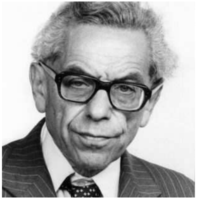
On the Erdős-Szekeres convex polygon problem

Andrew Suk

November 6, 2016

Origins of Ramsey theory

"A combinatorial problem in geometry," by Paul Erdős and George Szekeres (1935)



Erdős' first combinatorial paper, previous 9 were in number theory.

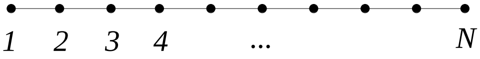
Origins of Ramsey theory

"On a problem of formal logic," by Frank Ramsey (1930)

Informal definition: Every sufficiently large system contains a large well-organized subsystem. In other words, complete disorder is impossible.

Ramsey theory

Formal definition: For any integers $k \geq 1$, $s, n \geq k$, there is a minimum $R_k(s, n) = N$, such that for every red/blue coloring of the k-tuples of $\{1, 2, \ldots, N\}$,



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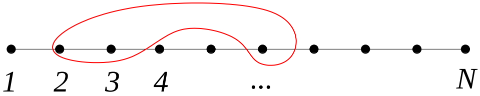

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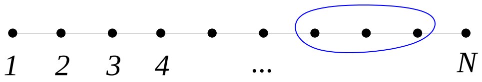

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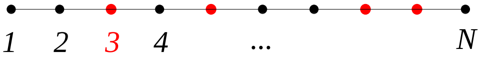

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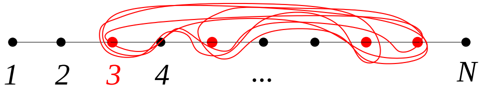

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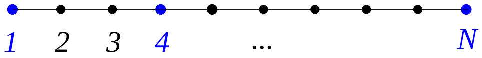

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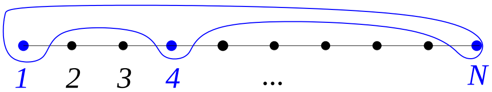

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$R_k(s, n)$ = Ramsey numbers

Original paper, 1935

A Combinatorial Problem in Geometry

by

P. Erdös and G. Szekeres
Manchester

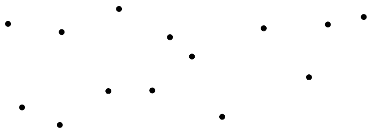
INTRODUCTION.

Our present problem has been suggested by Miss Esther Klein in connection with the following proposition.

From 5 points of the plane of which no three lie on the same straight line it is always possible to select 4 points determining a convex quadrilateral.

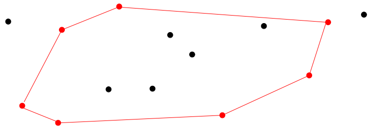
We present E. Klein's proof here because later on we are going to make use of it. If the least convex polygon which encloses the points is a quadrilateral or a pentagon the theorem is trivial. Let therefore the enclosing polygon be a triangle ABC. Then the two remaining points D and E are inside ABC. Two of the given points (say A and C) must lie on the same side of the connecting straight line $\overline{DE}$. Then it is clear that $AEDC$ is a convex quadrilateral.

1) Rediscovered Ramsey's theorem, 2) Monotone Subsequence theorem, 3) Convex polygon theorem.



Problem (Esther Klein 1933)

Given an integer n, is there a minimal integer $ES(n)$, such that any set of at least $ES(n)$ points in the plane in general position, contains n members in convex position?



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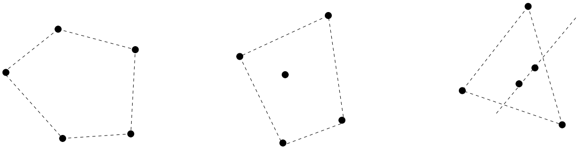
Given an integer n, is there a minimal integer $ES(n)$, such that any set of at least $ES(n)$ points in the plane in general position, contains n members in convex position?

First proof

Theorem (Erdős-Szekeres, 1935)

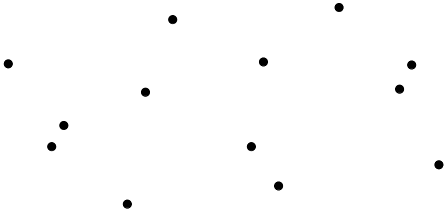
For every $n \geq 3$, $ES(n)$ exists.



First proof

Theorem (Erdős-Szekeres, 1935)

For every $n \geq 3$, $ES(n) \leq R_4(5, n)$. Set $N = R_4(5, n)$.



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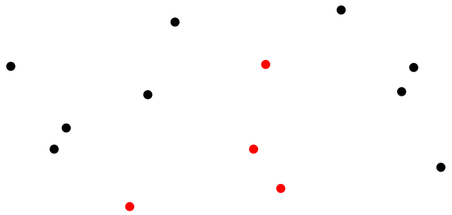

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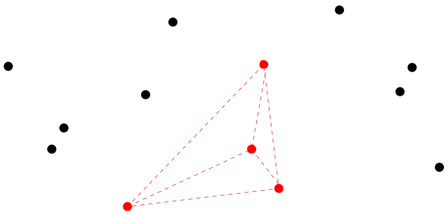

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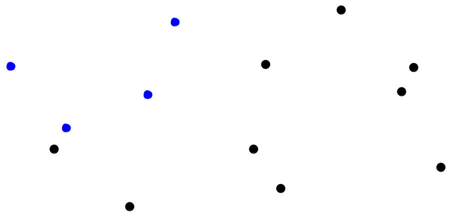

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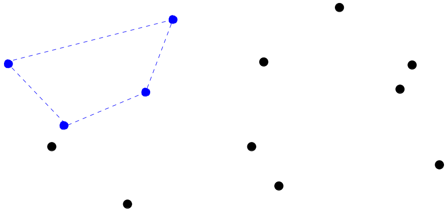

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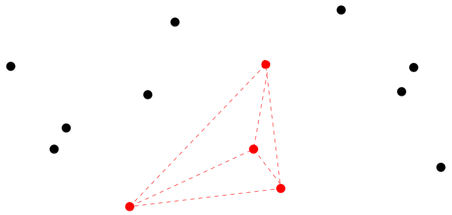

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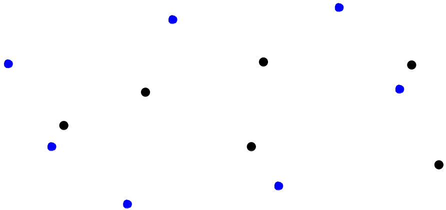

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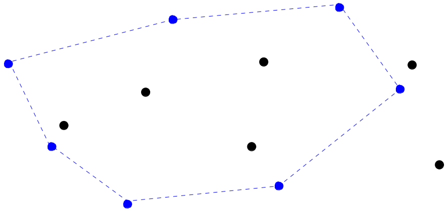


First Proof

Theorem (Erdős-Rado, 1952)

$$ES(n) \leq R_4(5, n) < 2^{2^{cn^4}}.$$

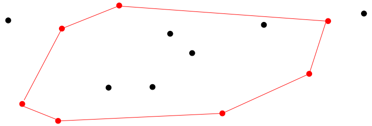
Theorem (Colon-Fox-Sudakov, 2008)

$$ES(n) \leq R_4(5, n) < 2^{2^{cn^2 \log n}}.$$

Theorem (Mubayi-S., 2015)

$$R_4(5, n) > 2^{n^{c \log \log n}}.$$

Second Proof



Theorem (Erdős-Szekeres 1935, 1960)

$$2^{n-2} + 1 \leq ES(n) \leq \binom{2n-4}{n-2} + 1 = O(4^n/\sqrt{n}).$$

Conjecture: $ES(n) = 2^{n-2} + 1,\ n \geq 3.$

Exact values

Conjecture: $ES(n) = 2^{n-2} + 1$

Trivial: $ES(3) = 3$

Klein (1933): $ES(4) = 5$

Makai and Turán (1935): $ES(5) = 9$

Szekeres-Peters (2006): $ES(6) = 17$.

