Quasiplanar graphs, string graphs, and Erdős-Gallai

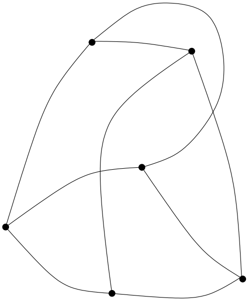
Andrew Suk (UC San Diego)

Graph Drawing 2022

Topological Graph $G = (V, E)$

$V =$ points in the plane.

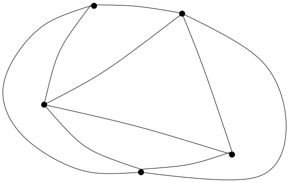
$E =$ curves connecting the corresponding points (vertices).



Quasi-planar graphs

Theorem (Euler)

Every n-vertex topological graph with no crossing edges has at most $3n - 6$ edges.



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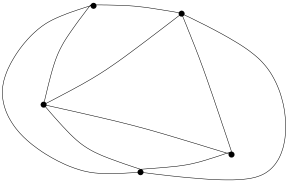

A topological graph is called k-**quasi-planar**, if there are no k pairwise crossing edges.

Conjecture

Every n-vertex k-quasi-planar graph has at most $O_k(n)$ edges.

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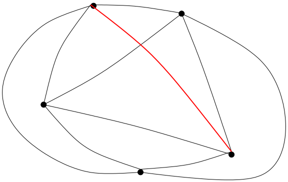

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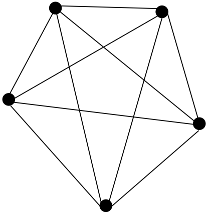
Best known bounds

- $k \geq 5$, $n(\frac{c \log n}{\log k})^{O(\log k)}$, Fox-Pach 2014.
- $k \geq 5$, $O(n \log^{4k-16})$, Pach-Radoicic-Toth 2006.
- Straight-line edges, $O(n \log n)$ Valtr 1997.

Natural approach: Intersection graphs

Conjecture

Every n-vertex k-quasi-planar graph has at most $O_k(n)$ edges.



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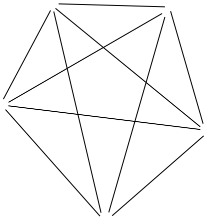

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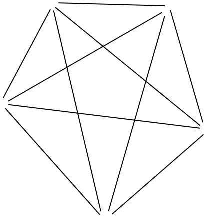

Conjecture (Erdős)

Given a family of segments in the plane with no k pairwise crossing members, can we properly color them with $f(k)$ colors?

Natural approach: Intersection graphs

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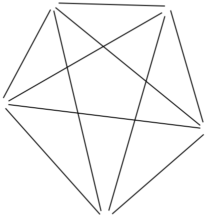

Conjecture (Erdős)

Every K_k-free intersection graph of segments in the plane has chromatic number at most $f(k)$.

Natural approach: Intersection graphs

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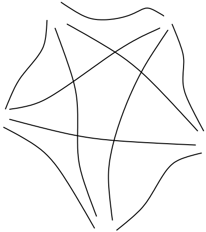
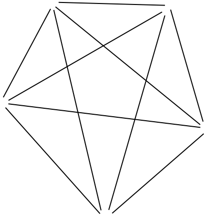


Conjecture (Erdős)

Every K_k-free **string graph** *has chromatic number at most $f(k)$.*

Natural approach: Intersection graphs



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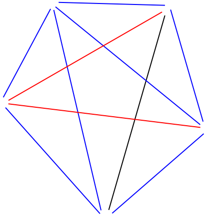

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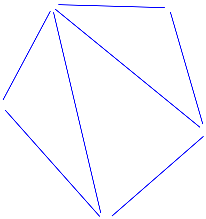

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$$\frac{|E(G)|}{f(k)}$$

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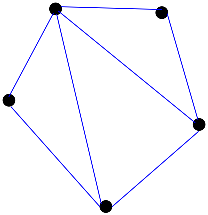

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$$\frac{|E(G)|}{f(k)} \leq 3(n-6) \quad \Rightarrow \quad |E(G)| \leq O_k(n)$$

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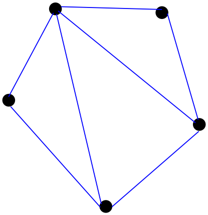

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Every K_k-free intersection graph of segments in the plane has chromatic number at most $f(k)$.

