# k-quasi-planar graphs

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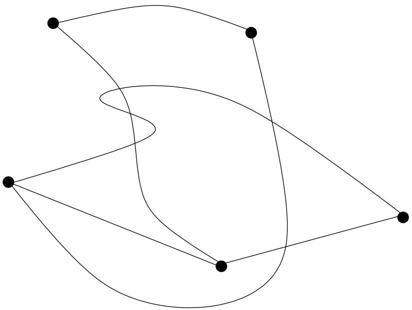
## Definition

A *topological graph* is a graph drawn in the plane with vertices represented by points and edges represented by curves connecting the corresponding points. A topological graph is *simple* if every pair of its edges intersect at most once.



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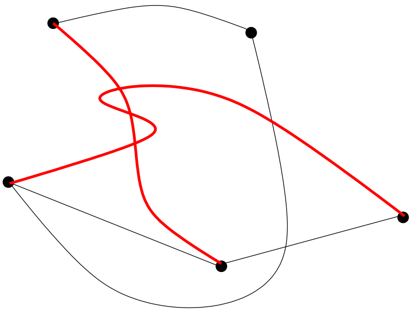
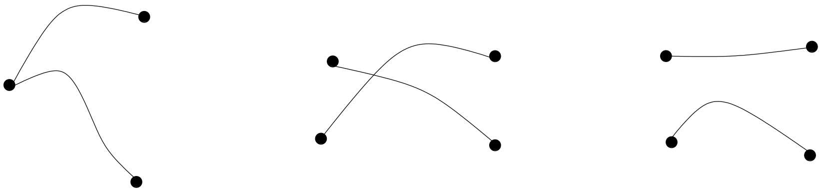


We will first only consider *simple* topological graphs.



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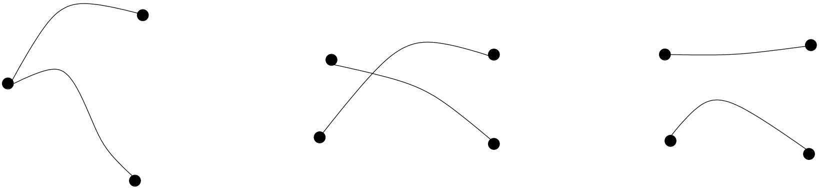


**Definition**

Two edges *cross* of their interiors share a point in common.

**Theorem**

*Every $n$-vertex topological graph with no crossing edges contains at most $3n - 6 = O(n)$ edges.*

Relaxation of planarity.

**Conjecture**

*Every $n$-vertex topological graph with no $k$ pairwise crossing edges contains at most $O(n)$ edges.*

All such graphs are called *$k$-quasi-planar*.

# Special Cases

For $k = 3, 4$.

**Theorem (Agarwal, Aronov, Pach, Pollack, Sharir 1997 and Pach, Radoičić, Tóth 2004)**

*Every $n$-vertex 3-quasi-planar graph has at most $O(n)$ edges.*

Also Ackerman and Tardos 2007.

**Theorem (Ackerman 2009)**

*Every $n$-vertex 4-quasi-planar graph has at most $O(n)$ edges.*

Edges drawn with $x$-monotone curves.

**Theorem (Valtr 1997)**

*Every $n$-vertex simple $k$-quasi-planar graph with edges drawn as $x$-monotone curves has at most $O(n \log n)$ edges.*

# General Bounds

**Theorem (Pach, Shahrokhi, Szegedy 1994)**

*Every n-vertex simple k-quasi-planar graph has at most* $O(n\log^{4k-16} n)$ *edges.*

**Theorem (Fox and Pach 2008)**

*Every n-vertex simple k-quasi-planar graph has at most* $n(\log n)^{O(\log k)}$ *edges.*

**Theorem (Main Result, Suk 2011)**

*Every n-vertex simple k-quasi-planar graph has at most* $(n\log^2 n)\cdot 2^{\alpha^{c_k}(n)}$ *edges.*

# Main tool: generalized Davenport Schinzel sequences

Definition: The sequence $s_1, s_2, ..., s_{l \cdot t}$ is said to be of type $up(l,t)$ if the first $l$ terms are pairwise different and for $i = 1, 2, ..., l$

$$s_i = s_{i+l} = s_{i+2l} = \cdots = s_{i+(t-1)l}$$

Example: $a, b, c, a, b, c$ is of type $up(3,2)$

Example: $h, w, h, w, h, w$ is of type $up(2,3)$

# Extremal question

Do "long enough" sequences over $n$ symbols always contain a subsequence of type (say) $up(3,2)$ as a subsequence?

