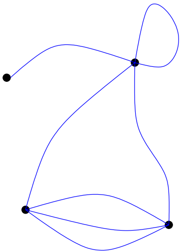
On the number of edges of separated multigraphs

Andrew Suk (UC San Diego)

September 16, 2021

Multigraph drawings

- Loops
- Multiple edges



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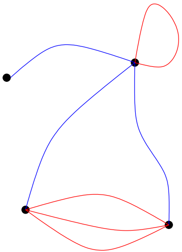

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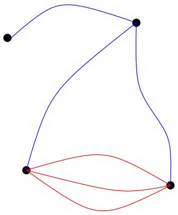
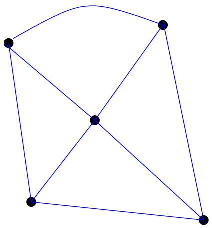


Crossing lemma for multigraphs

Theorem (Euler)

Every n-vertex planar multigraph with edge multiplicity at most m has at most $(3n-6)m$ edges.



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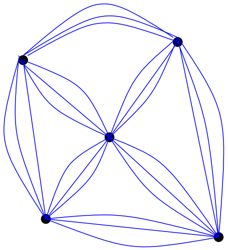

Crossing lemma for multigraphs

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Theorem

Let G be an n-vertex multigraph with e edges and edge multiplicity at most m. Then

$$\mathrm{cr}(G) \geq \Omega\left(\frac{e^3}{m \cdot n^2}\right) - O(m^2 n).$$

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Question (Kaufmann) Can we improve this crossing lemma for multigraphs with no empty lenses?

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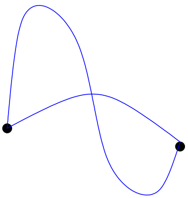
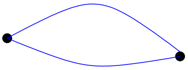


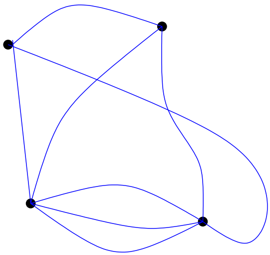
Lenses

Question (Kaufmann) How many edges can there be in a multigraph with no empty lenses?



Multigraph drawings

- No loops
- Multiple edges
- No two parallel edges cross



Multigraph drawings with no empty lenses

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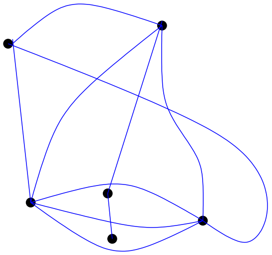

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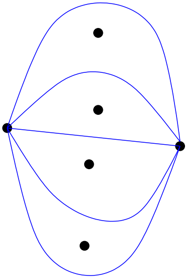

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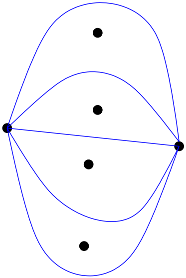

$$\binom{n}{2} \cdot (n-1) = O(n^3).$$

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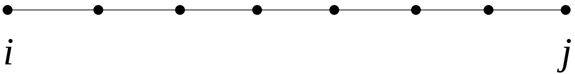

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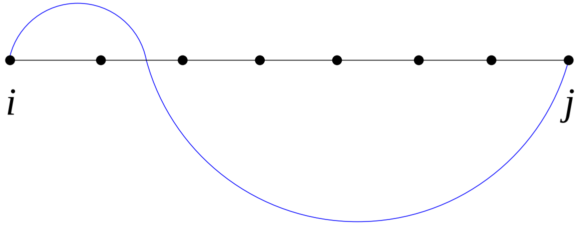