Conjecture is False!

Problems with this approach

Theorem (Pawlik, Kozik, Krawczyk, Larson, Micek, Trotter, Walczak)

For every positive integer m, there is a K_3-free intersection graph of m segments in the plane whose chromatic number at least $\Omega(\log\log m)$.

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No large independent set

Theorem (Walczak)

For every positive integer m, there is a K_3-free intersection graph of m segments in the plane whose independence number is $O(\frac{m}{\log \log m})$.

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Best hope for k-quasi-planar graphs: $|E(G)| \leq O_k(n \log \log n)$

Small hope: Erdős-Gallai, Erdős-Rogers type results

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Problem. Given a K_5-free intersection graph of m segments in the plane, is there an induced K_4-free subgraph on $\Omega(m)$ vertices?

$$|E(G)| = O(n).$$

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Problem. Given a K_k-free intersection graph of m segments in the plane, is there an induced K_{k-1}-free subgraph on $\Omega(m)$ vertices?

$$|E(G)| = O_k(n).$$

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Problem

Given a K_k-free intersection graph of m segments in the plane, is there an induced K_{k-1}-free subgraph on $\Omega(m)$ vertices?

Note true for $k = 3$, but **Open** for $k > 3$.

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Does not generalize to large cliques

New results on string graphs

String graph: Intersection graph of curves in the plane.

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Every n-vertex string graph contains a clique or independent set of size n^{ϵ}.

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Erdős-Gallai, Erdős-Rogers type result

Theorem (Fox-Pach-S. 2022)

For $s > q \geq 2$, every K_{2^s}-free string graph on m vertices has a K_{2^q}-free induced subgraph on $m\left(\frac{c_s}{\log m}\right)^{2s-2q}$ vertices

Application: Back to k-quasi-planar graphs

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Theorem (Fox-Pach-S. 2022)

For $s \geq 3$, every 2^s-quasi-planar graph on n vertices has at most $c_s n (\log n)^{2s-4}$ edges.

Fox-Pach 2014: $n(\log n)^{O(s)}$, $O(s) \approx 50s$.

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Theorem (Fox-Pach-S. 2022)

Every 8-quasi-planar graph on n vertices has at most $O(n \log^2 n)$ edges.

Pach-Radoicic-Toth 2006, Ackerman 2009: $O(n \log^{16} n)$.

Sketch proof

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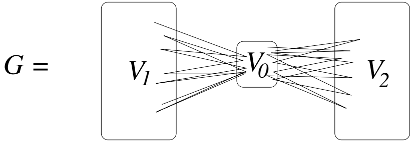
Key ingredient: Separator theorem

$G = (V, E)$, $V = V_0 \cup V_1 \cup V_2$, V_0 separator, $|V_1|, |V_2| < \frac{2}{3}|V|$.



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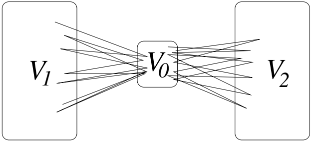
$G =$ 

Theorem (Lee 2016)

Every string graph on m vertices and e edges has a separator of size $O(\sqrt{e})$.

Matousek 2013: $O(\sqrt{e} \log e)$

Key ingredient: Extending a result of Tomon

Lemma

Let $G = (V, E)$ be a string graph with m vertices and at least αm^2 edges. Then there are disjoint subsets $V_1 \cup \cdots \cup V_t \subset V$, $t \geq 2$, such that

1. V_i *is complete to* V_j*, and*
2. $|V_i| \geq c\alpha \frac{m}{t^2}$

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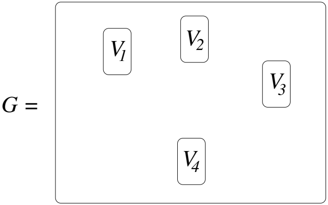

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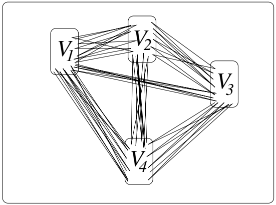

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Proof. Induction on s and m. G =string graph of m curves.