Example:

$$a, r, z, h, u, u, y, v, r, h, d, y, e, w, r, u, h$$

$$a, \underline{r}, z, \underline{h}, u, u, \underline{y}, v, \underline{r}, \underline{h}, d, \underline{y}, e, w, r, u, h$$

contains $r, h, y, r, h, y$.

## Problem

*What is the maximum length of a sequence over $n$ symbols that does not contain a subsequence of type $up(l, t)$ as a subsequence?*

Can be infinite: $a, a, a, a, a, a, a, a, a, \ldots\ldots$

## Definition

A sequence is $l$-regular if any $l$ consecutive terms in the sequence are pairwise different.

Not $l$-regular ($l > 1$)

$$a, a, a, a, a, a, a, a, .....$$

Example of 3-regular

$$a, g, e, h, q, w, a, h, d, e, n, t$$

Now the problem:

**Problem**

*Given fixed $l, t$, what is the maximum length of an $l$-regular sequence over $n$ symbols that does not contain a subsequence of type $up(l, t)$?*

Answer

**Theorem (Klazar 1993, Nivasch 2006)**

*Given fixed $l, t$, the maximum length of an $l$-regular sequence over $n$ symbols that does not contain a subsequence of type $up(l, t)$ is at most*

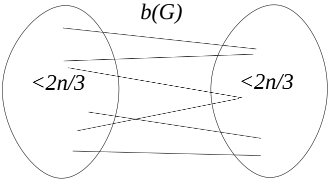
$$c_{l,t} n \cdot 2^{\alpha^{c_{l,t}}(n)}.$$

## Theorem (Main Result, Suk 2011)

*Every n-vertex simple k-quasi-planar graph has at most* $(n \log^2 n) \cdot 2^{\alpha^{c_k}(n)}$ *edges.*

**Proof of main theorem:** Suppose $G$ is $k$-quasi-planar with $m$ edges. Proceed by induction on $n$. **CASE 1.** If there are less than $O(m^2/\log^2 n)$ pairs of edges in $G$ that intersect, then use *Bisection Width* and inductive hypothesis.

$$b(G) = \min_{|V_1|,|V_2| \leq 2n/3} |E(V_1, V_2)|$$



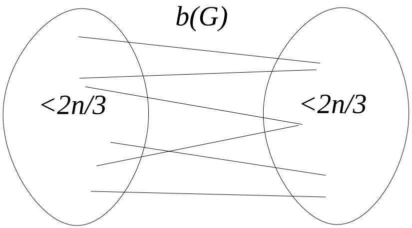
## Theorem (Pach, Shahrokhi, Szegedy 1996)

*Let G be a graph on n vertices and m edges. Then*

$$b(G) \leq 7\sqrt{cr(G)} + 3\sqrt{mn}$$

Since we assumed $cr(G) \leq O(m^2/\log^2 n)$, we have

$$b(G) \leq O\left(\frac{m}{\log n} + \sqrt{mn}\right)$$



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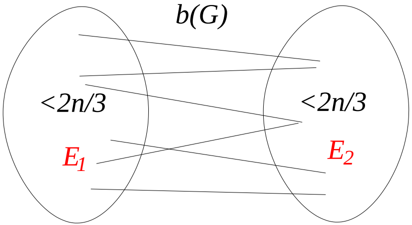
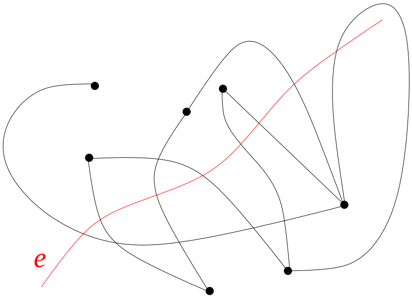