For $n \geq 7$, $ES(n)$ is still unknown.

Towards the conjecture $ES(n) = 2^{n-2} + 1$

1935, Erdős-Szekeres: $\binom{2n-4}{n-2} + 1$

1998, Chung-Graham: $\binom{2n-4}{n-2}$

1998, Kleitman-Pachter: $\binom{2n-4}{n-2} - 2n + 7$

1998, Tóth-Valtr: $\binom{2n-5}{n-2} + 2 \sim \frac{1}{2}\binom{2n-4}{n-2}$

2005, Tóth-Valtr: $\binom{2n-5}{n-2} + 1$

2015, Norin-Yuditsky and Mojarrad-Vlachos: $\limsup_{n\to\infty} \frac{ES(n)}{\binom{2n-4}{n-2}} \leq \frac{7}{16}$.

$2^{n-2} + 1 \leq ES(n) \leq 4^{n-o(n)}$.

Inspired many variants: Higher-dimensions, cyclic polytopes, lines, convex bodies, etc.

The original upper bound

$$ES(n) \leq \binom{2n-4}{n-2} + 1.$$

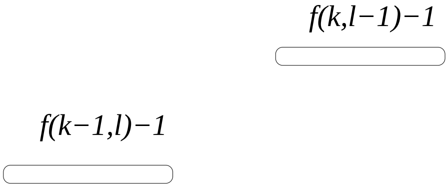
Theorem (Cups-Caps Theorem)

Let $f(k,\ell)$ be the smallest integer N such that any N-element point set in the plane in general position contains either a k-cup or an ℓ-cap. Then

$$f(k,\ell) = \binom{k+\ell-4}{k-2} + 1.$$

Cups-caps construction

$$f(k,\ell) \geq f(k-1,\ell) + f(k,\ell-1) - 1$$



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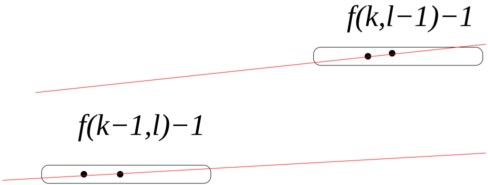

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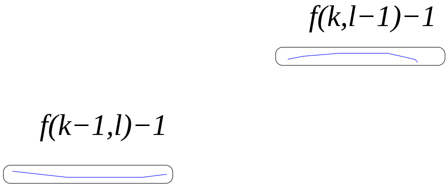
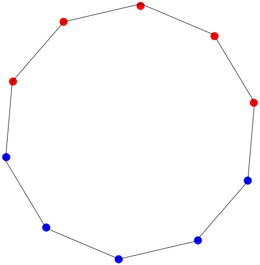


"typical" convex n-gon



Union of an $(n/2)$-cup and an $(n/2)$-cap. Note that:
$f(n/2, n/2) = 2^{n-o(n)}$

Question: Can we (somehow) combine the cups and caps from the cup-cap theorem?

Theorem (S. 2016)

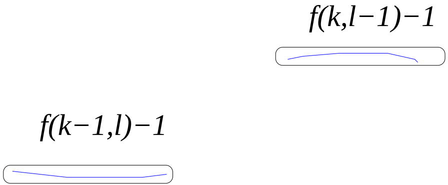
For $n \geq n_0$, where n_0 is a large absolute constant

$$ES(n) \leq 2^{n+2n^{3/4}}.$$

Conjecture: $ES(n) = 2^{n-2} + 1$

Erdős offered \$500 for a proof (Graham offered \$1000).

$$f(k,\ell) \geq f(k-1,\ell) + f(k,\ell-1) - 1$$



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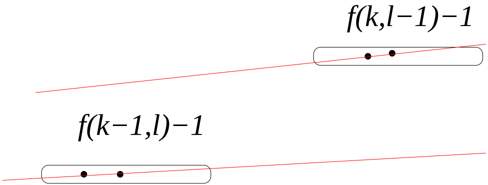
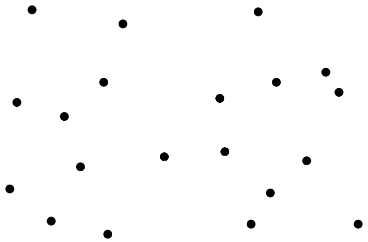


mutually avoiding sets

Theorem (Aronov-Erdős-Goddard-Kleitman-Klugerman-Pach-Schulman 1991)

Let P be an N-element planar point set in general position. Then there are subsets $A, B \subset P$ such that $|A|, |B| \geq \sqrt{N/12}$ and A and B are mutually avoiding.



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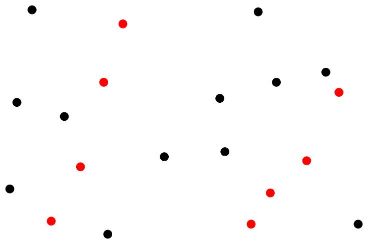

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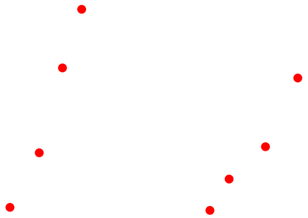

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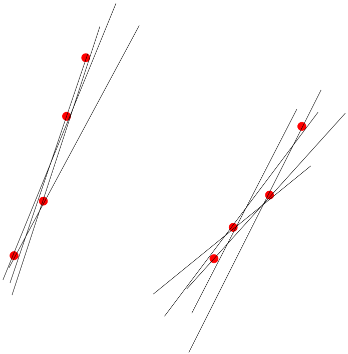

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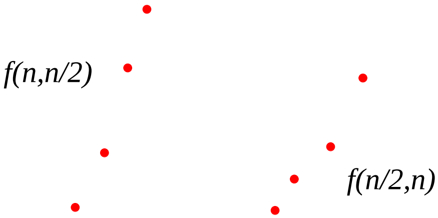

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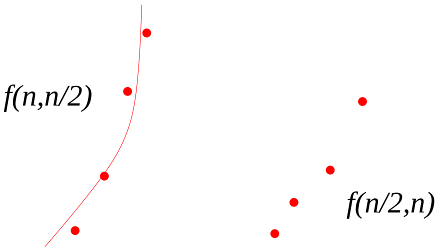

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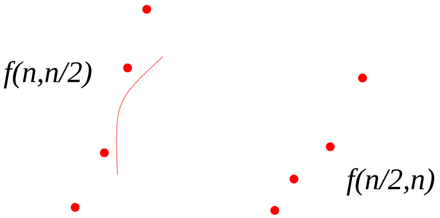

