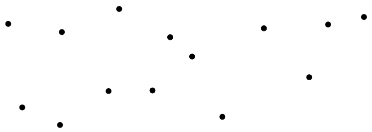
On the Erdős-Szekeres convex polygon problem

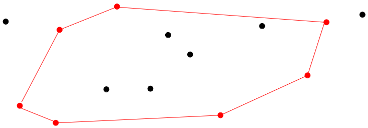
Andrew Suk

May 25, 2016



Problem (Esther Klein 1933)

Given an integer n, is there a minimal integer $ES(n)$, such that any set of at least $ES(n)$ points in the plane in general position, contains n members in convex position?



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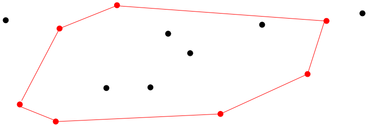


Theorem (Erdős-Szekeres 1935, 1960)

$$2^{n-2} + 1 \leq ES(n) \leq \binom{2n-4}{n-2} + 1 = O(4^n/\sqrt{n}).$$

Conjecture: $ES(n) = 2^{n-2} + 1$

Exact values

Conjecture: $ES(n) = 2^{n-2} + 1$

Klein (1933): $ES(4) = 5$

Makai (1935): $ES(5) = 9$

Szekeres-Peters (2006): $ES(6) = 17$.

For $n \geq 7$, $ES(n)$ is still unknown.

Towards the conjecture $ES(n) = 2^{n-2} + 1$

1935, Erdős-Szekeres: $\binom{2n-4}{n-2} + 1$

1998, Chung-Graham: $\binom{2n-4}{n-2}$

1998, Kleitman-Pachter: $\binom{2n-4}{n-2} - 2n + 7$

1998, Tóth-Valtr: $\binom{2n-5}{n-2} + 2$

2005, Tóth-Valtr: $\binom{2n-5}{n-2} + 1$

2015, Vlachos: $\limsup_{n\to\infty} \frac{ES(n)}{\binom{2n-5}{n-2}} \leq \frac{29}{32}$.

2015, Norin-Yuditsky and Mojarrad-Vlachos: $\limsup_{n\to\infty} \frac{ES(n)}{\binom{2n-5}{n-2}} \leq \frac{7}{8}$.

$= 4^{n-o(n)}$.

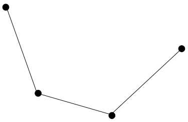
The original upper bound

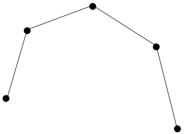
$$ES(n) \leq \binom{2n-4}{n-2} + 1.$$

Theorem (Cups-Caps Theorem)

Let $f(k,\ell)$ be the smallest integer N such that any N-element point set in the plane in general position contains either a k-cup or an ℓ-cap. Then

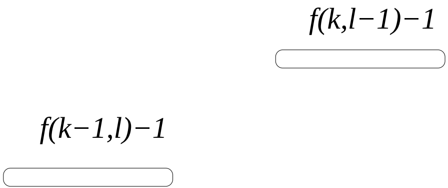
$$f(k,\ell) = \binom{k+\ell-4}{k-2} + 1.$$





Cups-caps construction

$$f(k,\ell) \geq f(k-1,\ell) + f(k,\ell-1) - 1$$



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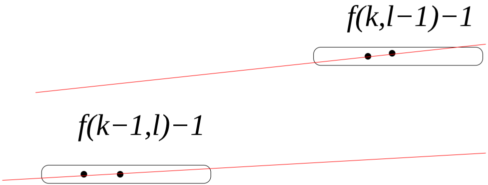

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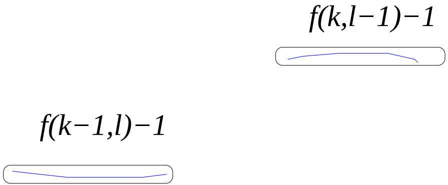
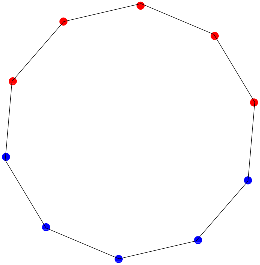


"typical" convex n-gon



Union of an $(n/2)$-cup and an $(n/2)$-cap. Note that:
$f(n/2, n/2) = 2^{n-o(n)}$

Question: Can we (somehow) combine the cups and caps from the cup-cap theorem?