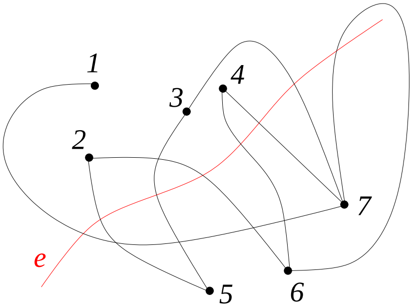

$$|E(G)| \le |E_1| + O\left(\frac{m}{\log n} + \sqrt{mn}\right) + |E_2| \le (n \log^2 n) \cdot 2^{\alpha^{c_k}(n)}$$

Case 2: There are at least $\Omega(m^2/\log^2 n)$ edges that cross. By a simple counting argument, there exists an edge $e$ that crosses at least $\Omega(m/\log^2 n)$ edges.




Let $E'$ denote the set of edges that cross $e$. $|E'| \geq \Omega(m/\log^2 n)$.

Define two sequence $S_1, S_2$




$$S_1 =$$

$$S_2 =$$

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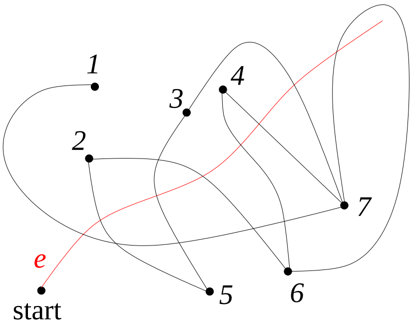


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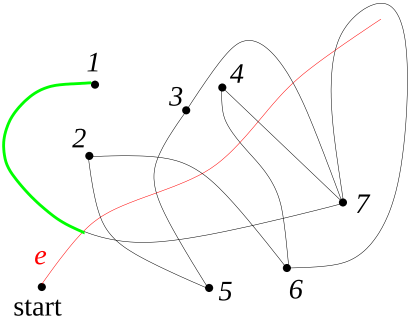
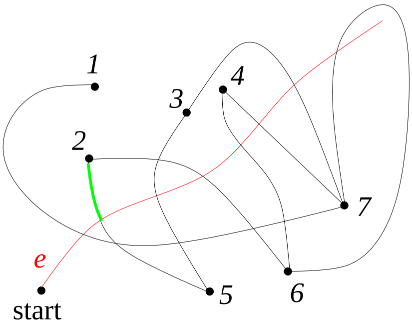