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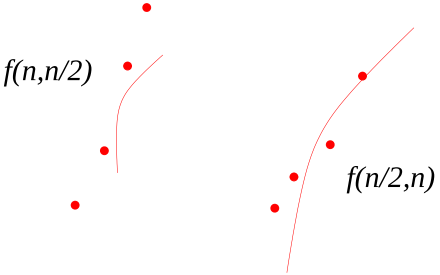

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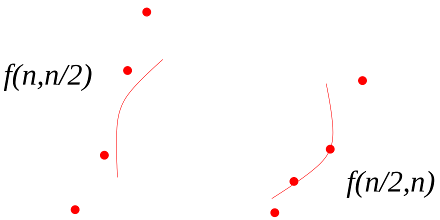

$$\sqrt{N} = f(n, n/2) \approx (2.6)^n \Longrightarrow N \approx (6.75)^n.$$

mutually avoiding sets

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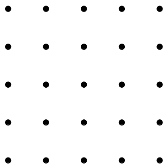
Let P be an N-element planar point set in general position. Then there are subsets $A, B \subset P$ such that $|A|, |B| \geq \sqrt{N/12}$ and A and B are mutually avoiding.

Theorem (Valtr 1994)

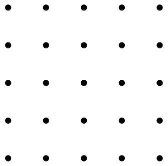
Any point set P with $|P| = N$ and with ratio $c\sqrt{N}$, contains no pair of mutually avoiding sets of size more than $c'\sqrt{N}$.



Grid-like point sets contain large cups and caps

Theorem (Alon, Katchalski, Pulleyblank 1989, Valtr 1994)

Any point set P with $|P| = N$ and with ratio $c\sqrt{N}$, contains $\Omega(N^{1/3})$ points in convex position.



Theorem (S. 2016)

For $n \geq n_0$, where n_0 is a large absolute constant

$$ES(n) \leq 2^{n+2n^{3/4}}.$$

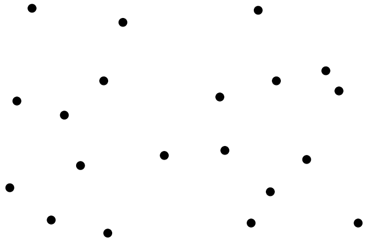
Basic Idea: Grid like ⇒ large cups and caps
Not grid like ⇒ large mutually avoiding sets.

Theorem (S. 2016)

For $n \geq n_0$, where n_0 is a large absolute constant

$$ES(n) = 2^{n+o(n)}.$$

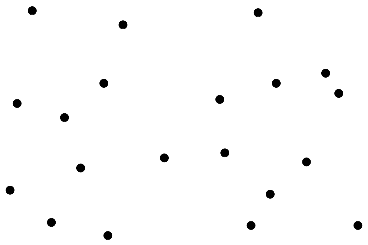
Proof. $|P| = 2^{n+2n^{3/4}}$.



Theorem (Pach-Solymosi 1998, Pór-Valtr 2002)

For $|P| \geq 16^k$, there is a k-element subset $X \subset P$ such that X is either a k-cup or a k-cap, and the regions $T_1, \ldots, T_{k-1}$ from the support of X satisfies $|T_i \cap P| \geq \frac{|P|}{2^{40k}}$.

Proof. $|P| = 2^{n+2n^{3/4}}$.



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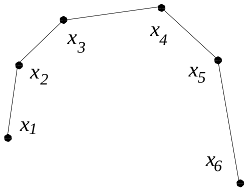
For $|P| \geq 16^k$, there is a k-element subset $X \subset P$ such that X is either a k-cup or a k-cap, and the regions $T_1, \ldots, T_{k-1}$ from the support of X satisfies $|T_i \cap P| \geq \frac{|P|}{2^{40k}}$.

Proof. $|P| = 2^{n+2n^{3/4}}$.



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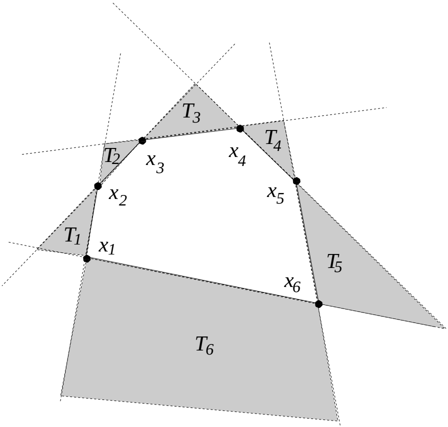

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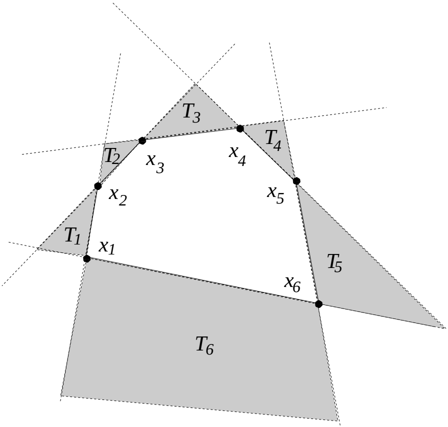

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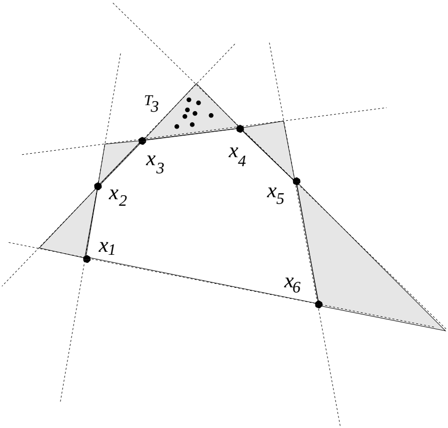
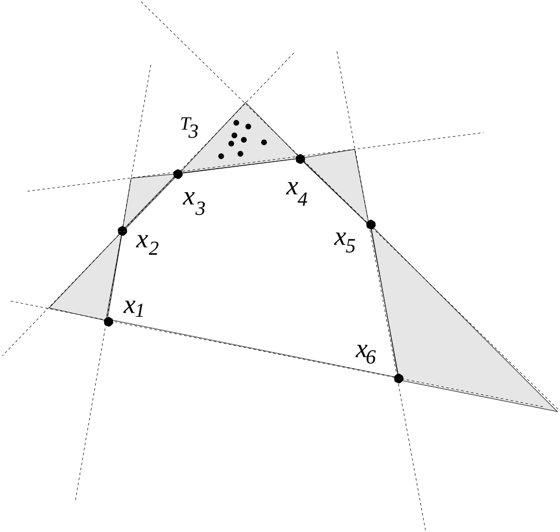


Define partial order of $P_i = T_i \cap P$, where $p \prec q$ iff $q \in conv(p \cup x_{i-1}x_{i+2})$