Theorem (S. 2016)

For $n \geq n_0$, where n_0 is a large absolute constant

$$ES(n) \leq 2^{n+2n^{3/4}}.$$

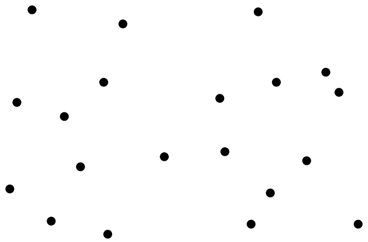
Conjecture: $ES(n) = 2^{n-2} + 1$

Erdős offered $500 for a proof (Graham offered $1000).

mutually avoiding sets

Theorem (Aronov-Erdős-Goddard-Kleitman-Klugerman-Pach-Schulman 1991)

Let P be an N-element planar point set in general position. Then there are subsets $A, B \subset P$ such that $|A|, |B| \geq \sqrt{N}$ and A and B are mutually avoiding.



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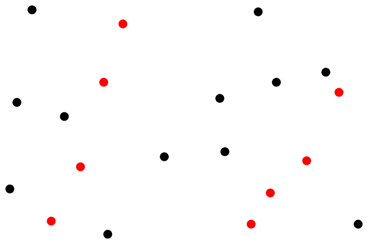

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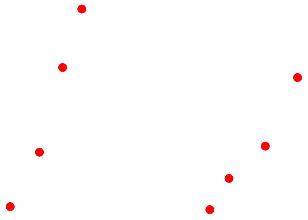

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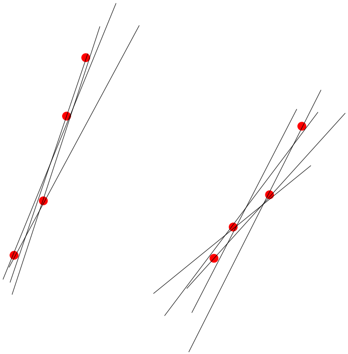

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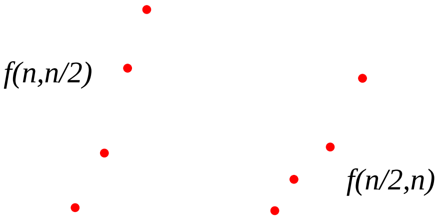

$$\sqrt{N} = f(n, n/2) \approx (2.6)^n \Longrightarrow N \approx (6.75)^n.$$

mutually avoiding sets

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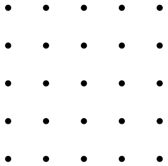
Let P be an N-element planar point set in general position. Then there are subsets $A, B \subset P$ such that $|A|, |B| \geq \sqrt{N}$ and A and B are mutually avoiding.

Theorem (Valtr 1994)

Any point set P with $|P| = N$ and with ratio $c\sqrt{N}$, contains no pair of mutually avoiding sets of size more than $c'\sqrt{N}$.

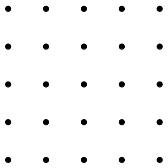


Grid-like point sets contain large cups and caps

Theorem (Valtr 1994)

Any point set P with $|P| = N$ and with ratio $c\sqrt{N}$, contains $\Omega(N^{1/3})$ points in convex position.

Basic Idea: Grid like $\Rightarrow$ large cups and caps
Not grid like $\Rightarrow$ large mutually avoiding sets.