$$S_1 = 1$$

$$S_2 =$$

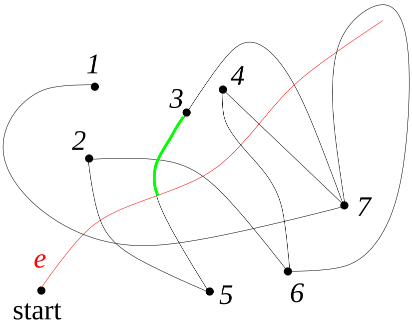
Define two sequence $S_1, S_2$




$$S_1 = 1, 2$$

$$S_2 =$$

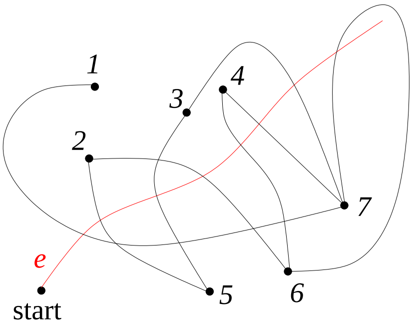
Define two sequence $S_1, S_2$




$$S_1 = 1, 2, 3$$

$$S_2 =$$

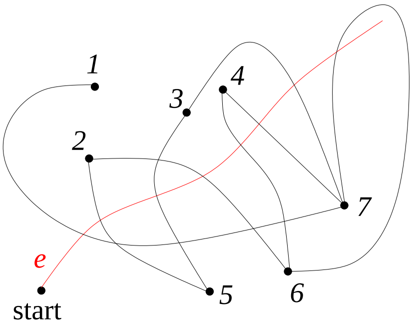
Define two sequence $S_1, S_2$




$$S_1 = 1, 2, 3, 2$$

$$S_2 =$$

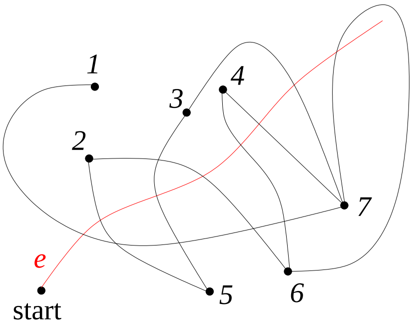
Define two sequence $S_1, S_2$




$$S_1 = 1, 2, 3, 2, 4$$

$$S_2 =$$

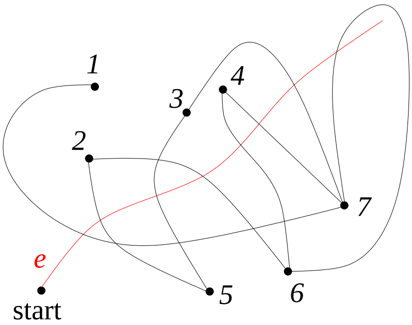
Define two sequence $S_1, S_2$




$$S_1 = 1, 2, 3, 2, 4, 4$$

$$S_2 =$$

Define two sequence $S_1, S_2$




$$S_1 = 1, 2, 3, 2, 4, 4, 3$$

$$S_2 =$$

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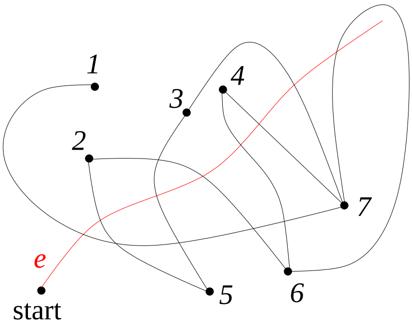
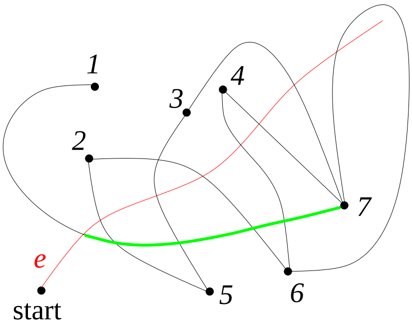



$$S_1 = 1, 2, 3, 2, 4, 4, 3, 6$$

$$S_2 =$$

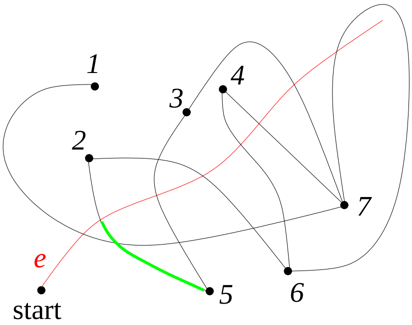
Define two sequence $S_1, S_2$




$$S_1 = 1, 2, 3, 2, 4, 4, 3, 6$$

$$S_2 = 7$$

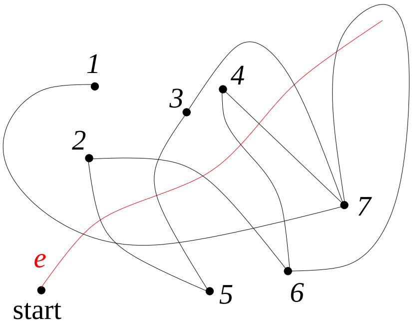
Define two sequence $S_1, S_2$




$$S_1 = 1, 2, 3, 2, 4, 4, 3, 6$$

$$S_2 = 7, 5$$

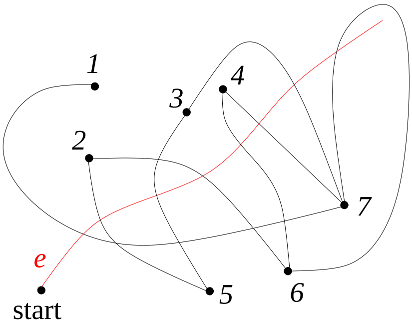
Define two sequence $S_1, S_2$




$$S_1 = 1, 2, 3, 2, 4, 4, 3, 6$$

$$S_2 = 7, 5, 5$$

Define two sequence $S_1, S_2$




$$S_1 = 1, 2, 3, 2, 4, 4, 3, 6$$

$$S_2 = 7, 5, 5, 6$$

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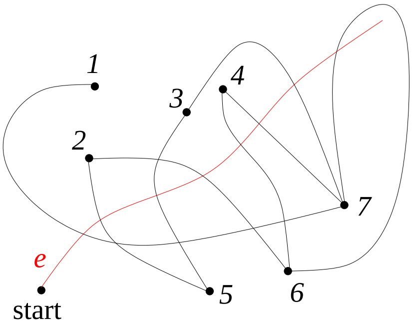