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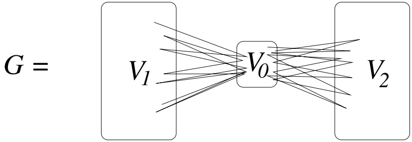
Case 1. G has less than $\varepsilon\frac{m^2}{\log^2 m}$ edges. Separator of size $c\sqrt{\varepsilon}\frac{m}{\log m}$.



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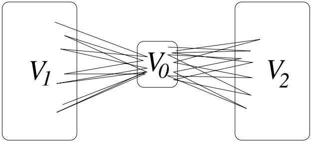

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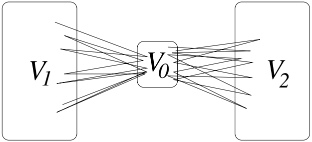
$G =$ 

$$|V_1|\left(\frac{c_s}{\log|V_1|}\right)^{2s-2q} + |V_2|\left(\frac{c_s}{\log|V_2|}\right)^{2s-2q}$$

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$$\geq (|V_1|+|V_2|)\left(\frac{c_s}{\log 2m/3}\right)^{2s-2q} \geq m\frac{(1-\frac{c\sqrt{\varepsilon}}{\log m})(c_s)^{2s-2q}}{(\log m - \log(3/2))^{2s-2q}}$$

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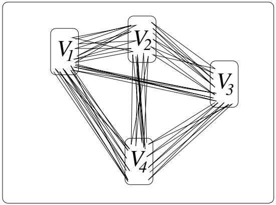
$$= m\left(\frac{c_s}{\log m}\right)^{2s-2q}\frac{1 - \frac{c\sqrt{\varepsilon}}{\log m}}{\left(1 - \frac{\log(3/2)}{\log m}\right)^{2s-2q}}$$

Sketch proof

Case 2. G has at least $\varepsilon \frac{m^2}{\log^2 m}$ edges. $t = 2^p$

$V_1, \ldots, V_t \subset V$, V_i is complete to V_j. $|V_i| \geq c\varepsilon \frac{m}{t^2 \log^2 m}$

$G =$ 

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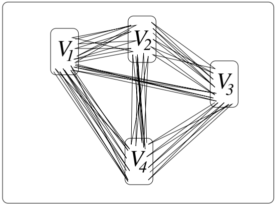

One part V_i is $K_{2^{s-p}}$-free.

Sketch proof

One part V_i is $K_{2^{s-p}}$-free.

Case 2.a. If $2^{s-p} \leq 2^q$, then V_i is K_{2^q}-free. $t = 2^p \geq 2^{s-q}$

$$|V_i| \geq c\varepsilon \frac{m}{t^2 \log^2 m}$$

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$$|V_i| \geq c\varepsilon \frac{m}{t^2 \log^2 m}$$

$$\geq c\varepsilon \frac{m}{2^2 \log^2 m} \geq m \left(\frac{c_s}{\log m} \right)^{2s-2q}$$

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One part V_i is $K_{2^{s-p}}$-free.

Case 2.b. If $2^{s-p} > 2^q$, then $t \leq 2^{s-q}$.

$$|V_i| \geq c\varepsilon \frac{m}{t^2 \log^2 m}$$

Apply induction on V_i to find a K_{2^q}-free subset

$$|V_i| \left(\frac{c_{s-p}}{\log |V_i|} \right)^{2(s-p)-2q} \geq c\varepsilon \frac{m}{t^2 \log^2 m} \left(\frac{c_{s-p}}{\log m} \right)^{2(s-p)-2q}$$

$$\geq m \left(\frac{c_s}{\log m} \right)^{2s-2q} \quad \square$$

Concluding remarks

Erdős-Gallai, Erdős-Rogers type problems

Problem. Given a K_k-free intersection graph of m segments in the plane, is there an induced K_{k-1}-free subgraph on $\Omega(m)$ vertices? False for $k = 3$, but open for $k > 3$.

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Problem. Given a K_k-free intersection graph of m segments in the plane, is there an induced K_{k-1}-free subgraph on $\Omega(m)$ vertices? False for $k = 3$, but open for $k > 3$.

Problem. Given a K_k-free intersection graph of **chords of a circle**, is there an induced K_{k-1}-free subgraph on $\Omega(m)$ vertices?

Thank you!