed $s \geq 4$,

$$2^{csn} < r_3(s,n) < 2^{c'n^{2s-4}}.$$

Theorem (Conlon-Fox-Sudakov 2010)

For fixed $s \geq 4$,

$$2^{csn\log n} < r_3(s,n) < 2^{c'n^{s-2}\log n}.$$

$$2^{cn^2} < r_3(n,n) < 2^{2^{c'n}}$$

Off-diagonal hypergraph Ramsey numbers

4-uniform hypergraphs: Exponential tower gap.

Theorem (Erdős-Rado, Conlon-Fox-Sudakov)

For fixed $s \geq 5$,

$$r_4(s,n) < 2^{2^{n^c}}.$$

Conjecture (Erdős-Hajnal 1972)

For all $s \geq 5$, $r_4(s,n) > 2^{2^{cn}}$.

Conjecture (Erdős-Hajnal 1972)

For all $s \geq k+1$, $r_k(s,n) > \mathrm{twr}_{k-1}(cn)$.

Tower growth rate for $r_4(5,n)$ is unknown.

Theorem (Erdos-Hajnal)

$$r_4(7, n) > 2^{2^{cn}}.$$

Corollary (Mubayi-S., Conlon-Fox-Sudakov 2017)

$$r_k(k+3, n) > \mathrm{twr}_{k-1}(cn).$$

Erdős-Hajnal (1972): $r_4(5, n), r_4(6, n) > 2^{cn}$

Towards the Erdős-Hajnal conjecture

Conjecture (Erdos-Hajnal)

$$r_4(5, n), r_4(6, n) > 2^{2^{cn}}$$

Erdős-Hajnal (1972): $r_4(5, n), r_4(6, n) > 2^{cn}$

Mubayi-S. (2017): $r_4(5, n) > 2^{n^2}$ $\quad r_4(6, n) > 2^{n^{c \log n}}$

New lower bounds for off-diagonal hypergraph Ramsey numbers

Theorem (Mubayi-S., 2017+)

$$r_4(5, n) > 2^{n^{c \log n}} \qquad r_4(6, n) > 2^{2^{cn^{1/5}}}.$$

for fixed $k > 4$

$$r_k(k+1, n) > \mathrm{twr}_{k-2}(n^{c \log n}) \qquad r_k(k+2, n) > \mathrm{twr}_{k-1}(cn^{1/5}).$$

Diagonal Ramsey problem ($500 Erdős):

$$2^{cn^2} < r_3(n,n) < 2^{2^{cn}}.$$

Off-diagonal Ramsey problem:

$$2^{n^{c \log n}} < r_4(5,n) < 2^{2^{cn}}.$$

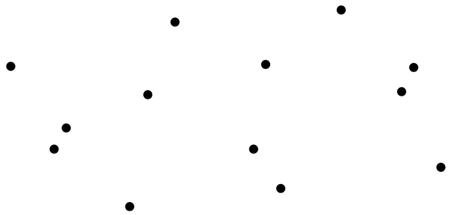
Theorem (Mubayi-S. 2017)

Showing $r_3(n,n) > 2^{2^{cn}}$ *implies that* $r_4(5,n) > 2^{2^{c'n}}$.

Convex polygon theorem, first proof

Theorem (Erdős-Szekeres, 1935)

For every $n \geq 3$, $ES(n) \leq r_4(5, n)$. Set $N = r_4(5, n)$.



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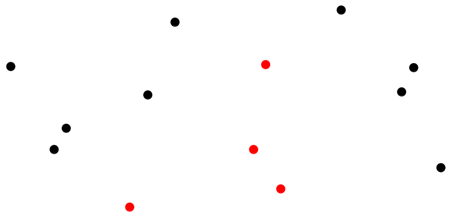

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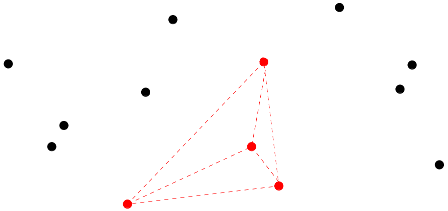

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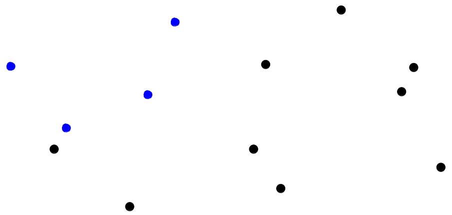