$$S_1 = 1, 2, 3, 2, 4, 4, 3, 6$$

$$S_2 = 7, 5, 5, 6, 6$$

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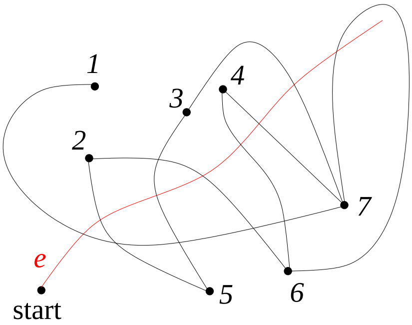
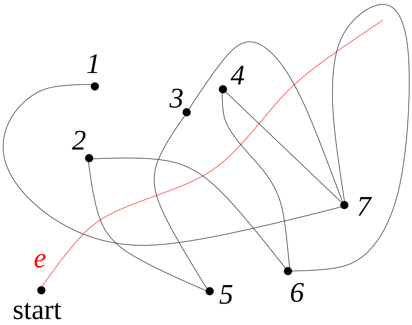



$$S_1 = 1, 2, 3, 2, 4, 4, 3, 6$$

$$S_2 = 7, 5, 5, 6, 6, 7$$

Define two sequence $S_1, S_2$




$$S_1 = 1, 2, 3, 2, 4, 4, 3, 6$$

$$S_2 = 7, 5, 5, 6, 6, 7, 7$$

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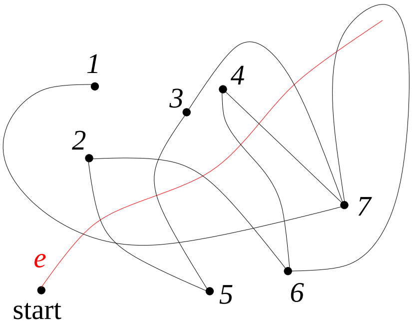


$$S_1 = 1, 2, 3, 2, 4, 4, 3, 6$$

$$S_2 = 7, 5, 5, 6, 6, 7, 7, 7$$

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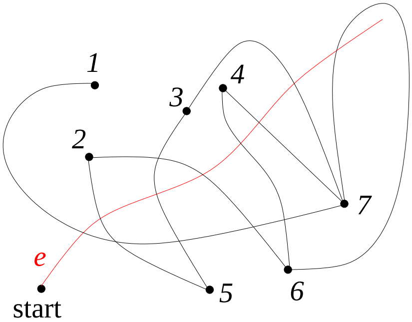
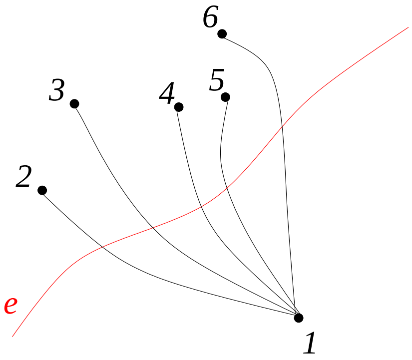



$$S_1 = 1, 2, 3, 2, 4, 4, 3, 6 \geq \Omega\left(\frac{m}{\log^2 n}\right)$$

$$S_2 = 7, 5, 5, 6, 6, 7, 7, 7 \geq \Omega\left(\frac{m}{\log^2 n}\right)$$

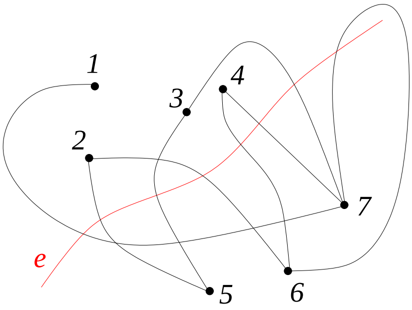
Need to make $S_1$ or $S_2$ $(2^{k^2+k})$-regular.
Observation:




$S_2 = ...., 1, 1, 1, 1, 1, .....$ is bad but $S_1 = ...., 2, 3, 4, 5, 6, ....$

## Theorem (Valtr 1997)

*For fixed $l$ and $S_1, S_2$ defined as above, either $S_1$ or $S_2$ (say $S_1$) has an $l$-regular subsequence $S_1'$ of length $\Omega(|S_1|/l^2) = \Omega(|E'|/l^2)$.*




Say $S_1$ has a $2^{k^2+k}$-regular subsequence of length $\Omega(|E'|/c_k) \geq \Omega\left(\frac{m}{\log^2 n}\right)$.