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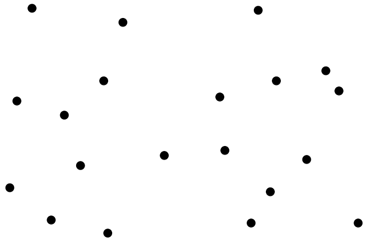
Proof. $|P| = 2^{n+2n^{3/4}}$.



Theorem (Pach-Solymosi 1998, Pór-Valtr 2002)

For $|P| \geq 16^k$, there is a k-element subset $X \subset P$ such that X is either a k-cup or a k-cap, and the regions $T_1, \ldots, T_{k-1}$ from the support of X satisfies $|T_i \cap P| \geq \frac{|P|}{2^{40k}}$.

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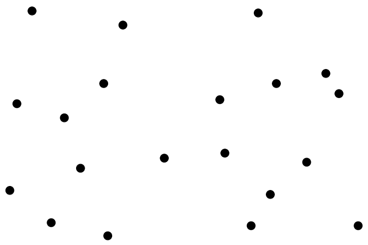

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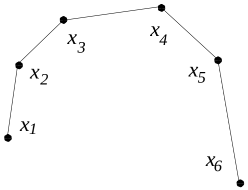

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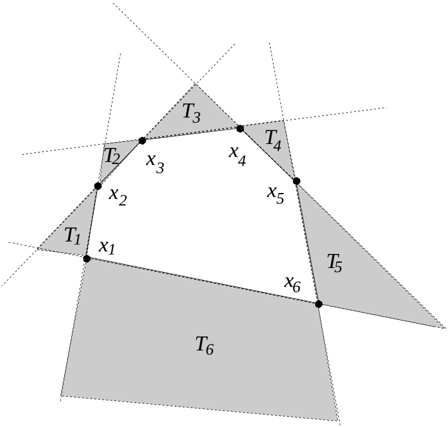

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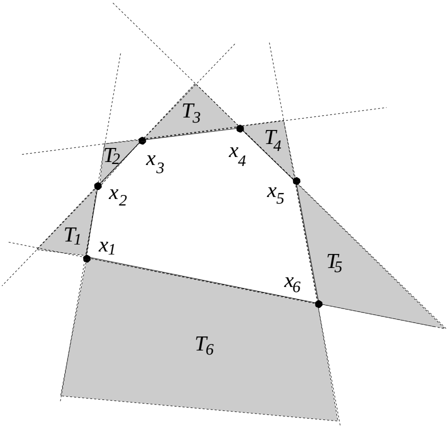
Proof. $|P| = 2^{n+2n^{3/4}}$. $k = n^{2/3}$, $|T_i \cap P| \geq 2^{n+2n^{3/4}-40n^{2/3}}$.



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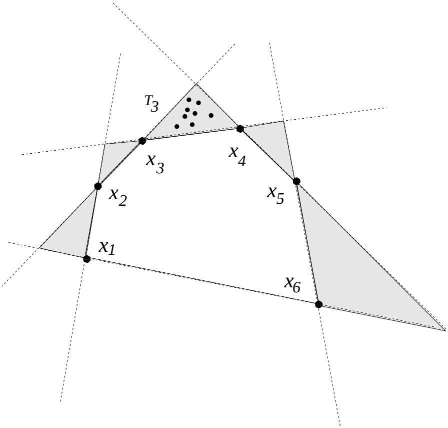
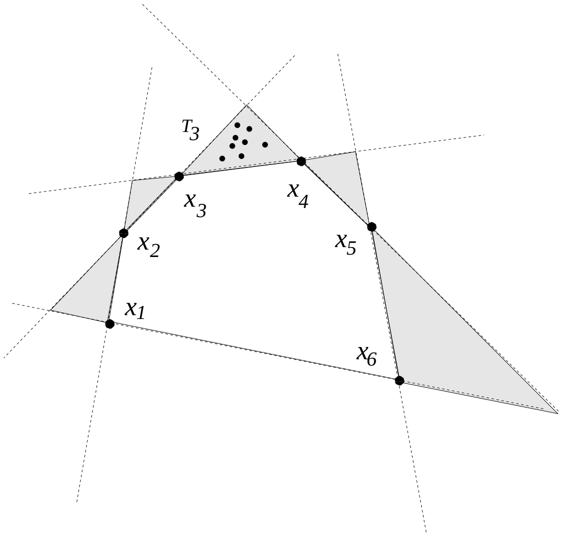