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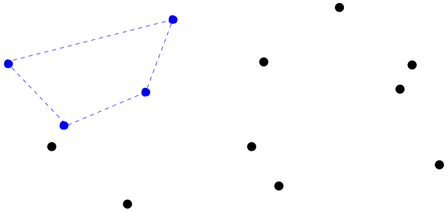

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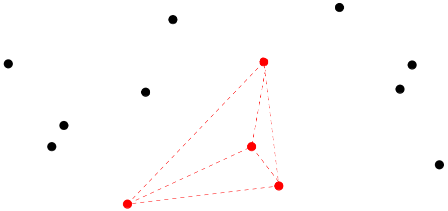

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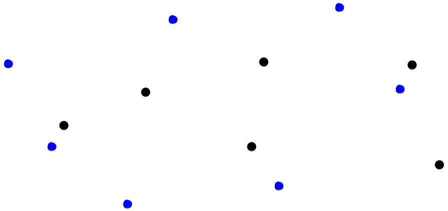

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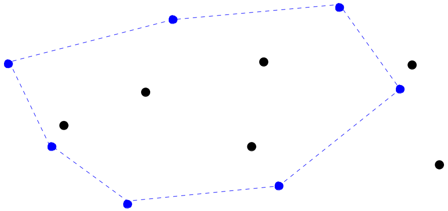

Applications of Ramsey numbers in geometry

$OT_d(n)$ = minimum N such that every set of N points in $\mathbb{R}^d$ in general position contains n members that form the vertices of a cyclic polytope.

Theorem

$$OT_d(n) \leq r_{d+1}(n, n)$$

Many more applications of Ramsey numbers.

Convex polygon theorem, first proof

Theorem (Colon-Fox-Sudakov, 2008)

$$ES(n) \leq r_4(5, n) < 2^{2^{cn^2 \log n}}.$$

$V = \{N \text{ points in the plane in general position}\}$

$E = \{4\text{-tuples in convex position}\}$

Better bounds for hypergraphs defined geometrically?

Definition of semi-algebraic hypergraphs

We say that $H = (V, E)$ is a **semi-algebraic k-uniform hypergraph in d-space** if

$V = \{n \text{ points in } \mathbb{R}^d\}$

E defined by polynomials $f_1, ..., f_t$ and a Boolean formula Φ such that

$$(p_{i_1}, ..., p_{i_k}) \in E$$

$$\Leftrightarrow \Phi(f_1(p_{i_1}, ..., p_{i_k}) \geq 0, ..., f_t(p_{i_1}, ..., p_{i_k}) \geq 0) = \text{yes}$$

Complexity of H is at most C if d, t and degree of the f_i-s is at most C (constant).

Definition

We define the *semi-algebraic Ramsey number* $r_k^{semi}(s,n)$ to be the minimum integer N such that any N-vertex k-uniform **semi-algebraic** hypergraph H (in $\mathbb{R}^d$) contains either a clique of size s or an independent set of size n.

Problem: Estimate $r_k^{semi}(s,n)$.

$r_k^{semi}(s,n) \leq r_k(s,n) = N$

Diagonal Semi-algebraic Ramsey numbers

Theorem (Alon, Pach, Pinchasi, Radoičić, Sharir 2005)

$$r_2^{semi}(n,n) \leq n^{c_1}.$$

Theorem (Conlon, Fox, Pach, Sudakov, S. 2012)

$$2^{cn} < r_3^{semi}(n,n) < 2^{n^{c'}}$$

For $k \geq 4$

$$t_{k-1}(c_2 n) \leq r_k^{semi}(n,n) \leq t_{k-1}(n^{c_1}).$$

Classical Ramsey numbers:

$$2^{n/2} \leq r_2(n,n) \leq 2^{2n} \qquad 2^{cn^2} \leq r_3(n,n) \leq 2^{2^{cn}}.$$

$$t_{k-1}(n^2) \leq r_k(n,n) \leq t_k(cn).$$

Off diagonal semi-algebraic Ramsey numbers

$$r_k^{semi}(s, n) \ll r_k^{semi}(n, n)$$

Theorem (Alon, Pach, Pinchasi, Radoičić, Sharir 2005)

$$r_2^{semi}(n, n) \leq n^{c_1}.$$

Conjecture

For fixed $s > 3$, $r_2^{semi}(s, n) = O(n)$

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Theorem (Fox-Pach-Suk 2017+)

$$r_2^{semi}(10, n) \geq n^{4/3}$$

Off diagonal semi-algebraic Ramsey numbers

Theorem (Conlon-Fox-Pach-Sudakov-S. 2013)

$$n^{c'} \leq r_3^{semi}(s,n) \leq 2^{n^c}.$$

$$\mathrm{twr}_{k-2}(n^{c'}) \leq r_k^{semi}(s,n) \leq r_k^{semi}(n,n) \leq \mathrm{twr}_{k-1}(n^c).$$

Classical version: $r_3(s,n) \leq 2^{n^c}$

Conjecture (Conlon-Fox-Pach-Sudakov-S.)

$$r_3^{semi}(s,n) \leq n^c$$

.