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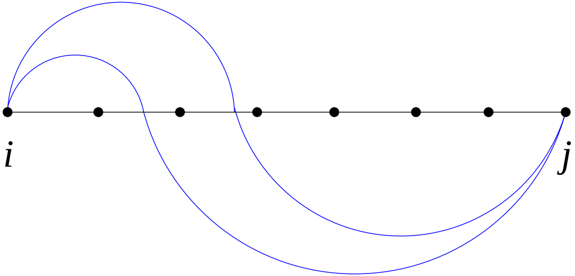

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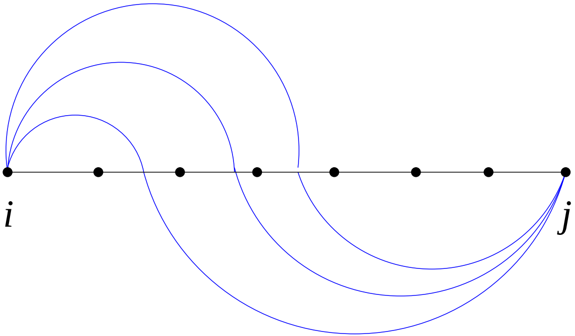

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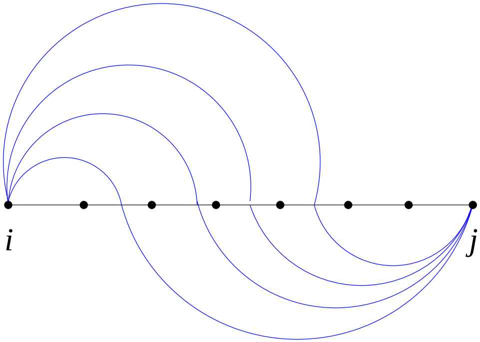

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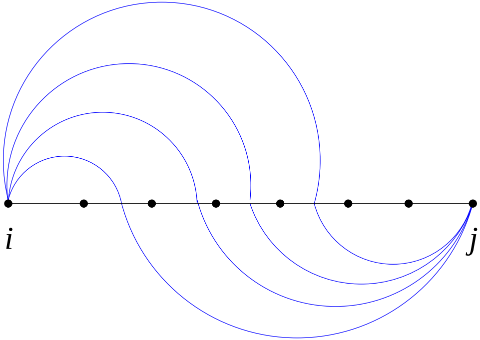

Any two edges cross at most twice. $\Omega(n^3)$.

Multigraph drawings with no empty lenses

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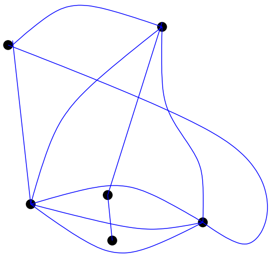
- Two edges cross at most once.



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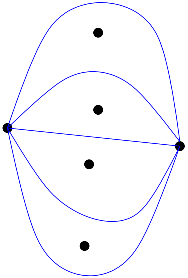
- Two edges cross at most once. Trivial: $O(n^3)$



Multigraph drawings with no empty lenses

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- Two edges cross at most once.
- Dependent edges are non-crossing

Theorem (Pach-Tóth, 2018)

The maximum number of edges in an n-vertex multigraph that can be drawn in the plane with the rules above is $O(n^2)$

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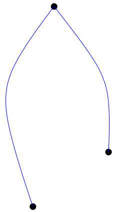

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Proof. Probabilistic Method + Thrackles

Concluding remarks

- Multiple edges
- No two parallel edges cross
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- Two edges cross at most once (including dependent edges).

Theorem (Fox-Pach-Suk)

Every n-vertex multigraph that can be drawn in the plane with the properties described above has at most $O(n^2 \log n)$ edges.

Corollary (Fox-Pach-Suk)

Let G be an n-vertex multigraph with e edges that can be drawn in the plane with the properties described above. Then

$$cr(G) \geq \Omega\left(\frac{e^3}{n^2 \log n}\right) - O(n)$$

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Every n-vertex multigraph that can be drawn in the plane with the properties described above has at most $O(n^2 \log n)$ edges.

Open problem: Is the log factor necessary?

Thank you!