Define partial order of $P_i = T_i \cap P$, where $p \prec q$ iff $q \in conv(p \cup x_{i-1}x_{i+2})$



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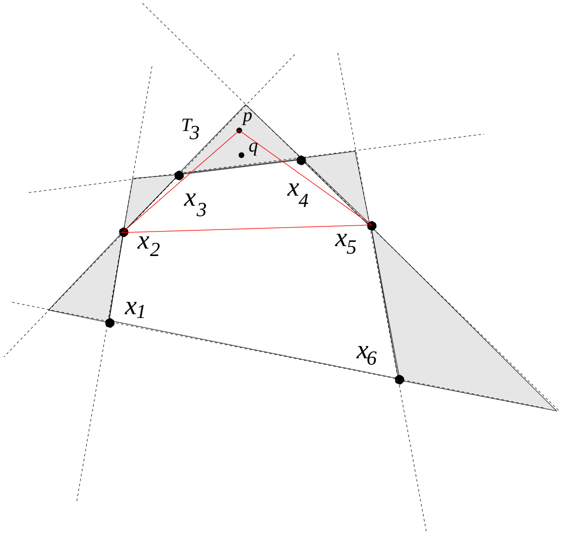
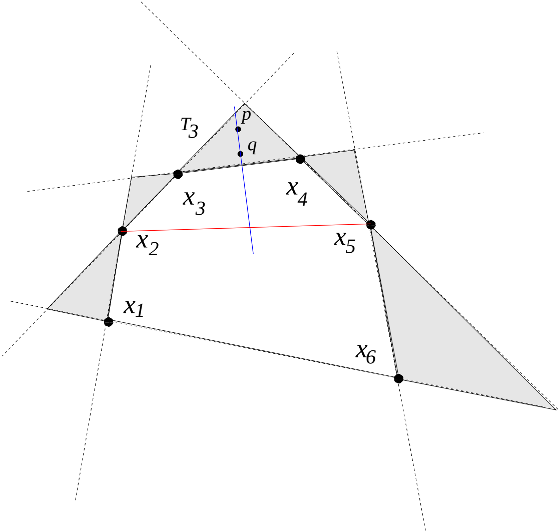
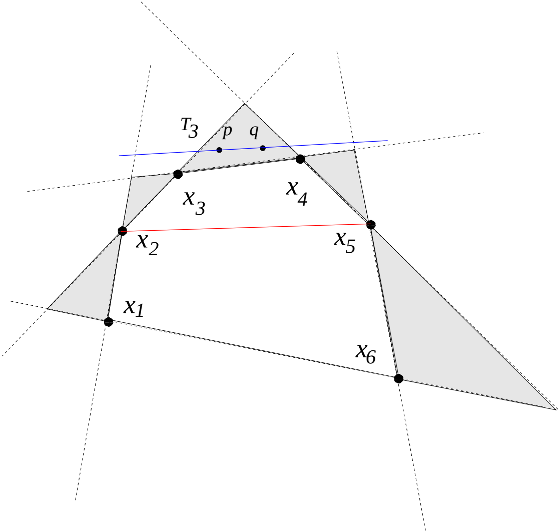