Theorem (S. 2016)

$$r_3^{semi}(s,n) < 2^{n^{o(1)}}.$$

$o(1) = \frac{1}{\log\log n}$.

Conjecture (Conlon-Fox-Pach-Sudakov-S.)

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Exponential tower improvement

Theorem (Conlon-Fox-Pach-Sudakov-S. 2012)

For semi-algebraic graphs in $\mathbb{R}$, $r_3^{1-semi}(s,n) \leq 2^{\log^C n}$

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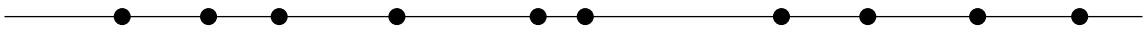
Proof. $V = \{N = 2^{\log^C n}$ points in $\mathbb{R}\}$, no RED $K_4^{(3)}$.
$E = \{(v_1, v_2, v_3) \in \binom{V}{3} : f(v_1, v_2, v_3) > 0\}$.



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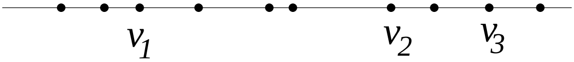

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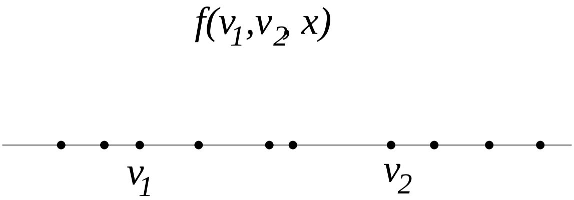

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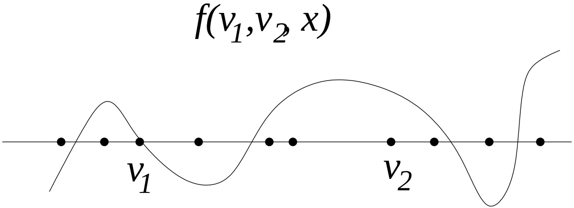

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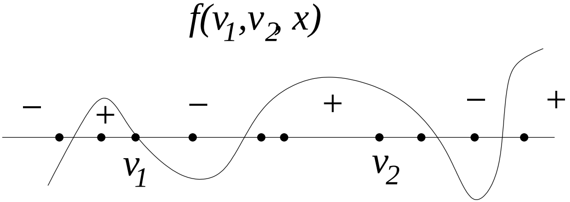

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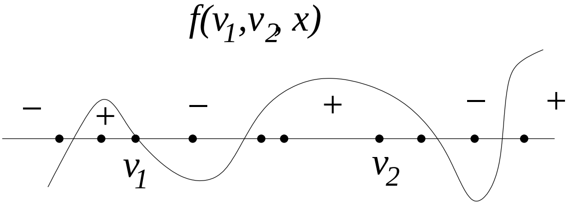
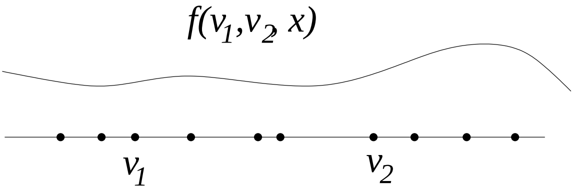


Key idea: Apply induction on the number of roots.

Induction on the number of roots

Base Case: Every pair gives rise to a polynomial with no roots within the interval



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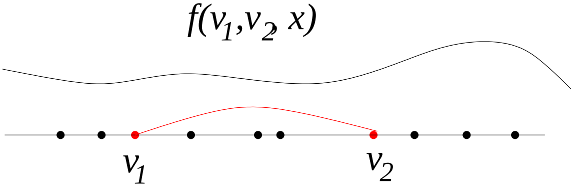
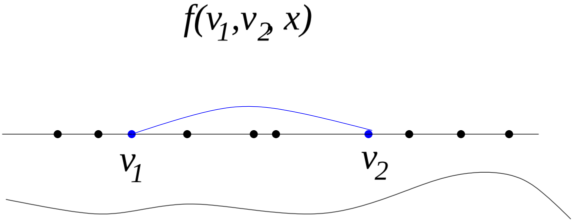


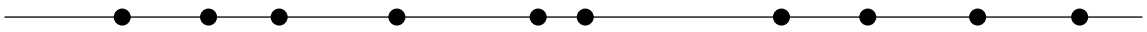
Induction on the number of roots

Base Case: Every pair gives rise to a polynomial with no roots within the interval



Induction on the number of roots

Inductive Step: Every pair gives rise to a polynomial with at most t roots.