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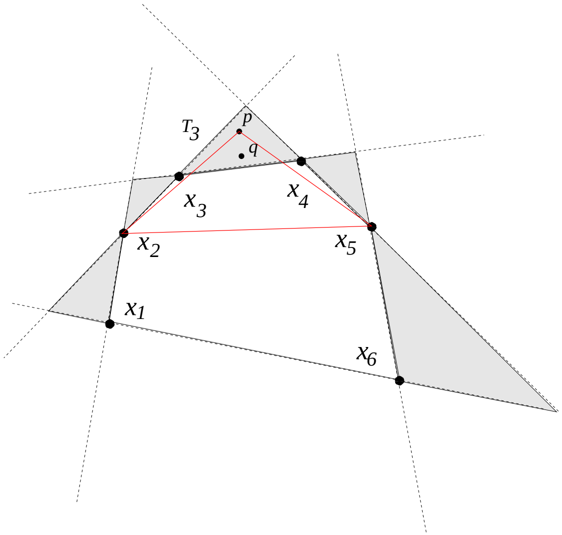

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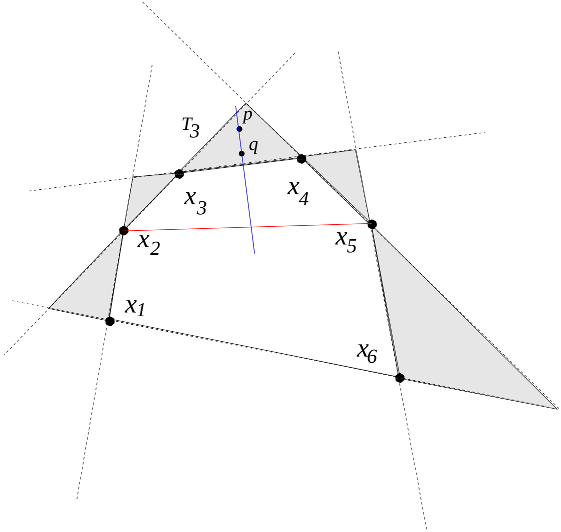

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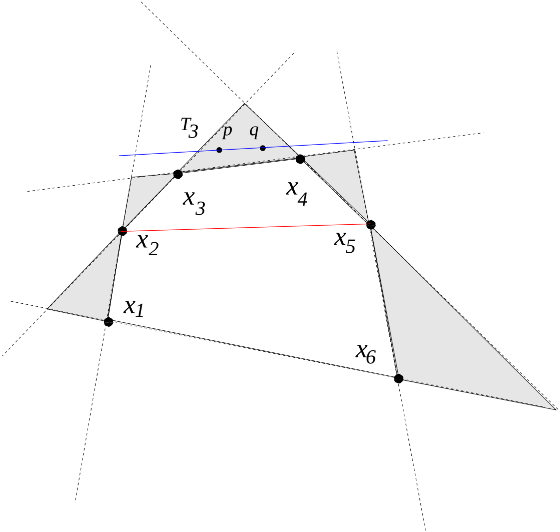

$N = 2^{n+2n^{3/4}}$, $k = n^{2/3}$. Set $\alpha = n^{-1/4}$.

Dilworth's Theorem: Each P_i contains either

1. Antichain size $\left(\frac{N}{2^{40k}}\right)^{\alpha} = 2^{n^{3/4}+2n^{1/2}-40n^{5/12}}$
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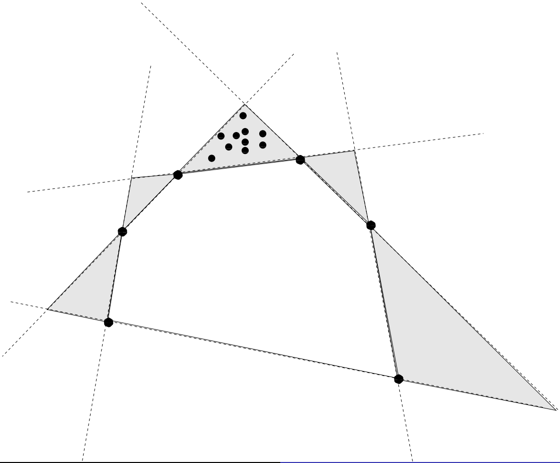

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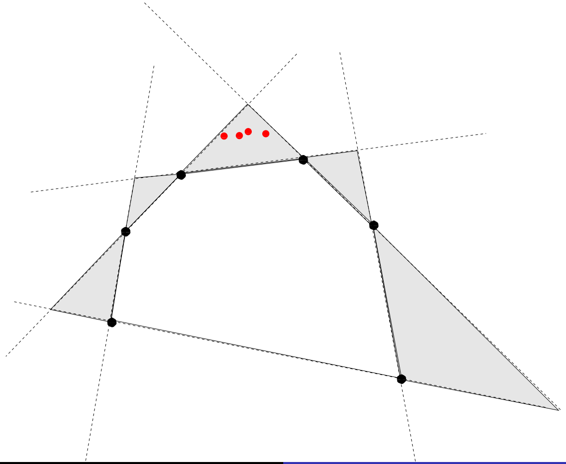

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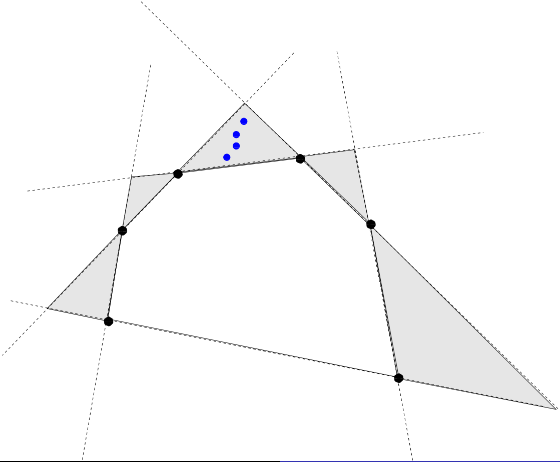

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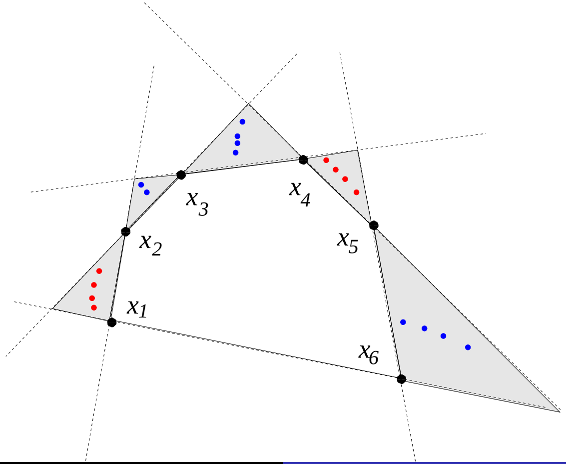
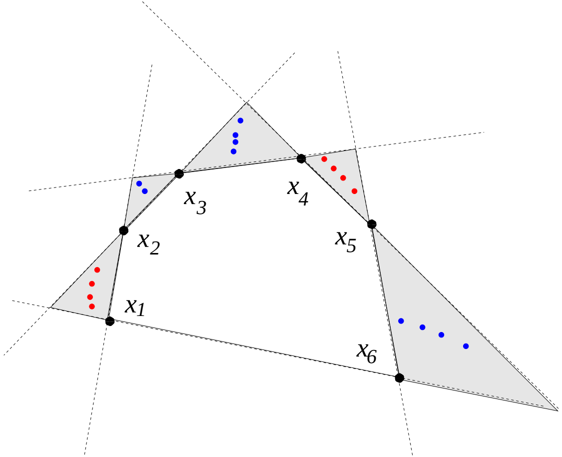


$k = n^{2/3}$ regions.