$p \prec q$



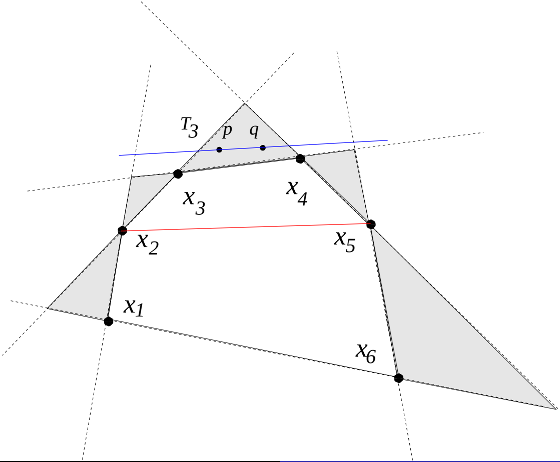
$p \not\prec q$



$N = 2^{n+2n^{3/4}}$, $k = n^{2/3}$. Set $\alpha = n^{-1/4}$.

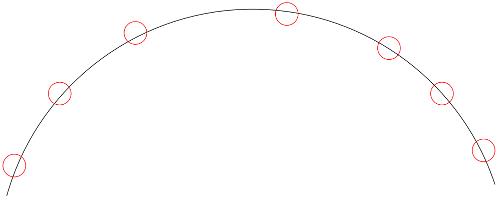
Dilworth's Theorem: Each P_i contains either

1. Antichain size $\left(\frac{N}{2^{40k}}\right)^{\alpha} = 2^{n^{3/4}+2n^{1/2}-40n^{5/12}}$
2. Chain size $\left(\frac{N}{2^{40k}}\right)^{1-\alpha} = 2^{n+n^{3/4}-40n^{2/3}-2n^{1/2}}$



Case 1: $\frac{\sqrt{k}}{2} = \frac{n^{1/3}}{2}$ of the P_i-s are nonadjacent antichains.

1. Antichain size $\left(\frac{N}{2^{40k}}\right)^{\alpha} = 2^{n^{3/4}+2n^{1/2}-40n^{5/12}} \geq f(n, 2n^{2/3})$



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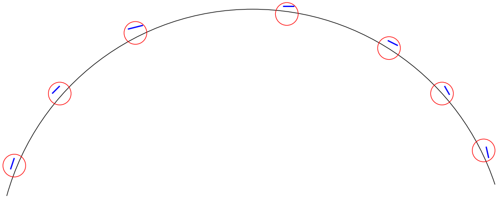

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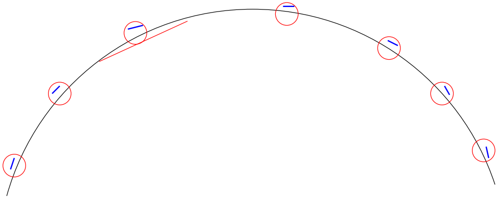

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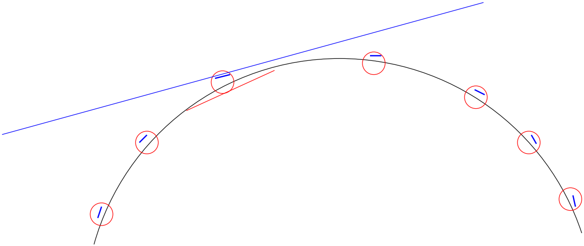
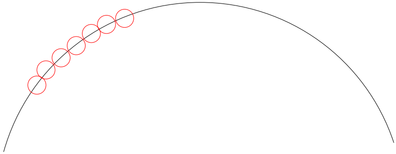


Union of all small caps is a cap (done).

Many large mutually avoiding sets

Case 2: $\sqrt{k} = n^{1/3}$ of the P_i-s are consecutive chains.