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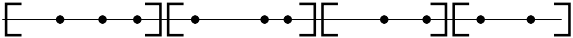

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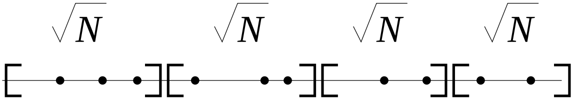

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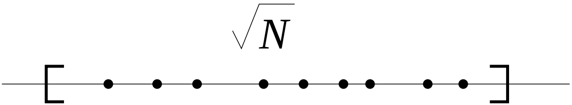

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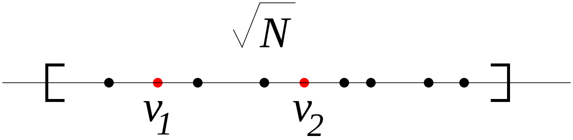

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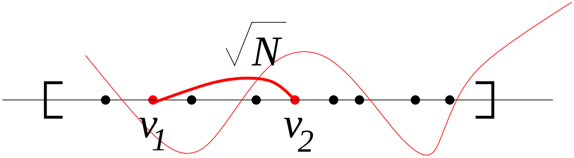

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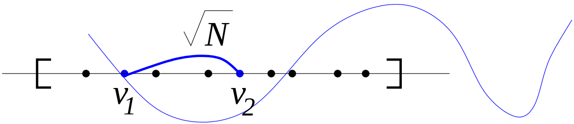

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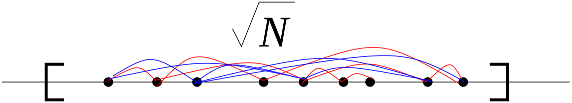

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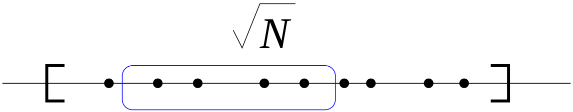

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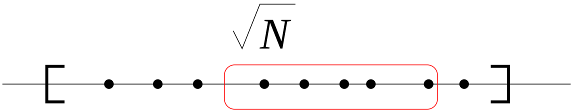

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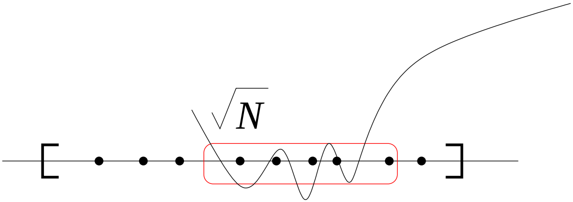

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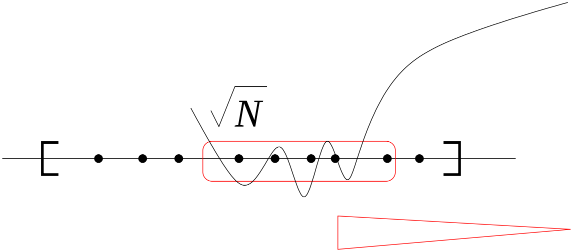

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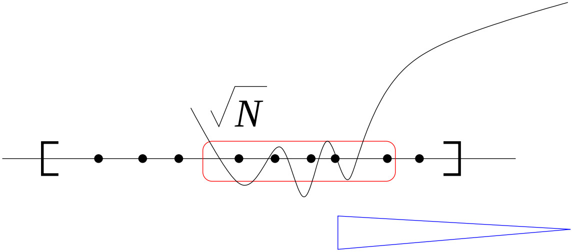

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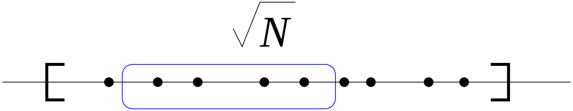

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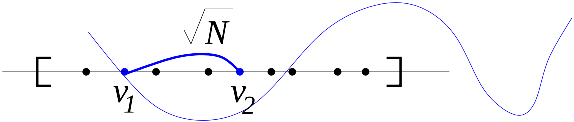

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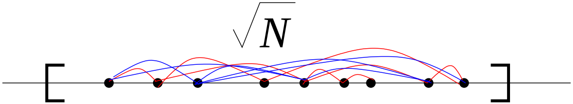


Find a monochromatic subset of size N^{ϵ}.