By Pigeonhole:

1. $\frac{\sqrt{k}}{2}$ nonadjacent regions have antichains of size $\left(\frac{N}{2^{40k}}\right)^{\alpha}$.
2. $\sqrt{k}$ consecutive regions have chains of size $\left(\frac{N}{2^{40k}}\right)^{1-\alpha}$



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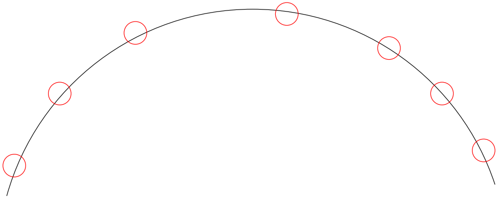

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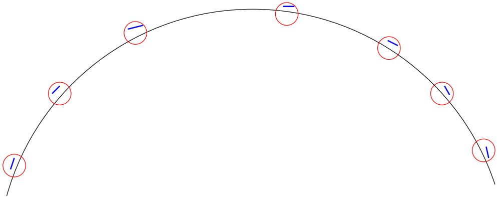

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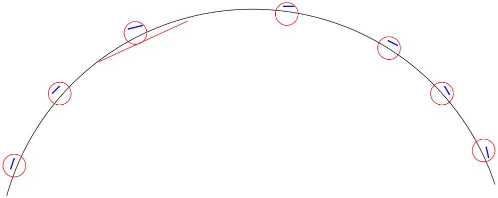

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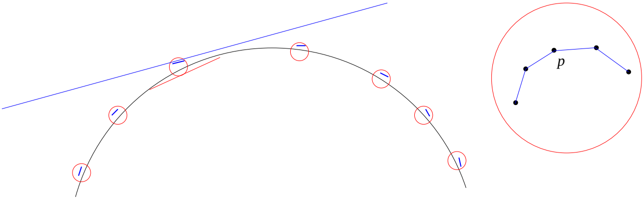

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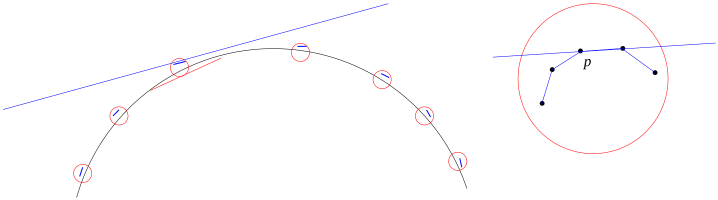

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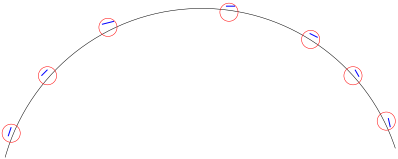

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Many large mutually avoiding sets

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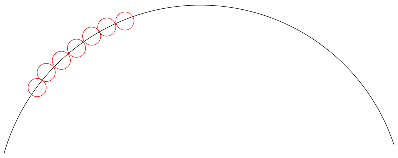

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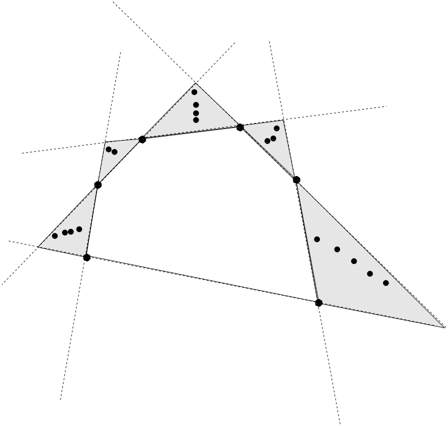

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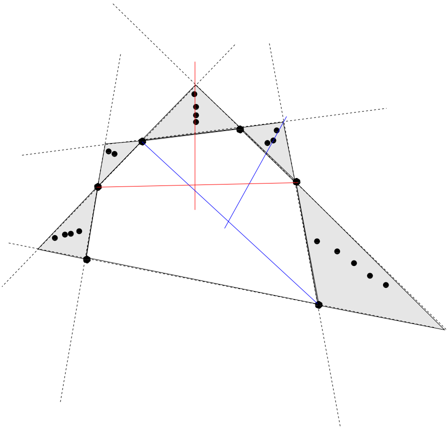

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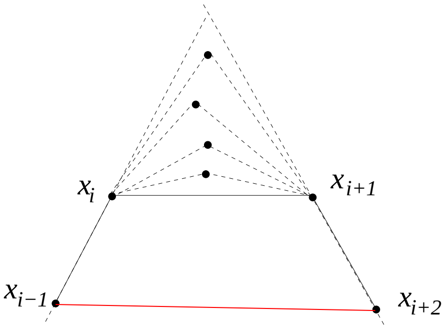

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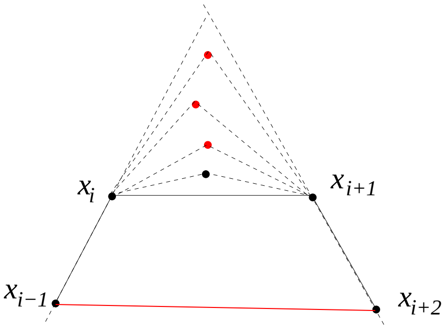

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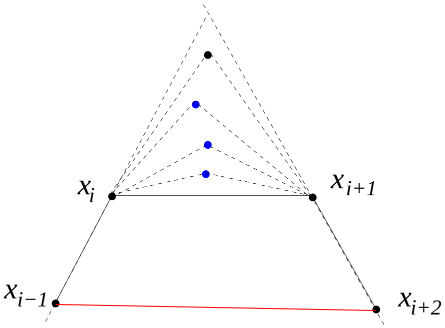

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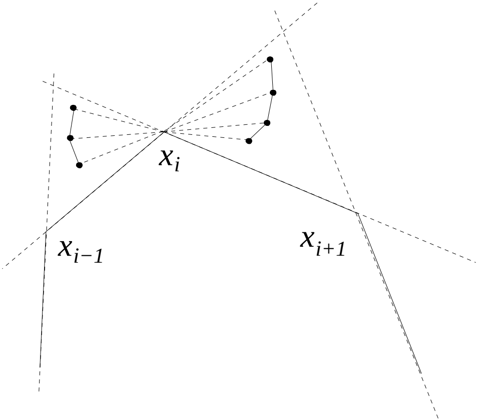

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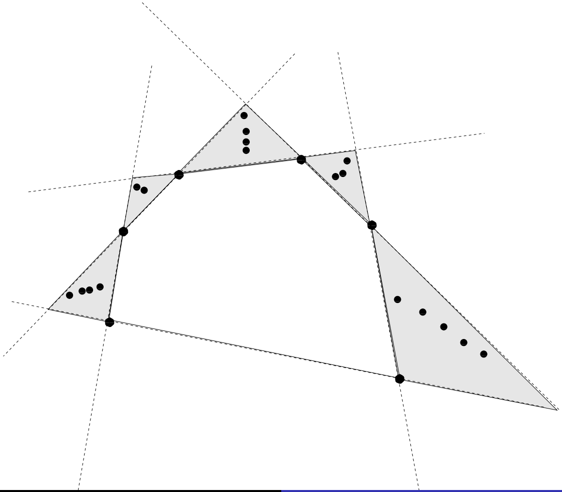