2. Chain of size $\left(\frac{N}{2^{40k}}\right)^{1-\alpha} = 2^{n+n^{3/4}-40n^{2/3}-2n^{1/2}}$



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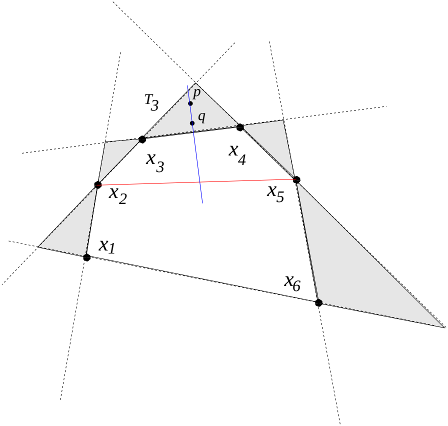

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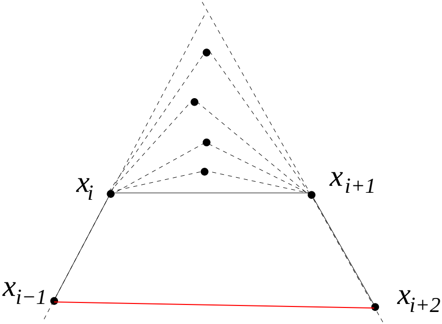

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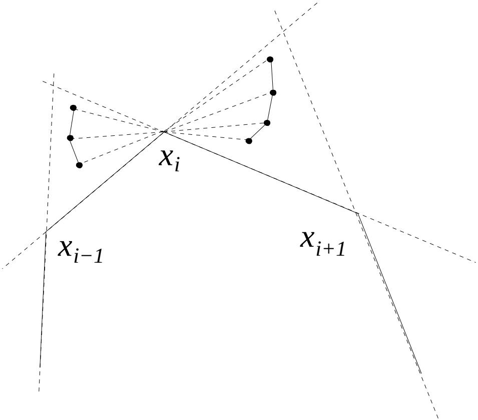

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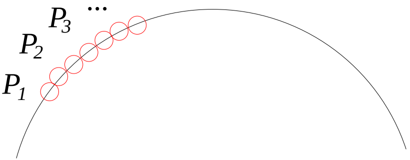
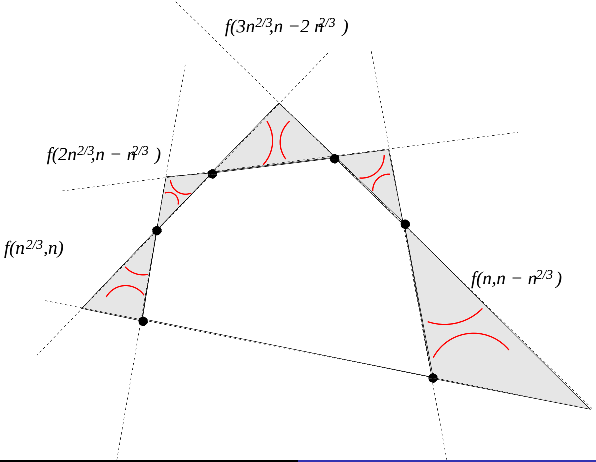


Many large mutually avoiding sets

Case 2: $\sqrt{k} = n^{1/3}$ chains of size $2^{n+n^{3/4}-40n^{2/3}-2n^{1/2}}$

$$f(in^{2/3}, n - in^{2/3} + n^{2/3}) = \binom{n + n^{2/3} - 4}{in^{2/3} - 2} + 1 \leq 2^{n+2n^{2/3}}$$



Higher dimensions

$ES_d(n)$ =smallest integer such that any set of $ES_d(n)$ points in $\mathbb{R}^d$ in general position contains n members in convex position.

Theorem (Károlyi 2001)

$$ES_d(n) \leq ES_{d-1}(n-1) + 1.$$

$$ES_d(n) \leq ES(n-d+2) + d - 2 \leq 2^{n+o(n)}.$$

Conjecture

Füredi: $ES_3(n) = 2^{c\sqrt{n}}$.

$ES_d(n) = 2^{c_d n^{1/(d-1)}}$.

Higher dimensions

Conjecture

Füredi: $ES_3(n) = 2^{c\sqrt{n}}$.