Sturm's theorem

$$E = \{(v_1, v_2, v_3) \in \binom{V}{3} : f(v_1, v_2, v_3) > 0\}$$

Definition (Sturm sequence)

Set $g_0(z) = f(x, y, z)$. Then define $g_1(z) = g_0'(z)$, $g_2(z) = -rem(g_0, g_1)$,

$$g_i(z) = -rem(g_{i-2}, g_{i-1}),$$

where $rem(g_{i-2}, g_{i-1})$ is the remainder when computing the long-division $\frac{g_{i-2}}{g_{i-1}}$.

Theorem (Sturm)

Given a Sturm sequence $g_0, g_1, \ldots, g_k$, let $\sigma(a)$ denote the number of sign changes in the sequence $g_0(a), g_1(a), \ldots, g_k(a)$. Then the number of distinct roots of $g(z) = f(x, y, z)$ inside the interval $[a, b]$ is $\sigma(a) - \sigma(b)$.

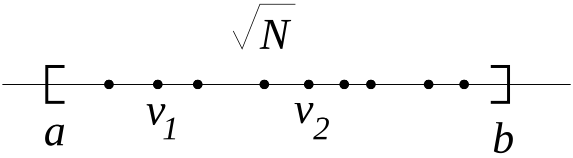
Semi-algebraic graph

$\chi(v_1, v_2) =$ *blue* if fewer than t roots.

$\chi(v_1, v_2) =$ *red* if t roots.

$g_0(a), g_1(a), \dots, g_k(a) = f(x, y, a), f_1(x, y, a), \dots, f_k(x, y, a)$

$g_0(b), g_1(b), \dots, g_k(b) = f(x, y, b), f_1(x, y, b), \dots, f_k(x, y, b)$



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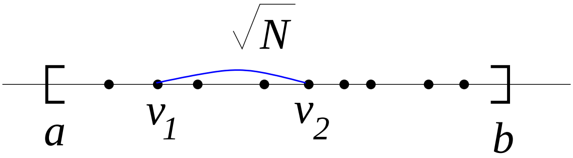

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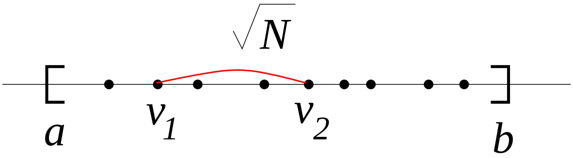

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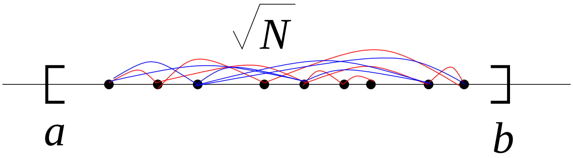

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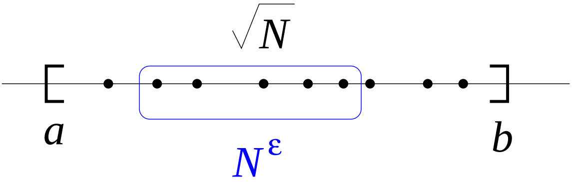

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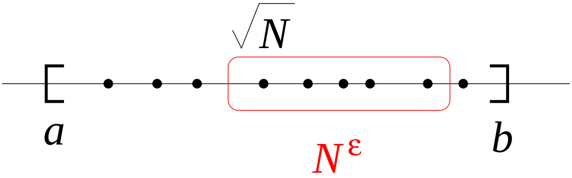

Open combinatorial problems

Problem (Erdős \$500)

Close the gap for $r_3(n, n)$.

$$2^{cn^2} < r_3(n, n) < 2^{2^{cn}}$$

Problem (Erdős-Hajnal 1972)

Close the gap for $r_4(5, n)$.

$$2^{n^{c \log n}} < r_4(5, n) < 2^{2^{cn}}$$

Open geometric problems

Conjecture (Conlon-Fox-Pach-Sudakov-S. 2012, S. 2016)

Close the gap for $r_3^{semi}(4, n)$

$$n^C < r_3^{semi}(4, n) < 2^{n^{o(1)}}$$

Theorem (Erdős-Szekeres, Conlon-Fox-Pach-Sudakov-S)

$$2^{2n} < r_3^{semi}(n, n) < 2^{n^C}$$

Problem

$$2^{cn^2} < r_3^{semi}(n, n)$$

Thank you!