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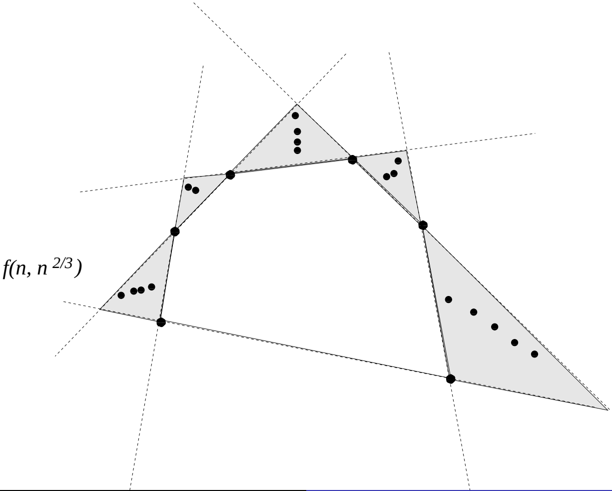

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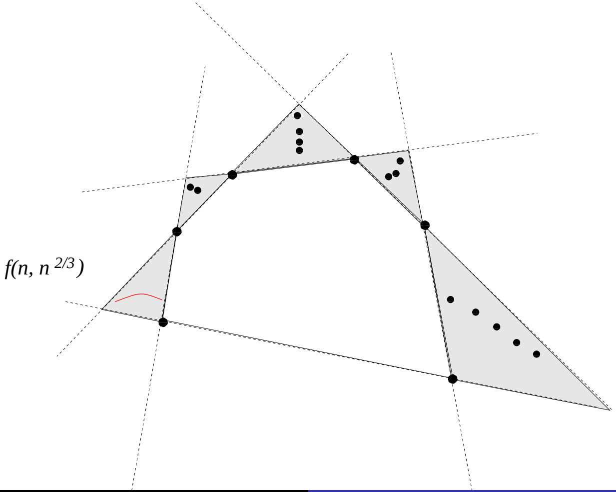

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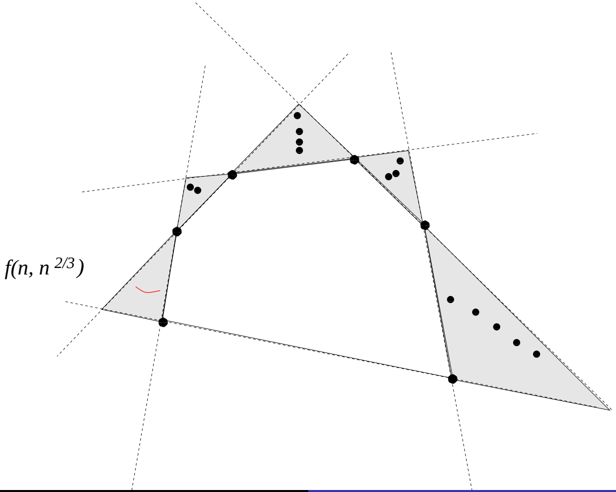

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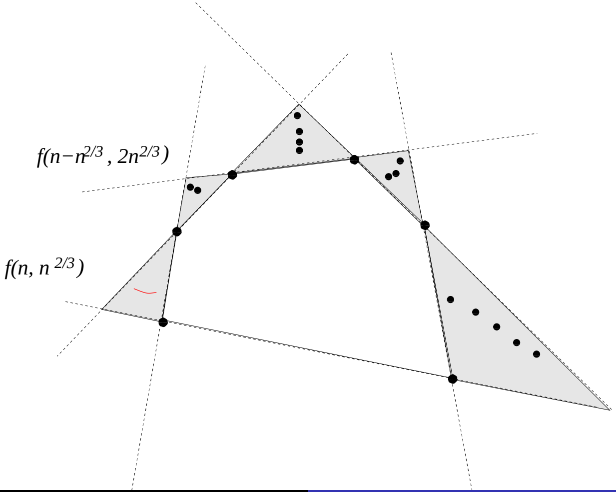

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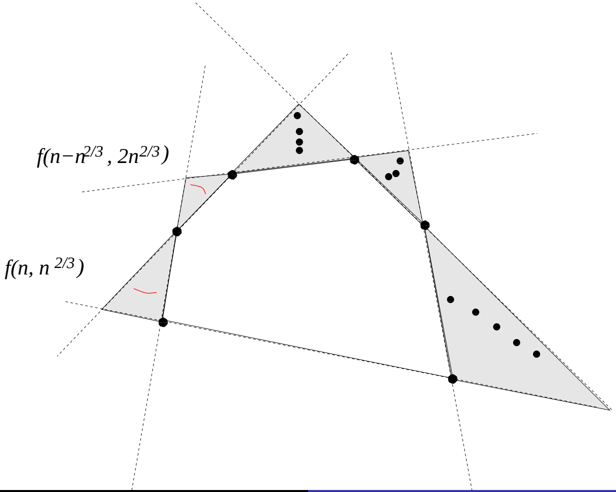

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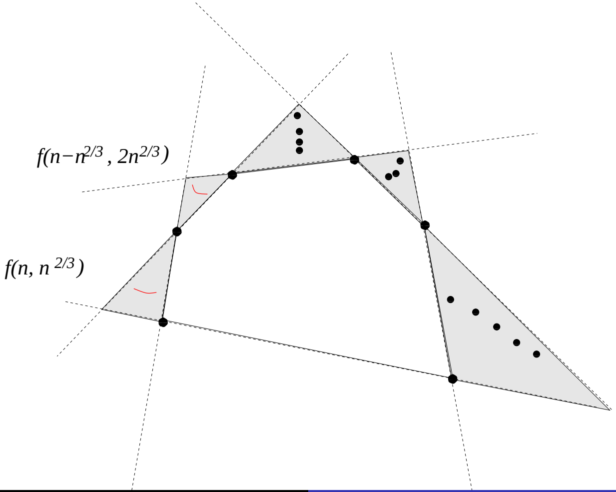

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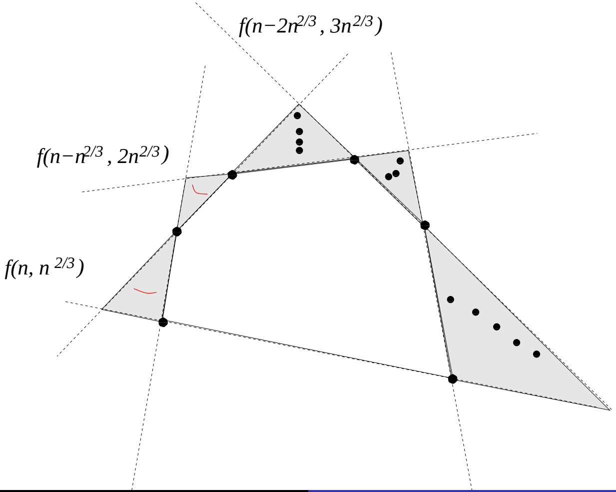