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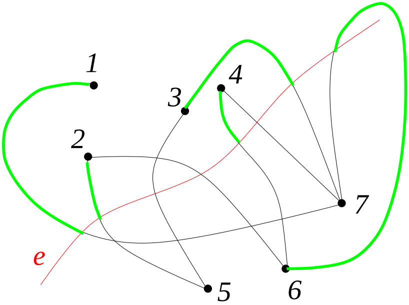
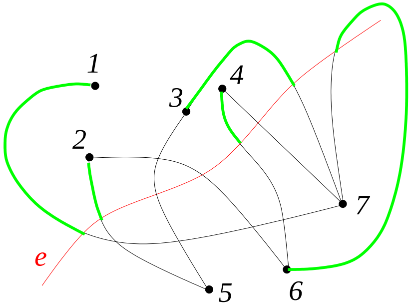


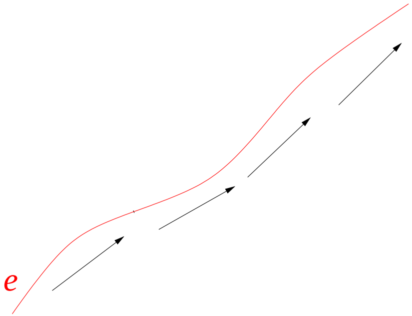

Say $S_1$ has a $2^{k^2+k}$-regular subsequence of length $\Omega(|E'|/c_k) \geq \Omega\left(\frac{m}{\log^2 n}\right)$.

Claim: $S_1'$ does not contain a subsequence of type $up(2^{k^2+k}, 2^k)$.

$$\Omega\left(\frac{m}{\log^2 n}\right) \leq |S_1'| \overset{Klazar}{\leq} c_k n 2^{\alpha^{c_k}(n)}.$$



Indeed, an $up(2^{k^2+k}, 2^k)$ subsequence creates $k$ pairwise crossing edges.




$2^k$ blocks, each block corresponds to $2^{k^2+k}$ distinct vertices.

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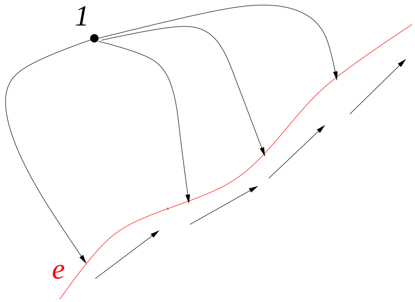


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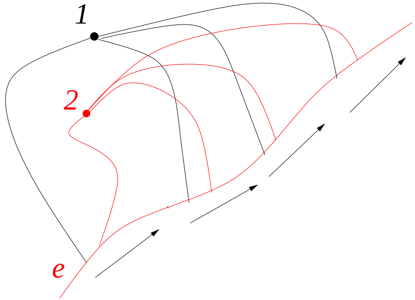


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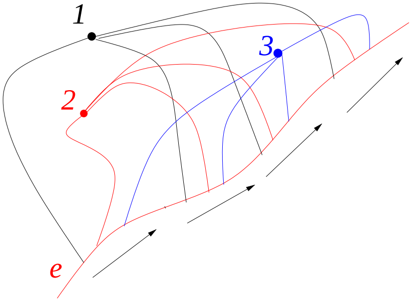


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# Other problems

1) Maximum unit distance among $n$ points in convex position. Conjecture $O(n)$ (Erdős). Best known $O(n \log n)$ by Füredi.

2) Maximum number of edges in a simple topological graph with no $k$ pairwise disjoint edges. Conjecture $O(n)$. Best known $O(n \log^{5k} n)$ by Pach and Tóth.

**Thank you!**