(???) $ES_d(n) = 2^{c_d n^{1/(d-1)}}$.

Theorem (Károlyi-Valtr 2003)

$$ES_d(n) \geq 2^{cn^{1/(d-1)}}.$$

$ES_d(n) \leq 2^{n+o(n)}$.

Mutually avoiding sets in $\mathbb{R}^d$

Theorem (Aronov-Erdős-Goddard-Kleitman-Klugerman-Pach-Schulman 1991)

Let P be an N-element point set in general position in $\mathbb{R}^d$. Then there are subsets $A, B \subset P$ such that $|A|, |B| \geq N^{\frac{1}{d^2-d+1}}$ and A and B are mutually avoiding.

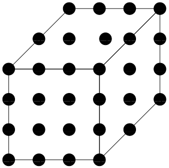
Mutually avoiding sets in $\mathbb{R}^d$

Theorem (Aronov-Erdős-Goddard-Kleitman-Klugerman-Pach-Schulman 1991)

Let P be an N-element point set in general position in $\mathbb{R}^d$, $d \geq 3$. Then there are subsets $A, B \subset P$ such that $|A|, |B| \geq N^{\frac{1}{d^2-d+1}}$ and A and B are mutually avoiding.

Theorem (Valtr 1994)

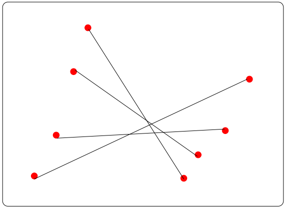
There is an N-element point set P in $\mathbb{R}^d$ in general position that does not contain a pair of mutually avoiding sets of size more than $cN^{1-\frac{1}{d}}$.



Back in the plane

Theorem (Aronov-Erdős-Goddard-Kleitman-Klugerman-Pach-Schulman 1991)

Every complete n-vertex geometric graph contains $\sqrt{n}$ pairwise crossing edges.

$K_n =$ 

Conjecture

Every complete n-vertex geometric graph contains $n^{1-\epsilon}$ pairwise crossing edges.

Ramsey approach

Conjecture

Füredi: $ES_3(n) = 2^{c\sqrt{n}}$.

(???) $ES_d(n) = 2^{c_d n^{1/(d-1)}}$.

$V = \{N \text{ points in } \mathbb{R}^d \text{ in general position}\}$
$E = \{(d+2)\text{-tuples NOT in convex position}\}$.

Theorem (Motzkin 1963)

Any set of $d+3$ vertices (points) in H induces 0,2,4 hyperedges.

$r_k(k+1, t; n) =$ smallest integer N such that every N-vertex k-uniform hypergraph H contains either $k+1$ vertices with t edges, or an independent set of size n.

$$ES_d(n) \leq r_{d+2}(d+3, 5; n).$$

$r_k(k+1, t; n)$ = smallest integer N such that every N-vertex k-uniform hypergraph H contains either $k+1$ vertices with t edges, or an independent set of size n.

Conjecture (Erdős-Hajnal 1964)

$$r_k(k+1, 5; n) = \mathrm{twr}_4(cn) = 2^{2^{2^{cn}}}.$$

$$r_k(k+1, t; n) = \mathrm{twr}_{t-1}(cn)$$

Ramsey approach

Not a good approach: $ES_d(n) \leq r_{d+2}(d+3, 5; n)$.

Theorem (Mubayi-S. 2016)

For $k \geq t+2$

$$r_k(k+1, t; n) = \mathrm{twr}_{t-1}(n^{k-t+1+o(1)})$$

$r_k^*(n) =$ smallest integer N such that every N-vertex k-uniform hypergraph H with the property that every $k+1$ vertices induces $0, 2, 4$ edges, contains an independent set of size n.

$$ES_d(n) \leq r^*_{d+2}(n) \leq r_{d+2}(d+3, 5; n).$$

Not much is known about $r_k^*(n)$.

Thank you!