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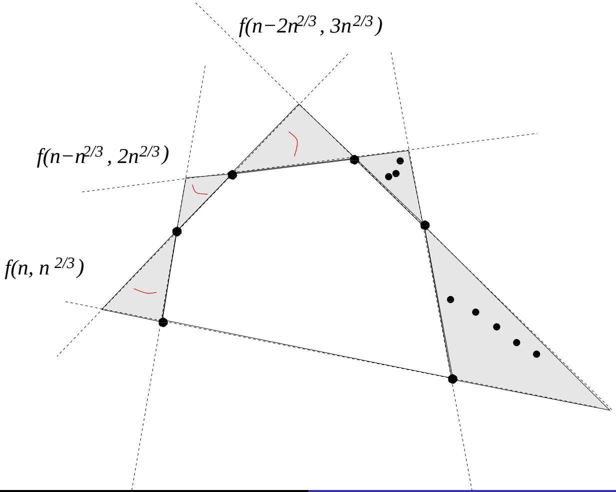

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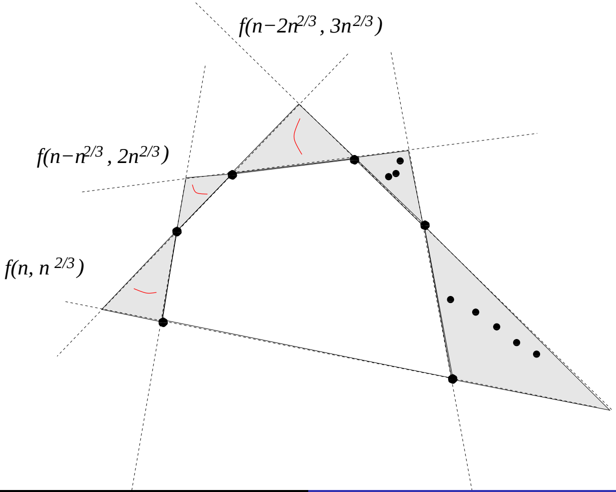

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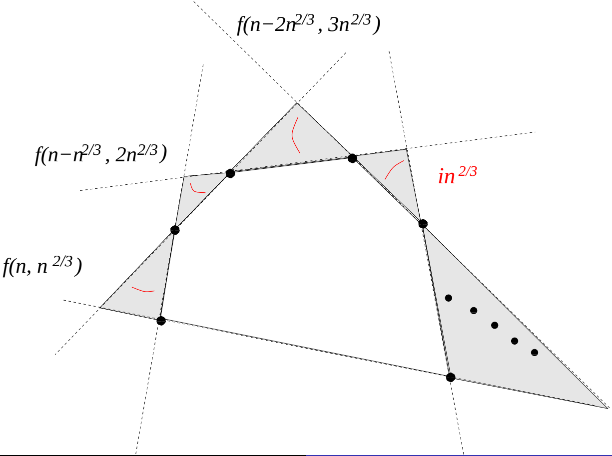

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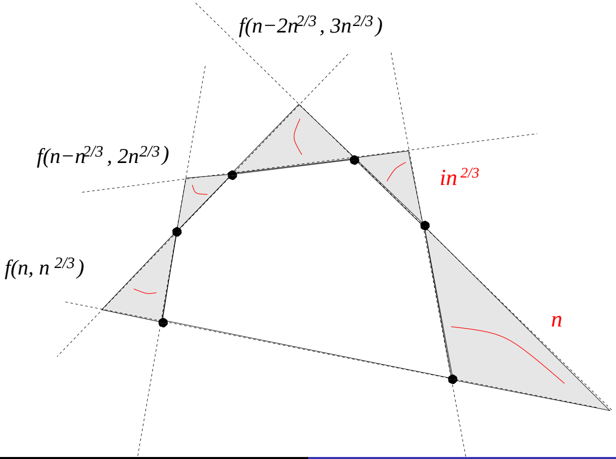

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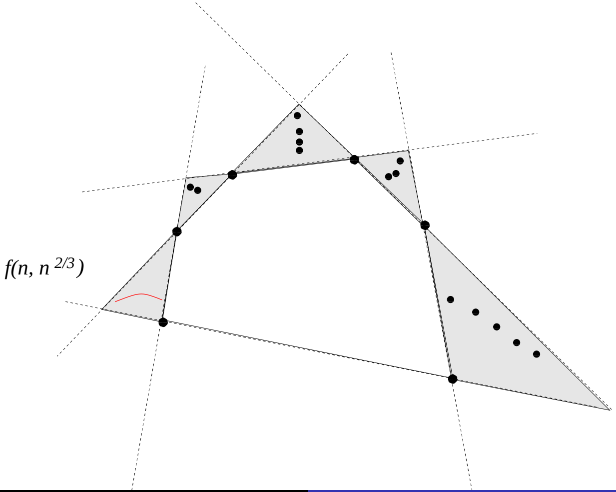

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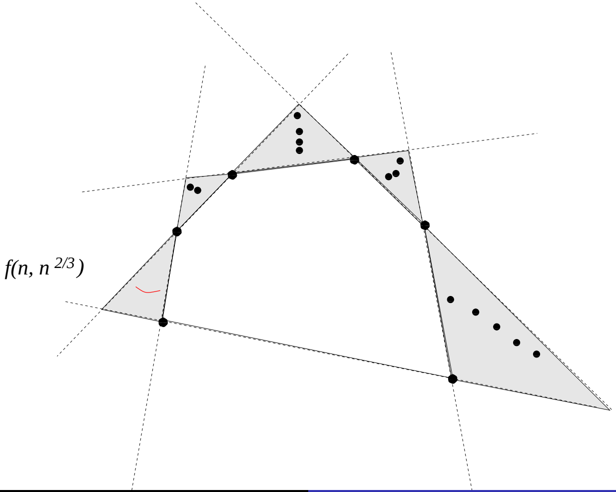

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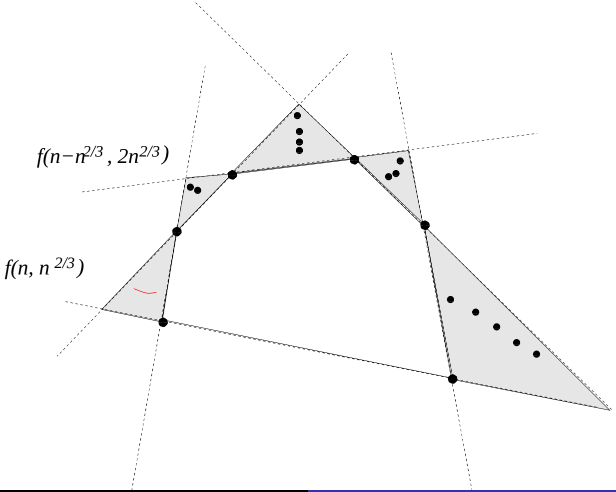

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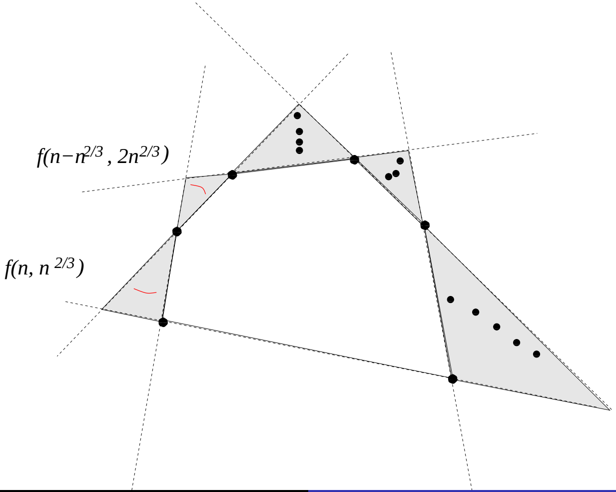
$\sqrt{k} = n^{1/3}$ of the P_i-s are consecutive chains.

2. Chain of size $\left(\frac{N}{2^{40k}}\right)^{1-\alpha} = 2^{n+n^{3/4}-40n^{2/3}-2n^{1/2}}$



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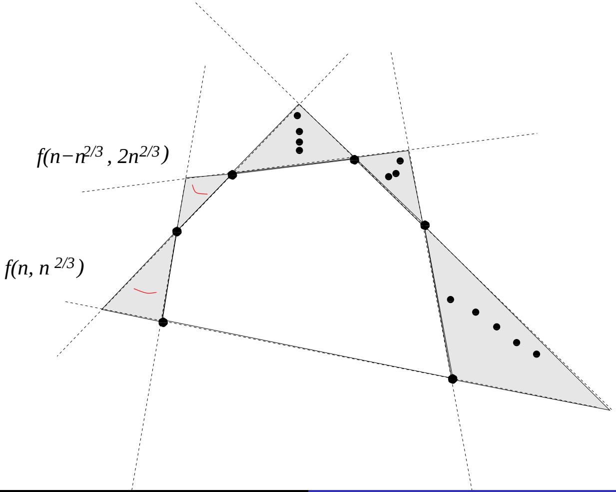

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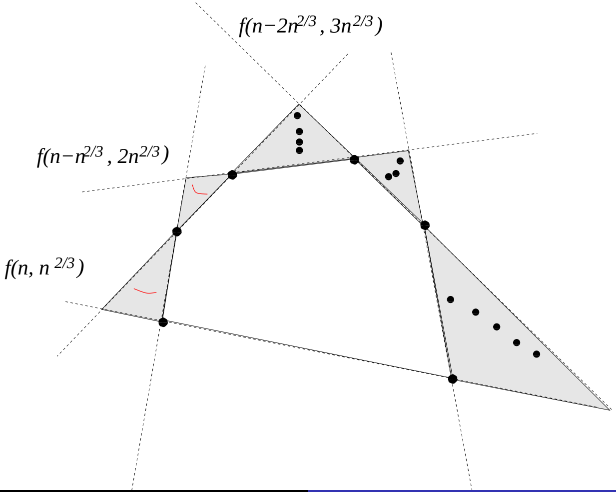

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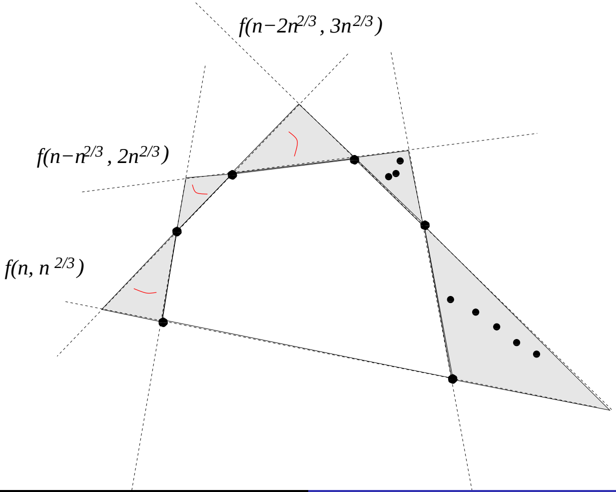

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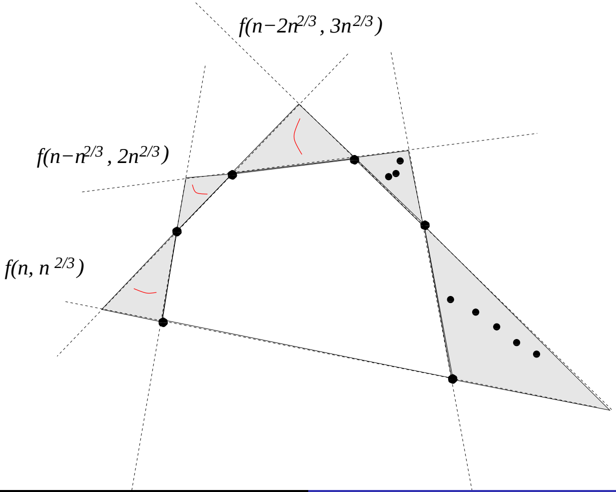

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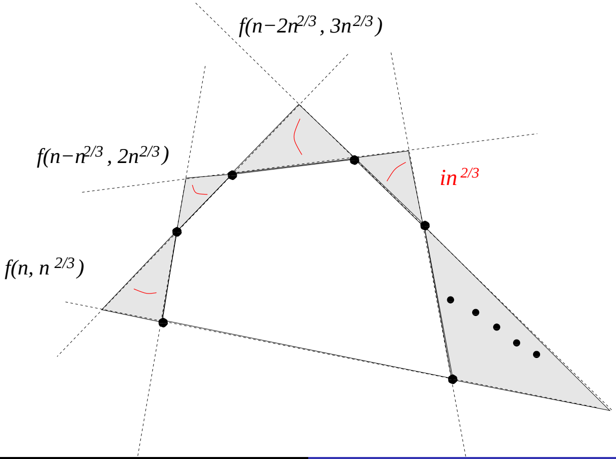

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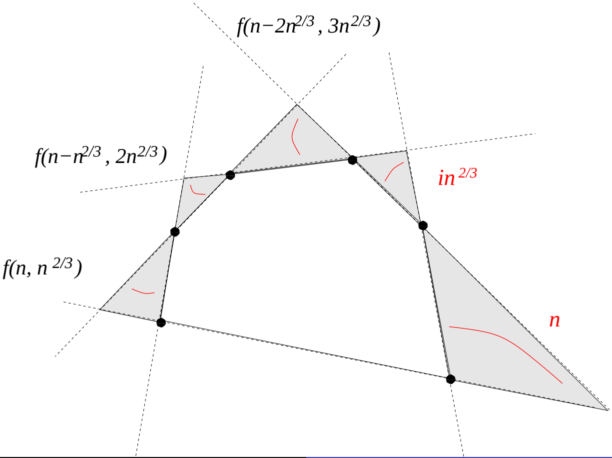

Higher dimensions

$ES_d(n)$ =smallest integer such that any set of $ES_d(n)$ points in $\mathbb{R}^d$ in general position contains n members in convex position.

Theorem (Károlyi 2001)

$$ES_d(n) \leq ES_{d-1}(n-1) + 1.$$

$$ES_d(n) \leq ES(n-d+2) + d - 2 \leq 2^{n+o(n)}.$$

Conjecture (Füredi)

$ES_3(n) = 2^{\Theta(\sqrt{n})}$.

$ES_d(n) = 2^{\Theta(n^{1/(d-1)})}$.

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Theorem (Károlyi-Valtr 2003)

$$ES_d(n) \geq 2^{cn^{1/(d-1)}}.$$

$ES_d(n) \leq 2^{n+o(n)}$.

Thank you!